Name: ANSWERS Lehman ID: ______________

MAT 226
Midterm Exam
March 29-30, 2020

Instructions (PLEASE READ CAREFULLY):
Turn off and put away your cell phone.
Please write your Name and Lehman ID # on the top of this page.
Please sign and date the pledge below to comply with the Code of Academic Integrity.
No consultation material, calculators, or electronic devices are allowed during the exam.
If anything is unclear, send an email to `r.bettiol@lehman.cuny.edu` *for clarifications.*
The amount of time you have to complete the exam is 100 minutes, unless you have a recognized disability. **You must show all of your work! No credit will be given for unsupported answers.** *Please try to be as organized, objective, and logical as possible in your answers.*
Submit your completed exam by 11:59pm on Monday, March 30 through the Blackboard Assignment "Midterm Exam" and be sure to attach images of all the pages.

#	**Points**	**Score**
1	15	
2	15	
3	10	
4	15	
5	10	
6	10	
7	10	
8	5	
9	10	
Total	100	

My signature below certifies that I complied with the CUNY Academic Integrity Policy and the Lehman College Code of Academic Integrity in completing this examination.

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Signature Date

Problem 1 (15 pts): Consider the vectors $\vec{v} = (2, 1, 5)$ and $\vec{w} = (-1, 0, 1)$.
Compute the following quantities:

a) **(5 pts)** $2\,\vec{v} + 4\,\vec{w}$

$$= 2\,(2,1,5) + 4\,(-1,0,1)$$

$$= (4,2,10) + (-4,0,4)$$

$$= \boxed{(0,2,14)}$$

b) **(5 pts)** $\langle \vec{v}, \vec{w} \rangle$

$$= \langle (2,1,5),(-1,0,1) \rangle$$

$$= -2 + 0 + 5 = \boxed{3}$$

c) **(5 pts)** $\vec{v} \times \vec{w}$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 5 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= (1, -2-5, 1) = \boxed{(1,-7,1)}$$

Problem 2 (15 pts): Consider the curve $\vec{r}(t) = t^4\,\mathbf{i} + t^3\,\mathbf{j} + t^2\,\mathbf{k}$

a) **(5 pts)** Compute the velocity vector of this curve, that is, the derivative $\vec{r}\,'(t)$

$$\vec{r}\,'(t) = 4t^3\,\hat{i} + 3t^2\,\hat{j} + 2t\,\hat{k}$$

b) **(5 pts)** What is the equation of the **tangent line** to this curve at $t = 1$?

$$\big(x(t), y(t), z(t)\big) = \vec{r}(1) + t\,\vec{r}\,'(1)$$

$$= (1,1,1) + t(4,3,2)$$

$$= \boxed{(4t+1,\ 3t+1,\ 2t+1)}$$

c) **(5 pts)** What is the equation of the **normal plane** to this curve at $t = 1$?

$$\langle (x-1, y-1, z-1), \vec{r}\,'(1) \rangle = 0$$

$$\langle (x-1, y-1, z-1), (4,3,2) \rangle = 0$$

$$4x - 4 + 3y - 3 + 2z - 2 = 0$$

$$\boxed{4x + 3y + 2z - 9 = 0}$$

Problem 3 (10 pts): Consider $\gamma(t) = \big(t\sin t + \cos t,\, \sin t - t\cos t,\, \sqrt{6}\,t^2\big), \quad t \geq 0.$

a) **(5 pts)** Compute (and simplify!) the length $\|\gamma'(t)\|$ of the tangent vector of $\gamma(t)$.

$$\gamma'(t) = \left(\sin t + t\cos t - \sin t,\ \cos t - \cos t + t\sin t,\ 2\sqrt{6}\,t\right)$$

$$= \left(t\cos t,\ t\sin t,\ 2\sqrt{6}\,t\right)$$

$$\|\gamma'(t)\| = \sqrt{(t\cos t)^2 + (t\sin t)^2 + (2\sqrt{6}\,t)^2}$$

$$= \sqrt{t^2\,(\underbrace{\cos^2 t + \sin^2 t}_{1}) + 4.6\,t^2}$$

$$= \sqrt{t^2 + 24t^2}$$

$$= \boxed{5t} \qquad (t \geq 0)$$

b) **(5 pts)** Use the above to compute the arclength of $\gamma(t)$ from $t = 0$ to $t = \pi$.

