Name: ANSWERS Lehman ID: ____________

MAT 176
Midterm 1
April 1, 2019

Instructions:
Turn off and put away your cell phone.
Please write your Name and Lehman ID # on the top of this page.
Please sign and date the pledge below to comply with the Code of Academic Integrity.
No consultation material, calculators, or electronic devices are allowed during the exam.
If any question is unclear, raise your hand to ask for clarifications.
The regular amount of time you have to complete the exam is 100 minutes.
You must show all of your work! *No credit will be given for unsupported answers.*
Please try to be as organized, objective, and logical as possible in your answers.

#	Points	Score
1	20	
2	20	
3	15	
4	15	
5	10	
6	10	
7	10	
Total	100	

My signature below certifies that I complied with the CUNY Academic Integrity Policy and the Lehman College Code of Academic Integrity in completing this examination.

Signature ____________ Date ____________

Problem 1 (20 pts): Compute the following integrals:

a) **(5 pts)** $\int x\cos(x)\,dx$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = \boxed{x\sin x + \cos x + C}$$

b) **(5 pts)** $\int \frac{e^{3t}}{e^{3t}+1}\,dt$

$$\int \frac{e^{3t}}{e^{3t}+1}\,dt = \int \frac{1}{u}\cdot\frac{du}{3} = \frac{1}{3}\ln u + C = \boxed{\frac{1}{3}\ln\left(e^{3t}+1\right)+C}$$

$u = e^{3t}+1$

$du = 3e^{3t}\,dt$

c) **(5 pts)** $\displaystyle\int \frac{x}{\sqrt{9-x^2}}\,\mathrm{d}x$

$$\int \frac{x}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{u}}\left(-\frac{du}{2}\right) = -\frac{1}{2}\,\frac{u^{1/2}}{1/2} + C$$

$u = 9-x^2$

$du = -2x\,dx$

$$= \boxed{-\sqrt{9-x^2} + C}$$

d) **(5 pts)** $\displaystyle\int y\ln(y)\,\mathrm{d}y$

$$\int y \ln y\, dy = \frac{y^2}{2}\ln y - \int \frac{y^2}{2}\,\frac{1}{y}\,dy$$

$$= \frac{y^2}{2}\ln y - \frac{1}{2}\int y\,dy = \frac{y^2}{2}\ln y - \frac{1}{2}\,\frac{y^2}{2} + C$$

$$= \boxed{\frac{y^2}{2}\ln y - \frac{y^2}{4} + C}$$

Problem 2 (20 pts): Compute the integral if it converges, or show it diverges:

a) **(10 pts)** $\int_0^{+\infty} 2xe^{-x^2}\,dx$

$$\int_0^{+\infty} 2xe^{-x^2}\,dx = \lim_{b\to\infty} \int_0^b 2xe^{-x^2}\,dx = \cdots$$

$$\left(\int 2xe^{-x^2}\,dx = \int e^u(-du) = -e^u + C = -e^{-x^2} + C \right)$$

$u = -x^2$

$du = -2x\,dx$

$$\cdots = \lim_{b\to\infty} -e^{-x^2}\Big|_0^b = \lim_{b\to\infty} -\underbrace{e^{-b^2}}_{\to 0} + 1 = \boxed{1}$$

(converges)

b) **(10 pts)** $\int_{-\infty}^{+\infty} \frac{x^3}{x^4+1}\,dx$

$$\int_{-\infty}^{+\infty} \frac{x^3}{x^4+1}\,dx = \lim_{a\to-\infty} \frac{1}{4}\int_a^0 \frac{4x^3dx}{x^4+1} + \lim_{b\to+\infty} \frac{1}{4}\int_0^b \frac{4x^3dx}{x^4+1}$$

$$= \lim_{a\to-\infty} \frac{1}{4}\ln(x^4+1)\Big|_a^0 + \lim_{b\to\infty} \frac{1}{4}\ln(x^4+1)\Big|_0^b$$

