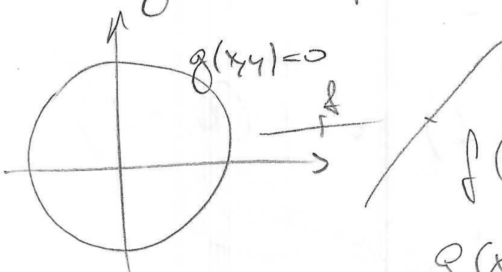
Today: Office Hours: 3 – 4pm ⚠

Reminder: Midterm Exam in 1 week (next Monday 10/28!)

Last topic: Lagrange Multipliers (Sec 13.10)

Q: What is the maximum of $f(x,y) = x^2 + 4y^2 + 2y$ among the points (x,y) that satisfy $x^2+y^2=1$?

constraint



$f(x,y) = x^2 + 4y^2 \leftarrow$ target/objective fct.

$g(x,y) = x^2+y^2-1 \leftarrow$ constraint fct.

Lagrange Multiplier Method: Find all points (x,y) such that $\exists \lambda \in \mathbb{R}$ with $\nabla f(x,y) = \lambda \cdot \nabla g(x,y)$. Evaluate $f(x,y)$ at these points. The largest / smallest values are the max/min of $f(x,y)$ restricted to the levelset $g(x,y) = c$.

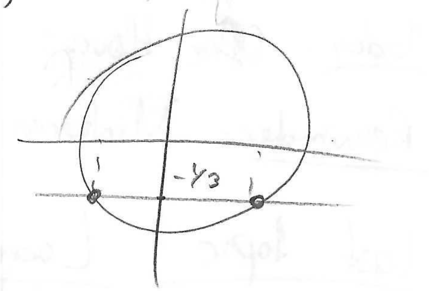
$$\left.\begin{array}{l}\nabla f(x,y) = (2x, 8y+2) \\ \nabla g(x,y) = (2x, 2y)\end{array}\right\} \quad \nabla f(x,y) = \lambda \nabla g(x,y)$$

$$\Updownarrow$$

$$\begin{cases} 2x = \lambda \cdot 2x \\ 8y+2 = \lambda \cdot 2y \end{cases} \Longleftrightarrow \begin{cases} (\lambda - 1)x = 0 \\ 4y+1 = \lambda y \end{cases}$$

$\Rightarrow \boxed{\lambda = 1}$ or $\boxed{x=0}$

$\boxed{\lambda = 1} \longrightarrow \quad 4y + 1 = y \Rightarrow 3y = -1 \Rightarrow y = -1/3$

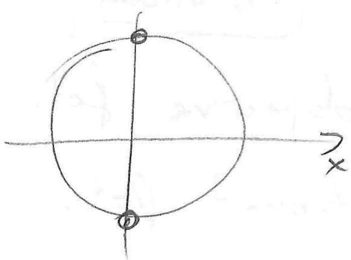


$$x^2 + \left(\frac{1}{9}\right) = 1$$

$$x^2 = \frac{8}{9} \Rightarrow x = \pm \frac{2\sqrt{2}}{3}$$

Candidates: $\left(-\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$ and $\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$

$\boxed{x = 0} \longrightarrow \quad x^2 + y^2 = 1 \Rightarrow y = \pm 1$



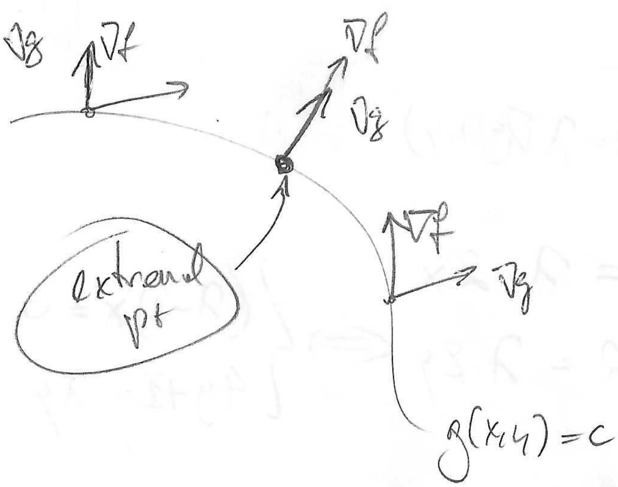
candidates: $(0,1)$ and $(0,-1)$

$$f(0,1) = 4(1)^2 + 2 = \boxed{6} \quad \text{max}$$

$$f(0,-1) = 4(-1)^2 - 2 = 2$$

$$f\left(\pm\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right) = \frac{8}{9} + 4\left(\frac{1}{9}\right) - \frac{2}{3} = \frac{12}{9} - \frac{6}{9} = \frac{6}{9} = \boxed{2/3} \quad \text{min}$$

Why does it work?



- ∇g always normal to levelsets $g(x,y) = c$
- ∇f points in the direction of max. increase.
- Constrained optimization: need ∇f and ∇g pointing in same direction!

EX: (Business). The Cobb-Douglas production function for a certain company is $f(x,y) = 100\, x^{3/4} y^{1/4}$,

"output elasticities"

where $\begin{cases} x = \text{units of labor } (\$150/\text{unit}) \\ y = \text{units of capital } (\$250/\text{unit}) \end{cases}$

total cost of labor and capital is limited to \$50,000.

