MAT 226 Lecture 19 4/6/2020

Multiple Integrals

$$\int y^{-2}\,dy = \frac{y^{-2+1}}{-2+1} = -\frac{1}{y}$$

$$\int_1^x \left(2x^2y^{-2}+2y\right)dy = \left(2x^2\left(-\frac{1}{y}\right)+\frac{\cancel{2}y^2}{\cancel{2}}\right)\Big|_1^x =$$

(x here is a constant; the variable of integration is y)

$$= \left(-\frac{2x^2}{y}+y^2\right)\Big|_1^x = \underbrace{\left(-\frac{2x^2}{x}+x^2\right)}_{\text{from } x} - \underbrace{\left(-\frac{2x^2}{1}+1^2\right)}_{\text{from } 1}$$

$$= -2x+x^2+2x^2-1 = 3x^2-2x-1.$$

$$\int_1^2 \left(\int_1^x \left(2x^2y^{-2}+2y\right)dy\right)dx = \int_1^2 3x^2-2x-1\,dx$$

(we did this one first, separately)

$$= \left(x^3-x^2-x\right)\Big|_1^2 = \underbrace{(8-4-2)}_{\text{from } 2} - \underbrace{(1-1-1)}_{\text{from } 1} = 2+1 = \boxed{3}$$

Upshot:

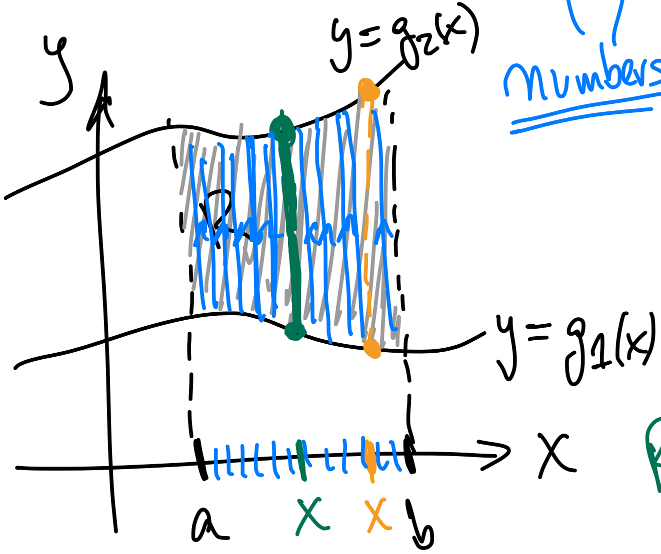
$$\int_1^2 \left(\int_1^x \left(2x^2y^{-2}+2y\right)dy\right)dx = 3$$

This is the final result, combining (iterating) of the above 2 integrals

Iterated integrals: $\boxed{\iint_R f(x,y)\, dA}$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx = \int_a^b \Bigg(\underbrace{\int_{g_1(x)}^{g_2(x)} f(x,y)\, dy}_{\text{function of } x \text{ alone (all } y\text{'s are gone once we integrate } y)} \Bigg) \underline{dx} = \text{number!}$$

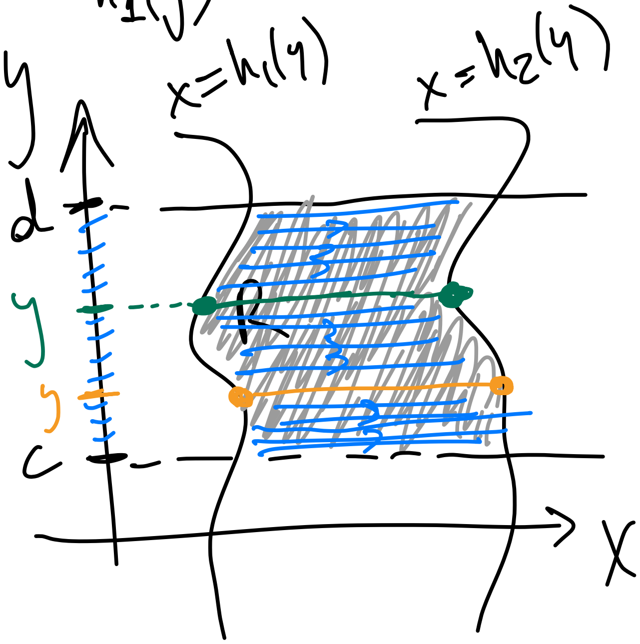
numbers!



