Critical points

Def: p is a critical point for $f(x)$ if $\nabla f(p)=0$ or if $f(x)$ is not differentiable at p.

Ex: $f(x,y)=2x^2+y^2+8x-6y+20$

$\nabla f(x,y)=(4x+8,\ 2y-6)=(0,0) \iff \begin{array}{l} x=-2 \\ y=3 \end{array}$.

$p=(-2,3)$ is the only critical point.

Geometrically: Tangent plane to graph(f) is parallel to xy-plane: $F(x,y,z)=z-f(x,y)$

$$\vec{n}=\left(-\frac{\partial f}{\partial x},\ -\frac{\partial f}{\partial y},\ 1\right)=(0,0,1)$$

if $\nabla f(p)=0$

Ex: $f(x,y,z)=x^2+y^2+2z^2+1$

$\nabla f(x,y,z)=(2x,\ 2y,\ 4z)=(0,0,0) \iff x=y=z=0$.

$p=\vec{0}$ is the only critical pt.

Ex: $f(x,y) = -x^3 + 4xy - 2y^2 + 1$

$\nabla f(x,y) = \left(-3x^2+4y,\ 4x-4y\right) = (0,0)$

$\iff \begin{cases} x = y \\ -3x^2+4x=0 \Rightarrow x=0 \text{ or } x = 4/3 \\ \qquad\qquad\qquad\quad\ y=0 \qquad\ \ y = 4/3 \end{cases}$

There are 2 critical points. $(0,0)$ and $\left(\frac{4}{3}, \frac{4}{3}\right)$.

Local extreme are always among critical points.

Q: How to decide if a crit. pt. is loc max/min, none?

Second Derivative Test.

Suppose $p=(x_0,y_0)$ is a critical point of $f(x,y)$.

$$\text{Hess } f(p) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

1. If $\det(\text{Hess } f(p)) > 0$ and $\frac{\partial^2 f}{\partial x^2}(p) > 0$, then p is a local min
2. If $\det(\text{Hess } f(p)) > 0$ and $\frac{\partial^2 f}{\partial x^2}(p) < 0$, then p is a local max
3. If $\det(\text{Hess } f(p)) < 0$, then p is a saddle.
4. If $\det \text{Hess } f(p) = 0$ then inconclusive

In the previous example:

$$\text{Hess } f = \begin{pmatrix} -6x & 4 \\ 4 & -4 \end{pmatrix}$$

$$(\text{Hess } f)(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & -4 \end{pmatrix} \qquad \det(\text{Hess } f)(0,0) = -16 < 0$$

$$\Rightarrow \boxed{(0,0) \text{ is a } \underline{\text{saddle point}}}$$

$$(\text{Hess } f)\left(\tfrac{4}{3}, \tfrac{4}{3}\right) = \begin{pmatrix} -8 & 4 \\ 4 & -4 \end{pmatrix} \qquad \det \text{Hess } f\left(\tfrac{4}{3}, \tfrac{4}{3}\right) = 32 - 16 = 16 > 0$$

$$\frac{\partial^2 f}{\partial x^2}(p) = -8 < 0$$

$$\Rightarrow \left(\tfrac{4}{3}, \tfrac{4}{3}\right) \text{ is a } \underline{\text{loc. max}}.$$

Ex: $f(x,y) = 4 + x^3 + y^3 - 3xy$ $\qquad (0,0) \leftarrow$ saddle

$\qquad (1,1) \leftarrow$ rel. min.

$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

$(0,0)$ rel max

$(0,2)$ rel min

$(-1,1)$ & $(1,1)$ saddle