SOLUTION TO PRACTICE PROBLEMS FOR MIDTERM 2

1\. a) $f(x) = x^4 - 4x + 1$

$f'(x) = 4x^3 - 4 = 0 \iff x^3 - 1 = 0$

$\iff x = 1$

$\boxed{\text{Critical point: } x = 1}$

b) $f(x) = x^3 - 12x + 2$

$f'(x) = 3x^2 - 12 = 0 \iff x^2 - 4 = 0 \iff x = \pm 2$

$\boxed{\text{Critical points: } x = -2,\ x = 2.}$

c) $f(x) = x^2 + x - 5$

$f'(x) = 2x + 1 = 0 \iff x = -\frac{1}{2}$

$\boxed{\text{Critical point: } x = -1/2}$

d) $f(x) = |x-2| = \begin{cases} x-2 & \text{if } x > 2 \\ -x+2 & \text{if } x < 2 \end{cases}$

$f'(x) = \begin{cases} -1 & \text{if } x < 2 \\ 1 & \text{if } x > 2 \end{cases}$, $f(x)$ not differentiable at $x = 2$.

(i.e., $f'(2)$ does not exist)

$\boxed{\text{Critical point: } x = 2.}$

e) $f(x) = \pi e^{x^2}$

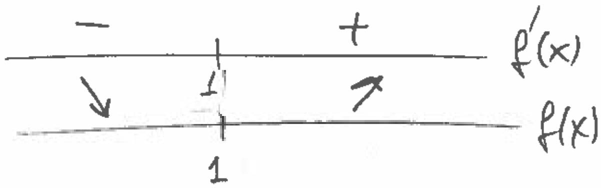
$f'(x) = \pi \cdot e^{x^2} \cdot 2x = 2\pi x e^{x^2} = 0 \iff x = 0$

$\boxed{\text{Critical point: } x = 0}$

f) $f(x) = 2^x = e^{x \ln 2}$

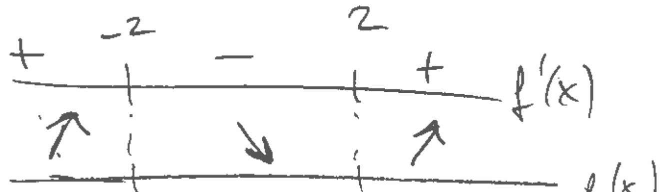
$f'(x) = e^{x \ln 2} \cdot \ln 2 \neq 0$ for all x $\Rightarrow$ $\boxed{\text{No critical points}}$

2. a) $f'(x) = 4x^3 - 4$



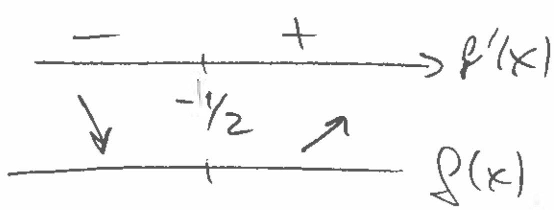
$\boxed{f(x) \text{ is decreasing on } (-\infty, 1) \text{ and increasing on } (1, +\infty)}$

b) $f'(x) = 3x^2 - 12$



$\boxed{f(x) \text{ is increasing on } (-\infty, -2) \cup (2, +\infty) \text{ and decreasing on } (-2, 2)}$

c) $f'(x) = 2x + 1$



$\boxed{f(x) \text{ is decreasing on } (-\infty, -\tfrac{1}{2}) \text{ and increasing on } (-\tfrac{1}{2}, +\infty)}$

d) $f'(x) = \begin{cases} -1 & \text{if } x < 2 \\ 1 & \text{if } x > 2 \end{cases}$

$\boxed{f(x) \text{ is decreasing on } (-\infty, 2) \text{ and increasing on } (2, +\infty)}$

e) $f'(x) = 2\pi x e^{x^2}$ $> 0 \iff x > 0$

$< 0 \iff x < 0$

$\boxed{f(x) \text{ is increasing on } (0, +\infty) \text{ and decreasing on } (-\infty, 0)}$

f) $f'(x) = e^{x\ln 2} \cdot \ln 2 > 0 \Rightarrow$ $\boxed{f(x) \text{ is } \underline{\text{always increasing}}.}$

