Name: ANSWERS ______________________ Lehman ID: ______________

MAT 226
Final Exam
May 18, 2020

Instructions (PLEASE READ CAREFULLY):

• *Please sign and date the pledge below to comply with the Code of Academic Integrity.*

• *This is an* **OPEN BOOK EXAM***, meaning that:*
– YOU ARE ALLOWED to consult any references, notes, and class materials.
– YOU ARE NOT ALLOWED to share the contents (in part or in full) of this exam in any forum or tutoring service (such as Chegg, Humbot, Skooli, math.stackexchange, etc.), nor to receive any kind of third-party assistance during the exam. Any violations of the academic integrity code will be fully investigated and penalized as appropriate.

• *Some questions in this exam require you to generate your OWN UNIQUE INPUT. This is a preventative safeguard to protect students that comply with the academic integrity code. If 2 or more students present the EXACT SAME INPUT to such questions, they will receive ZERO POINTS on that question, even if their answers are correct. Your ability to generate such inputs on your own is part of what is being evaluated.*

• *If anything is unclear, email me at* `r.bettiol@lehman.cuny.edu` *for clarifications.*

• *The amount of time you have to complete the exam is 100 minutes, unless you have a recognized disability.* **You must show all of your work! No credit will be given for unsupported answers.** *Please try to be as organized, objective, and logical as possible in your answers.*

• Submit your completed exam by 11:59pm on Monday, May 18 through the Blackboard Assignment "Final Exam" and be sure to attach clear images of all the pages.

My signature below certifies that I complied with the CUNY Academic Integrity Policy and the Lehman College Code of Academic Integrity in completing this examination.

______________________________ ______________
Signature Date

Problem 1 (20 pts): Let $\vec{r}\colon \mathbb{R} \to \mathbb{R}^3$ be the smooth curve given by

(Fill in with YOUR OWN UNIQUE INPUT, making sure it is a well-defined regular curve in $\mathbb{R}^3$)

$$\vec{r}(t) = \Big(\underbrace{\qquad\qquad\qquad}_{x(t)}, \underbrace{\qquad\qquad\qquad}_{y(t)}, \underbrace{\qquad\qquad\qquad}_{z(t)}\Big)$$

a) **(4 pts)** Compute the velocity vector of this curve, that is, the derivative $\vec{r}\,'(t)$.

$r'(t) = (x'(t), y'(t), z'(t))$

b) **(8 pts)** What is the equation of the **tangent line** to the curve $\vec{r}(t)$ at $t = 0$?

$\vec{r}(0) + t\vec{r}\,'(0)$

c) **(8 pts)** What is the equation of the **normal plane** to the curve $\vec{r}(t)$ at $t = 0$?

$\langle (x,y,z) - \vec{r}(0), \vec{r}\,'(0) \rangle = 0.$

Problem 2 (20 pts): Consider the smooth function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by

(Fill in with YOUR OWN UNIQUE INPUT, making sure it is a well-defined smooth function)

$f(x, y, z) =$

a) **(4 pts)** Compute the gradient vector field $\nabla f(x, y, z)$.

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

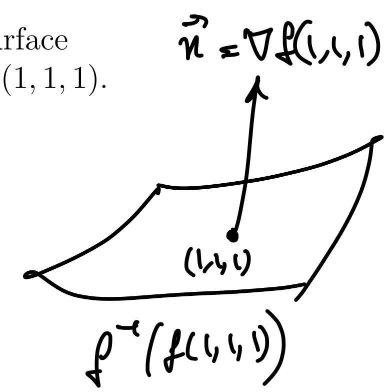
b) **(4 pts)** Find the value of the gradient vector at $(x_0, y_0, z_0) = (1, 1, 1)$.

$\nabla f(1, 1, 1) =$

Warning: If your answer to b) is $\nabla f(1, 1, 1) = (0, 0, 0)$, then you must go back and change your input $f(x, y, z)$, otherwise the next question will not be feasible!

c) **(12 pts)** Write the equation of the **tangent plane** to the levelset surface $S = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = f(1, 1, 1)\}$ at the point $(x_0, y_0, z_0) = (1, 1, 1)$.

$$\langle (x-1, y-1, z-1), \nabla f(1,1,1) \rangle = 0.$$



Problem 3 (20 pts): Consider the vector field

$$\vec{F}(x,y) = \left(e^{x+y} - 2x, e^{x+y} + 2\cos(2y)\right),$$

and the following points $P, Q \in \mathbb{R}^2$

(Fill in with YOUR OWN UNIQUE INPUT, making sure these are coordinates of points in $\mathbb{R}^2$*)*

$$P = \Big(\underbrace{\qquad\qquad}_{x_1}, \underbrace{\qquad\qquad}_{y_1}\Big)$$

$$Q = \Big(\underbrace{\qquad\qquad}_{x_2}, \underbrace{\qquad\qquad}_{y_2}\Big)$$

Compute the line integral $\int_\gamma \vec{F}\, d\gamma$, where γ is the straight line segment from P to Q.

Hint: Check if $\vec{F}$ is conservative.

$M = e^{x+y} - 2x$

$N = e^{x+y} + 2\cos 2y$

$$\begin{cases} \dfrac{\partial M}{\partial y} = e^{x+y} = \dfrac{\partial N}{\partial x} \\ \mathbb{R}^2 \text{ is simply-connected} \end{cases} \Rightarrow \vec{F} \text{ is conservative!}$$

Finding a potential $f: \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f = \vec{F}$.

$$f(x,y) = \int e^{x+y} - 2x \, dx + g(y) = e^{x+y} - x^2 + g(y)$$

$$\frac{\partial f}{\partial y} = e^{x+y} + g'(y) \stackrel{!}{=} e^{x+y} + 2\cos 2y \Rightarrow g'(y) = 2\cos 2y$$

$$\Rightarrow g(y) = \sin 2y + c.$$

$$\boxed{f(x,y) = e^{x+y} - x^2 + \sin 2y + c}$$

$$\int_\gamma \vec{F}\, d\gamma = f(Q) - f(P)$$

presumably will be chosen simply-connected.