$$\ell = \int_0^{\pi} \|\gamma'(t)\|\,dt = \int_0^{\pi} 5t\,dt = 5\,\frac{t^2}{2}\Big|_0^{\pi} = \boxed{\frac{5\pi^2}{2}}$$

Problem 4 (15 pts): Consider the function $f(x, y, z) = z^2y + 3\cos(xz) - xe^{2y}$.

a) **(5 pts)** Compute the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.

$$\frac{\partial f}{\partial x}(x, y, z) = -3\sin(xz)z - e^{2y}$$

$$\frac{\partial f}{\partial y}(x, y, z) = z^2 - 2xe^{2y}$$

$$\frac{\partial f}{\partial z}(x, y, z) = 2zy - 3\sin(xz)x$$

b) **(5 pts)** Compute the second partial derivatives $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial y \partial z}$, and $\frac{\partial^2 f}{\partial z^2}$.

$$\frac{\partial^2 f}{\partial y^2}(x, y, z) = -4xe^{2y}$$

$$\frac{\partial^2 f}{\partial y \partial z}(x, y, z) = 2z$$

$$\frac{\partial^2 f}{\partial z^2}(x, y, z) = 2y - 3x^2\cos(xz)$$

c) **(5 pts)** Compute the directional derivative $\frac{\partial f}{\partial \vec{v}}(0, 0, 0)$ in the direction of $\vec{v} = \left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$.

$$\frac{\partial f}{\partial \vec{v}}(0,0,0) = \left\langle \nabla f(0,0,0), \left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right) \right\rangle$$

$$= \left\langle (-1, 0, 0), \left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right) \right\rangle$$

$$= \boxed{-\frac{1}{2}}$$

Problem 5 (10 pts): A foundational result in telecommunications is the *Shannon–Hartley Law*, which determines the maximum rate at which information can be sent through a communication channel in presence of noise. It states that:

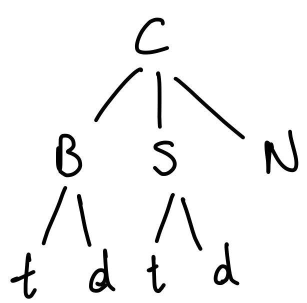
$$C = B\ \ln\left(1 + \frac{S}{N}\right),$$

where C is the channel capacity, B is the bandwidth of the channel, S is the (average) received signal power, and N is the (average) power of the noise and interference. Assume that the bandwidth and received signal power of a Wi-Fi router are functions of time t and distance d to the router's antenna, that is, $B = B(t, d)$ and $S = S(t, d)$, while the noise N is constant.

(In reality, $\log_2$ should be used instead of ln, but here we are using ln to simplify the computations.)

a) **(5 pts)** Compute the rate of change of channel capacity over time for this router. Your answer **must only involve** B, S, and their partial derivatives, and N.

$$\frac{\partial C}{\partial t} = \frac{\partial C}{\partial B}\frac{\partial B}{\partial t} + \frac{\partial C}{\partial S}\frac{\partial S}{\partial t}$$



$$= \ln\left(1 + \frac{S}{N}\right)\frac{\partial B}{\partial t} + B\,\frac{1}{1 + \frac{S}{N}} \cdot \frac{1}{N}\,\frac{\partial S}{\partial t}$$

$$\boxed{= \ln\left(1 + \frac{S}{N}\right)\frac{\partial B}{\partial t} + \frac{B}{N + S}\,\frac{\partial S}{\partial t}}$$

b) **(5 pts)** Compute the rate of change of channel capacity over distance for this router. Your answer **must only involve** B, S, and their partial derivatives, and N.

$$\frac{\partial C}{\partial d} = \frac{\partial C}{\partial B}\frac{\partial B}{\partial d} + \frac{\partial C}{\partial S}\frac{\partial S}{\partial d}$$

$$\boxed{= \ln\left(1 + \frac{S}{N}\right)\frac{\partial B}{\partial d} + \frac{B}{N+S}\,\frac{\partial S}{\partial d}}$$