R { For each $x \in [a,b]$, enter this region when $y = g_1(x)$, leave when $y = g_2(x)$.

$dA = dy\,dx$

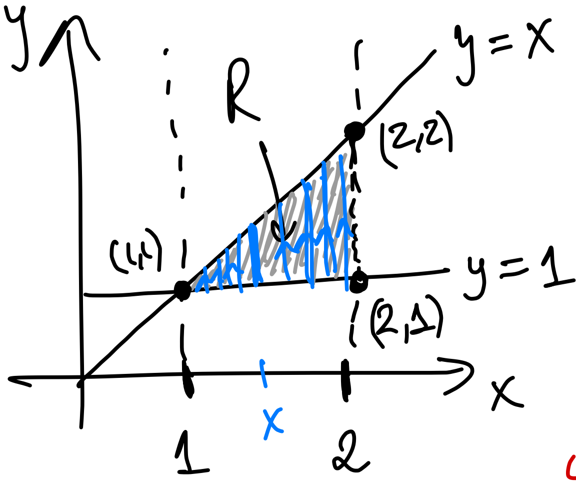
$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy = \int_c^d \Bigg(\underbrace{\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx}_{\text{function of } y \text{ alone (all } x\text{'s are gone)}} \Bigg) dy = \underline{\text{number}}$$



$dA = dx\,dy$

R { For each $y \in [c,d]$, enter region at $x = h_1(y)$ and leave when $x = h_2(y)$

Revisiting first example:



Upshot:

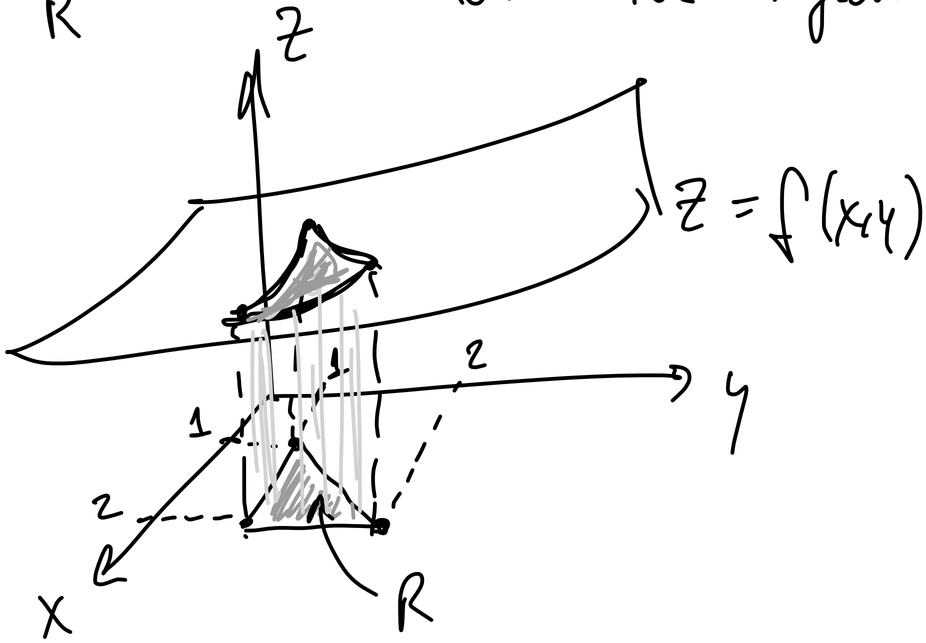
$$\int_1^2 \underbrace{\boxed{\left(\int_1^x \left(2x^2y^{-2}+2y\right)dy\right)}}_{=\,3x^2-2x-1.} dx = 3$$

$$\underbrace{\iint_R 2x^2y^{-2}+2y\,dA}_{\text{double integral of } f(x,y)=2x^2y^{-2}+2y \text{ over the region } R} = \int_1^2\int_1^x \left(2x^2y^{-2}+2y\right)\underbrace{dy\,dx}_{dA} = 3$$