Problem 4 (10 pts): Consider the smooth vector field $\vec{F}\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ given by

(Fill in with YOUR OWN UNIQUE INPUT, making sure it is a well-defined smooth vector field)

$$\vec{F}(x,y,z) = \Big(\underbrace{\qquad\qquad\qquad}_{M(x,y,z)}, \underbrace{\qquad\qquad\qquad}_{N(x,y,z)}, \underbrace{\qquad\qquad\qquad}_{P(x,y,z)} \Big)$$

a) **(5 pts)** Compute the curl $\nabla \times \vec{F}$ of the above vector field. Simplify your answer.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

b) **(5 pts)** Is the vector field $\vec{F}$ conservative? Justify.

Yes if $\nabla \times \vec{F} = (0,0,0)$, no otherwise.

(b/c Ω is simply-connected, $\vec{F}$ is conservative if and only if $\nabla \times \vec{F} = 0$.)

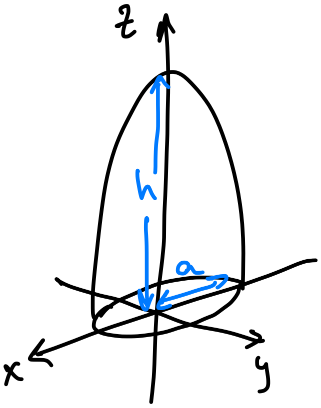
if Ω is not simply-connected, need to analyze case by case...

Problem 5 (20 pts): A chocolate factory is planning to launch a new chocolate truffle, which will have the shape of the region of $\mathbb{R}^3$ given by

$$0 \leq z \leq h\left(1 - \frac{x^2+y^2}{a^2}\right),$$

where the values of $a > 0$ and $h > 0$ are to be determined.

a) **(10 pts)** Sketch the above region, indicating the geometric meaning of h and a. Use a double or triple integral to find its volume $V(a,h)$.



$$V(a,h) = \int_0^{2\pi}\int_0^a\int_0^{h\left(1-\frac{r^2}{a^2}\right)} dz\, r\,dr\,d\theta$$

$$= 2\pi h \int_0^a r - \frac{r^3}{a^2}\,dr = 2\pi h\left(\frac{r^2}{2} - \frac{r^4}{4a^2}\right)\Big|_0^a$$

$$= 2\pi h\left(\frac{a^2}{2} - \frac{a^2}{4}\right) = \boxed{\frac{\pi h a^2}{2}}$$

b) **(10 pts)** Due to packaging restrictions, the dimensions of the chocolate truffle must satisfy $2a + h = 5$. Find the values of a and h that satisfy this constraint and maximize the volume $V(a,h)$ of the proposed chocolate truffle.

$$c(a,h) = 2a + h = 5 \Rightarrow \nabla c(a,h) = (2,1).$$

$$\nabla V(a,h) = \left(\pi h a, \frac{\pi a^2}{2}\right)$$

$$\nabla V = \lambda \cdot \nabla c \iff \begin{cases} \pi h a = 2\lambda \\ \frac{\pi a^2}{2} = \lambda \end{cases} \iff \frac{\lambda}{\pi} = \frac{a^2}{2} = \frac{ha}{2}$$

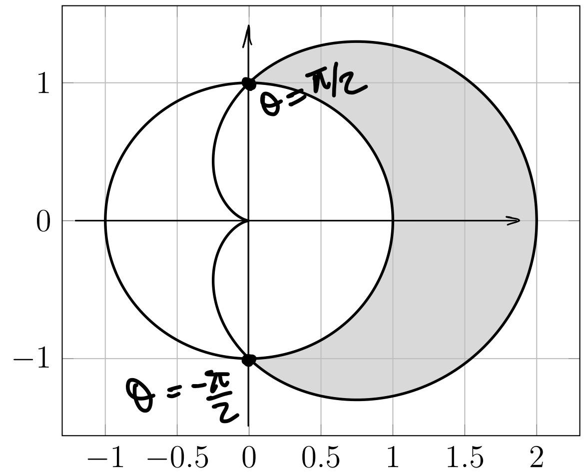
$$\overset{a>0}{\Rightarrow} \boxed{a = h}$$

$$3a = 5 \Rightarrow \boxed{a = h = \frac{5}{3}}$$

Max vol:

$$V\left(\tfrac{5}{3}, \tfrac{5}{3}\right) = \frac{\pi}{2}\left(\frac{5}{3}\right)^3 = \boxed{\frac{125\pi}{54}}$$

Problem 6 (10 pts): Use double integrals in polar coordinates to find the area outside the unit circle that lies inside the cardioid $r(\theta) = 1 + \cos(\theta)$, as shown below.



Unit circle: $r(\theta) = 1$

Cardioid: $r(\theta) = 1 + \cos\theta$

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} r\,dr\,d\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^2}{2}\Big|_{1}^{1+\cos\theta} d\theta = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta + 2\cos\theta \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\, d\theta + \left(\sin\theta\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left(\theta + \frac{\sin 2\theta}{2}\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + 2 = \frac{1}{4}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + 2$$

$$= \frac{\pi}{4} + 2 = \boxed{\frac{\pi + 8}{4}}$$