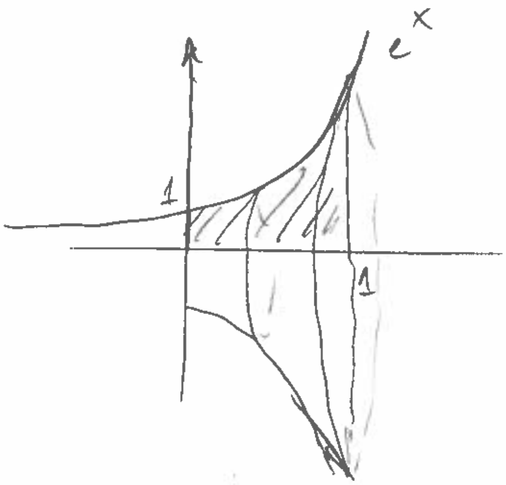
$$= \lim_{a\to-\infty} -\frac{1}{4}\ln(a^4+1) + \lim_{b\to+\infty} \frac{1}{4}\ln(b^4+1)$$

$$= -\infty \qquad + \infty$$

(Diverges)

Problem 3 (15 pts): Let R be the region bounded by the graph of $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = 1$. Sketch a picture of the region R and use it to write down **(but do not evaluate)** integrals that compute the volume of the solids obtained by revolving R about the following lines:

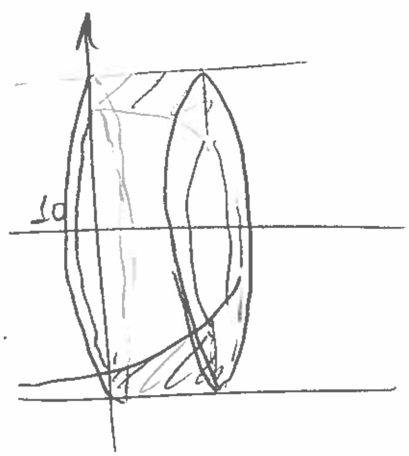
a) **(5 pts)** $y = 0$



Disk Method:

$$V = \pi \int_0^1 e^{2x}\,dx$$

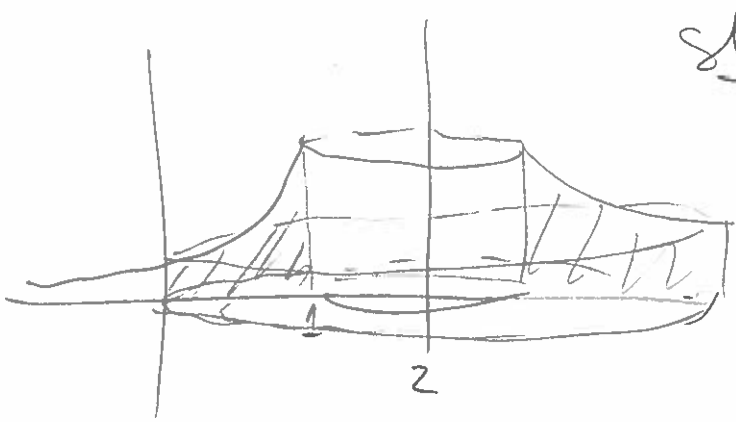
b) **(5 pts)** $y = 10$



Washer Method:

$$V = \pi \int_0^1 10^2 - (10 - e^x)^2\,dx$$

c) **(5 pts)** $x = 2$



Shell Method:

$$V = 2\pi \int_0^1 \underbrace{(2-x)}_{\text{shell radius}}\underbrace{e^x}_{\text{shell height}}\,dx$$

Problem 4 (15 pts): A dosage of 100mg of a certain antibiotic is given to a patient at 8:00am each day. Suppose 10% of the antibiotic remains in the body after one full day period (8:00am next day).

a) **(5 pts)** What is the amount of antibiotic in the body 2 days after the treatment started before the next dose is given at 8:00am?

From 1^{st} dose: $\frac{1}{10} \cdot \frac{1}{10} \cdot 100 = 1$ mg

From 2^{nd} dose: $\frac{1}{10} \cdot 100 = 10$ mg

Total $\boxed{11 \text{ mg}}$

b) **(5 pts)** What is the amount of antibiotic in the body 3 days after the treatment started before the next dose is given at 8:00am?

From 1^{st} dose: $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot 100 = 0.1$ mg

From 2^{nd} dose: $\frac{1}{10} \cdot \frac{1}{10} \cdot 100 = 1$ mg

From 3^{rd} dose: $\frac{1}{10} \cdot 100 = 10$ mg

total $\boxed{11.1 \text{ mg}}$

c) **(5 pts)** Use a geometric series to estimate the amount of antibiotic in the body after a very long time of treatment (measured before a new dose is given).