R is the triangle with vertices (1,1), (2,1), and (2,2)

GEOMETRIC INTERPRETATION:

$$\iint_R f(x,y)\,dA = \left(\text{Volume under the graph } z=f(x,y) \text{ over the region } R\right)$$



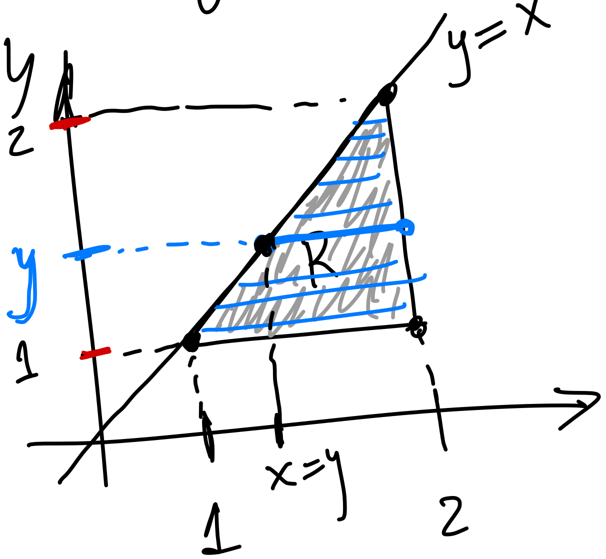
So: In the example above, the volume under the graph of $f(x,y)=\frac{2x^2}{y^2}+2y$ over the triangle R is $= 3$.

Switching order of Integration:

Upshot:

$$dA = dy\,dx$$

$$\int_1^2 \left(\int_1^x \left(2x^2y^{-2} + 2y\right) dy \right) dx = 3$$



$$\iint_R \left(2x^2y^{-2} + 2y\right) dA \qquad dA = dx\,dy$$

Describe region R:

$$\underbrace{1}_{c} \le y \le \underbrace{2}_{d}$$

$$\underbrace{y}_{h_1(y)} \le x \le \underbrace{2}_{h_2(y)}$$

$$\iint_R 2x^2y^{-2} + 2y\,dA = \int_{\underbrace{1}_{d}}^{\underbrace{2}_{c}} \int_{\underbrace{y}_{h_1(y)}}^{\underbrace{2}_{h_2(y)}} \left(2x^2y^{-2} + 2y\right) dx\,dy$$

$$= \int_1^2 \int_y^2 \left(\frac{2x^2}{y^2} + 2y\right) dx\,dy = \int_1^2 \left(\left(\frac{2}{y^2}\frac{x^3}{3} + 2yx\right)\Bigg|_y^2\right) dy$$

$$= \int_1^2 \left(\underbrace{\left(\frac{2}{y^2}\frac{2^3}{3} + 2y\cdot 2\right)}_{\text{from } x=2} - \underbrace{\left(\frac{2}{y^2}\frac{y^3}{3} + 2y^2\right)}_{\text{from } x=y}\right) dy$$

$$= \int_1^2 \frac{16}{3} y^{-2} + 4y - \frac{2}{3} y - 2y^2 \, dy$$

$$= \frac{16}{3}\left(-\frac{1}{y}\right)\Big|_1^2 + \left(4-\frac{2}{3}\right)\frac{y^2}{2}\Big|_1^2 - 2\,\frac{y^3}{3}\Big|_1^2$$

$$= \frac{16}{3}\Big(\underbrace{-\frac{1}{2}+1}_{=\frac{1}{2}}\Big) + \frac{10}{3}\Big(\underbrace{\frac{4}{2}-\frac{1}{2}}_{3/2}\Big) - 2\Big(\underbrace{\frac{8}{3}-\frac{1}{3}}_{7/3}\Big)$$

$$= \frac{8}{3} + 5 - \frac{14}{3} = \frac{8+15-14}{3} = \frac{9}{3} = \boxed{3}$$

⚠ Choosing the order of integration $dA = dx\,dy$ v. $dA = dy\,dx$ will not change the result, <u>but</u> it might make the computation easier/harder.