MAT 226

Solutions to HW 11

a) $\vec{F}(x,y) = (xy^2, x^2y)$, $\Omega = \mathbb{R}^2$ is simply-connected

$$\frac{\partial}{\partial y}(xy^2) = 2xy = \frac{\partial}{\partial x}(x^2y) \Rightarrow \underline{\vec{F} \text{ is conservative}}$$

$$f(x,y) = \int xy^2\,dx + g(y) = \frac{1}{2}x^2y^2 + g(y)$$

$$\frac{\partial f}{\partial y}(x,y) = x^2y + g'(y) = x^2y \Rightarrow g'(y) = 0 \Rightarrow g(y) = c.$$

$$\boxed{\underline{\text{Potential function:}}\quad f(x,y) = \frac{1}{2}x^2y^2 + c}$$

b) $\vec{F}(x,y,z) = (ye^x, 2y\sin z, x+z)$, $\Omega = \mathbb{R}^3$ is simply-connected

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^x & 2y\sin z & x+z \end{vmatrix}$$

$$= (0 - 2y\cos z, -1, -e^x) \neq 0$$

$\Rightarrow \vec{F}$ is <u>not</u> conservative.

c) $\vec{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$,

$\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$ is <u>not</u> simply-connected.

$$\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2} \right) = -\frac{2xy}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2} \right)$$

Since Ω is <u>not</u> simply-connected, it is (a priori) not clear if $\vec{F}$ is conservative. Let us try to find a potential anyways:

$$f(x,y) = \int \frac{x}{x^2+y^2}\, dx + g(y) = \frac{1}{2}\int \frac{2x}{x^2+y^2}\, dx + g(y)$$

$$= \frac{1}{2} \ln\left(x^2+y^2\right) + g(y)$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{2} \frac{2y}{x^2+y^2} + g'(y) = \frac{y}{x^2+y^2} \Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = C.$$

$$\boxed{f(x,y) = \frac{1}{2} \ln\left(x^2+y^2\right) + C}$$

Verify that it is a <u>potential</u>, that is, $\nabla f = \vec{F}$:

$$\nabla f(x,y) = \left(\frac{1}{2} \frac{2x}{x^2+y^2}, \frac{1}{2} \frac{2y}{x^2+y^2} \right) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right) = \vec{F}(x,y)$$

<u>So $\vec{F}$ is conservative</u> and has potential $f(x,y) = \frac{1}{2}\ln(x^2+y^2) + C$