How many units of labor & capital maximize production? $\boxed{\cancel{f(250,50) = 16{,}719} \text{ product units}}$

Sol. Target fct: $f(x,y) = 100\, x^{3/4} y^{1/4}$

Constraint fct: $g(x,y) = 150x + 250y = 50{,}000$

$$\nabla f(x,y) = \left(100 \cdot \frac{3}{4} x^{-1/4} y^{1/4},\ 100 \cdot \frac{1}{4} x^{3/4} y^{-3/4}\right)$$

$$= \left(75\, x^{-1/4} y^{1/4},\ 25\, x^{3/4} y^{-3/4}\right)$$

$$\nabla g(x,y) = (150, 250)$$

$$\nabla f(x,y) = \lambda \nabla g(x,y) \iff \begin{cases} 75 x^{-1/4} y^{1/4} = 150\lambda \\ 25 x^{3/4} y^{-3/4} = 250\lambda \end{cases}$$

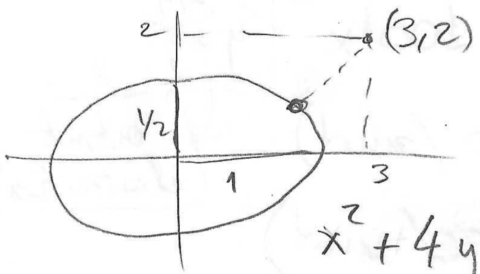
From 1st eqns: $\lambda = \frac{75}{150} x^{-1/4} y^{1/4} = \frac{x^{-1/4} y^{1/4}}{2}$

Plug in to 2nd eq: $\frac{25}{250} x^{3/4} y^{-3/4} = \lambda = \frac{x^{-1/4} y^{1/4}}{2} \implies \boxed{\frac{1}{5} x = y}$

$150x + 250y = 50{,}000 \implies 150x + 50x = 50{,}000 \implies x = \frac{50000}{200} = \underline{\underline{250}}$

$\boxed{\begin{matrix} x = 250 \\ y = 50 \end{matrix}}$

Ex: A planet has elliptical orbit $x^2 + 4y^2 = 1$.



$x^2 + 4y^2 = 1$

Aliens living in this planet want to capture a spaceship at the point $(3,2)$. What is the closest point in their orbit to the spaceship?

square of distance to $(3,2)$

Sol: Target fct: $f(x,y) = (x-3)^2 + (y-2)^2$

Constraint fct: $g(x,y) = x^2 + 4y^2 = 1$

$\nabla f(x,y) = \big(2(x-3),\ 2(y-2)\big) \qquad \nabla g(x,y) = (2x, 8y)$

$$\nabla f(x,y) = \lambda \nabla g(x,y) \iff \begin{cases} x-3 = \lambda \cdot x \\ y-2 = \lambda 4y \end{cases}$$

$(1-\lambda)x = 3 \Rightarrow \boxed{x = \frac{3}{1-\lambda}} \qquad (1-4\lambda)y = 2 \Rightarrow \boxed{y = \frac{2}{1-4\lambda}}$

$$x^2 + 4y^2 = 1 \Rightarrow \left(\frac{3}{1-\lambda}\right)^2 + 4\cdot\left(\frac{2}{1-4\lambda}\right)^2 = 1$$

$$\Rightarrow \frac{9}{(1-\lambda)^2} + \frac{16}{(1-4\lambda)^2} = 1$$

$$\frac{9(1-4\lambda)^2 + 16(1-\lambda)^2}{(1-\lambda)^2(1-4\lambda^2)} = 1 \Rightarrow \cdots \boxed{\text{see picture}}$$

min $f(x,y) = xy$

subject to $g(x,y) = \frac{x^2}{2} + \frac{y^2}{3} = 1$

$\nabla f = (y, x)$, $\nabla g = \left(x, \frac{2y}{3}\right)$

$$\begin{cases} y = \lambda x \\ x = \frac{2}{3}\lambda y \end{cases}$$

Substitute

$\Rightarrow x = \frac{2}{3}\lambda^2 x$

$x = 0$ or $\boxed{\lambda = \pm\sqrt{\frac{3}{2}}}$

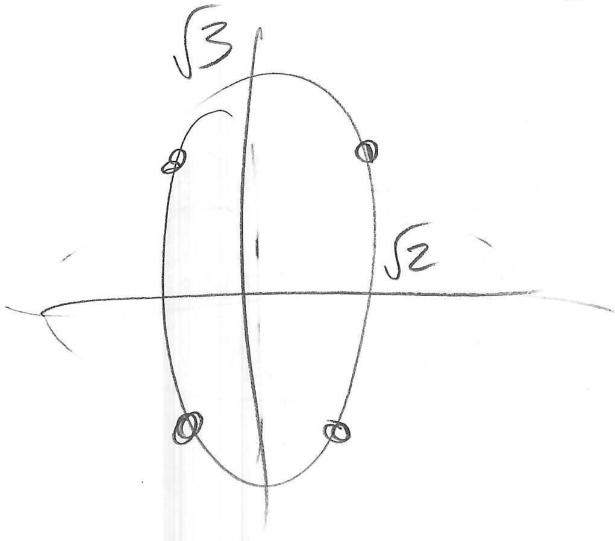
$\Downarrow$

$y = 0$

$\Rightarrow y = \pm\sqrt{\frac{3}{2}}\, x$

$\Rightarrow \frac{x^2}{2} + \frac{3}{2\cdot 3}x^2 = x^2 = 1 \Rightarrow x = \pm 1$

$y = \pm\sqrt{\frac{2}{3}}$



$\boxed{\left(1, \sqrt{\frac{3}{2}}\right), \left(-1, -\sqrt{\frac{3}{2}}\right)}$ ← max

$\boxed{\left(1, -\sqrt{\frac{3}{2}}\right), \left(-1, \sqrt{\frac{3}{2}}\right)}$ ← min