Problem 6 (10 pts): Consider the function $f(x,y) = x^3 - 3xy + y^3 + 1$

a) **(5 pts)** Find all the critical points of $f(x,y)$.

$$\nabla f(x,y) = \left(3x^2 - 3y,\ -3x - 3y^2\right) = (0,0) \iff$$

$$\iff \begin{cases} x^2 - y = 0 \implies y = x^2 \\ x + y^2 = 0 \end{cases} \longrightarrow x + x^4 = 0 \begin{cases} \boxed{x=0} \\ \text{or} \\ 1 + x^3 = 0 \\ x^3 = -1 \\ \boxed{x = -1} \end{cases}$$

If $x=0$: $y = 0^2 = 0$.

If $x=1$: $y = 1^2 = 1$

So the critical points are: $\boxed{(0,0) \text{ and } (1,1)}$

b) **(5 pts)** Classify these critical points into local minima, local maxima, and saddles.

$$\text{Hess} f(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & -6y \end{pmatrix}$$

$$\text{Hess} f(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \implies \det \text{Hess} f(0,0) = -9 < 0$$

$$\implies \boxed{(0,0) \text{ is a saddle}}$$

$$\text{Hess} f(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \implies \det \text{Hess} f(1,1) = 36 - 9 > 0$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = 6 > 0$$

$$\implies \boxed{(1,1) \text{ is a local minimum}}$$

Problem 7 (10 pts): Consider the paraboloid given by $z = 3x^2 + y^2$. Find the equation of the **tangent plane** to this paraboloid at the following points (x_0, y_0, z_0):

a) **(5 pts)** $(x_0, y_0, z_0) = (1, 0, 3)$

$F(x,y,z) = 3x^2 + y^2 - z \Rightarrow \nabla F(x,y,z) = (6x, 2y, -1)$

$\nabla F(1,0,3) = (6, 0, -1)$

Tangent plane at $(1,0,3)$: $\langle (x-1, y, z-3), (6,0,-1) \rangle = 0$

$6(x-1) - z + 3 = 0$

$\boxed{6x - z - 3 = 0}$

b) **(5 pts)** $(x_0, y_0, z_0) = (2, -1, 13)$

$\nabla F(2,-1,13) = (12, -2, -1)$

Tangent plane at $(2,-1,13)$:

$\langle (x-2, y+1, z-13), (12, -2, -1) \rangle = 0$

$12(x-2) - 2(y+1) - z + 13 = 0$

$12x - 24 - 2y - 2 - z + 13 = 0$

$\boxed{12x - 2y - z - 13 = 0}$

Problem 8 (5 pts): Compute the following limit, or explain why it does not exist: **(Remember you must justify your answer!)**

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} - \frac{y^3}{x^2+y^2}$$

Squeeze Theorem

$$= \lim_{(x,y)\to(0,0)} \underset{\substack{\downarrow \\ 0}}{x} \underbrace{\frac{x^2}{x^2+y^2}}_{\text{bounded}} - \underset{\substack{\downarrow \\ 0}}{y} \underbrace{\frac{y^2}{x^2+y^2}}_{\text{bounded}} = 0$$

Problem 9 (10 pts): Find the equation in polar coordinates, that is, in the form $r = r(\theta)$, for the curve given in Euclidean coordinates by

$$\left(x^2 + y^2\right)^{\frac{3}{2}} = 3\left(x^2 - y^2\right).$$

For 2 extra points: sketch a plot of this curve!

Substitute $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$x^2 + y^2 = r^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = r^2$$

$$x^2 - y^2 = r^2\underbrace{(\cos^2\theta - \sin^2\theta)}_{\cos 2\theta} = r^2\cos 2\theta$$

$$\Longrightarrow \quad (r^2)^{\frac{3}{2}} = 3(r^2\cos 2\theta)$$

$$\Longrightarrow \quad r^3 = 3r^2\cos 2\theta$$

$(r \neq 0)$ $\Longrightarrow$ $\boxed{r = 3\cos 2\theta}$