After n doses, the patient has in his/her body:

$$S_n = \underbrace{\left(\frac{1}{10}\right)^n 100}_{\text{From }1^{st}\text{ dose}} + \underbrace{\left(\frac{1}{10}\right)^{n-1} 100}_{\text{From }2^{nd}\text{ dose}} + \cdots + \underbrace{\left(\frac{1}{10}\right)^2 100}_{\text{From second to last dose}} + \underbrace{\left(\frac{1}{10}\right)^1 100}_{\text{From last dose}}$$

So after a long time, $\lim_{n\to\infty} S_n = \sum_{n=1}^{+\infty} \left(\frac{1}{10}\right)^n \cdot 100 = 100 \cdot \frac{1/10}{1-1/10} = \frac{100}{9}$

the patient has $\frac{100}{9} = 11.111\ldots$ mg of antibiotic in his/her body

Problem 5 (10 pts): Use a trigonometric substitution to compute the integral

$$\int \frac{x^2}{\sqrt{16-x^2}}\,\mathrm{d}x$$

Your answer must be a function of x, simplified as much as possible.

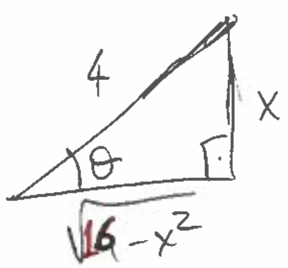
$$\int \frac{x^2\,dx}{\sqrt{16-x^2}} = \int \frac{(4\sin\theta)^2\; 4\cos\theta\, d\theta}{\sqrt{16(1-\sin^2\theta)}}$$

$x = 4\sin\theta$

$dx = 4\cos\theta\, d\theta$

$$= 16\int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos\theta} = 16\int \sin^2\theta\, d\theta$$

$$= 16\int \frac{1-\cos 2\theta}{2}\, d\theta = 8\left(\theta - \frac{\overset{2\sin\theta\cos\theta}{\sin 2\theta}}{2}\right) + C = 8\theta - 8\sin\theta\cos\theta + C$$



$$= 8\arcsin\left(\frac{x}{4}\right) - 8\cdot\frac{x}{4}\,\frac{\sqrt{16-x^2}}{4} + C$$

$$\boxed{= 8\arcsin\left(\frac{x}{4}\right) - \frac{x\sqrt{16-x^2}}{2} + C}$$

$\sin\theta = \frac{x}{4}$

$\cos\theta = \frac{\sqrt{16-x^2}}{4}$

Problem 6 (10 pts): Determine whether each of the following series converges or diverges. **You must carefully specify the convergence tests** that are being using.

a) **(5 pts)** $\sum_{n=1}^{+\infty} \frac{n}{n^2+1}$ Apply Limit comparison test:

$$a_n = \frac{n}{n^2+1}, \quad b_n = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 = L \quad 0 < L < +\infty$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty \text{ (hormonic series) diverges}$$

thus also $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n}{n^2+1} = +\infty$ $\boxed{\text{diverges}}$

b) **(5 pts)** $\sum_{n=1}^{+\infty} \frac{n+1}{n\,3^n}$ Apply Limit Comparison Test:

$$a_n = \frac{n+1}{n\cdot 3^n}, \quad b_n = \frac{1}{3^n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1 = L \quad 0 < L < +\infty$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1/3}{1-1/3} = \frac{1}{2} < \infty \text{ converges (geometric series)}$$

thus also $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+1}{n.3^n} < \infty$ $\boxed{\text{converges}}$

Problem 7 (10 pts): Find the solution $y = y(x)$ of the following initial value problem:

$$\begin{cases} \dfrac{dy}{dx} = \sin^3 x \cos^4 x \\ y(0) = 0 \end{cases}$$

$$\frac{dy}{dx} = \sin^3 x \cos^4 x \implies y(x) = \int \sin^3 x \cos^4 x \, dx$$

$$= \int (1 - \cos^2 x) \cos^4 x \cdot \sin x \, dx = -\int (1 - u^2) u^4 \, du$$

$u = \cos x$
$du = -\sin x \, dx$

$$= -\int u^4 - u^6 \, du = \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$$

$$y(0) = 0 \implies 0 = \frac{1}{7} - \frac{1}{5} + C \implies C = \frac{1}{5} - \frac{1}{7} = \frac{7-5}{35} = \frac{2}{35}$$

Thus $\boxed{y(x) = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + \frac{2}{35}}$