3. a) $f''(x) = 12x^2 > 0 \Rightarrow$ $\boxed{f(x) \text{ is always } \underline{\text{concave up}}}$

b) $f''(x) = 6x > 0 \Leftrightarrow x > 0 \therefore$ $\boxed{\begin{array}{l} f(x) \text{ is } \underline{\text{concave up}} \text{ on } (0, +\infty) \\ \underline{\text{concave down}} \text{ on } (-\infty, 0) \end{array}}$

c) $f''(x) = 2 > 0 \Rightarrow$ $\boxed{f(x) \text{ is always } \underline{\text{concave up}}.}$

d) $f''(x) = 0 \Rightarrow$ $\boxed{\begin{array}{l} f(x) \text{ is } \underline{\text{never}} \text{ concave up nor} \\ \text{concave down (it is piecewise linear!)} \end{array}}$

e) $f''(x) = 2\pi e^{x^2} + 2\pi x \cdot e^{x^2} \cdot 2x$

$= \underbrace{2\pi e^{x^2}}_{>0} \underbrace{(1 + 2x^2)}_{>0} > 0 \Rightarrow$ $\boxed{\begin{array}{l} f(x) \text{ is always} \\ \underline{\text{concave up}} \end{array}}$

f) $f''(x) = e^{x\ln 2} \cdot (\ln 2)^2 > 0 \Rightarrow$ $\boxed{\begin{array}{l} f(x) \text{ is always} \\ \underline{\text{concave up}} \end{array}}$

4. a) $x = 1$ is a <u>local minimum</u> since $f''(1) > 0$

b) $x = -2$ is a <u>local maximum</u> since $f''(-2) < 0$

$x = 2$ is a <u>local minimum</u> since $f''(2) > 0$

c) $x = -\frac{1}{2}$ is a <u>local minimum</u> since $f''(\frac{1}{2}) > 0$

(2nd derivative test)

d) $x = 2$ is a <u>local minimum</u> since $f(x)$ is decreasing if $x < 2$ and increasing if $x > 2$

(1st derivative test)

e) $x=0$ is a <u>local minimum</u> since $f''(0)>0$

f) There are no critical points.

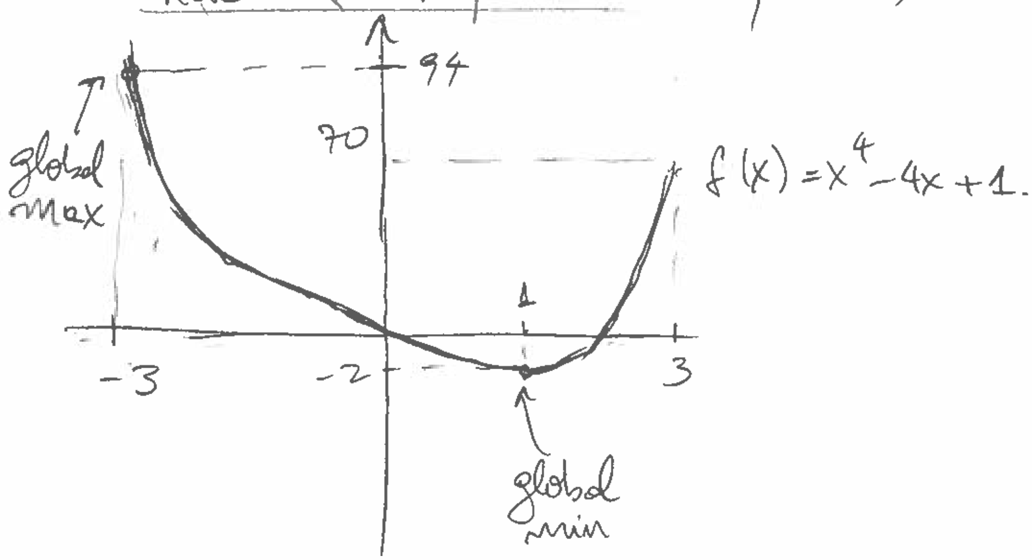
5. In order to find global min/max on closed interval $I=[-3,3]$, compare values of critical points in the interior and endpoints $(f(\pm 3))$.

a, $f(-3)=(-3)^4-4(-3)+1=81+12+1=94$
$f(3)=3^4-4.3+1=81-12+1=70$ } endpoints.

$f(1)=1-4+1=-2$ } crit. pt. interior

$$\boxed{\begin{array}{l}\text{Global maximum: } 94=f(-3)\\ \text{Global minimum: } -2=f(1)\end{array}}$$

Sketch of Graph (not required!)



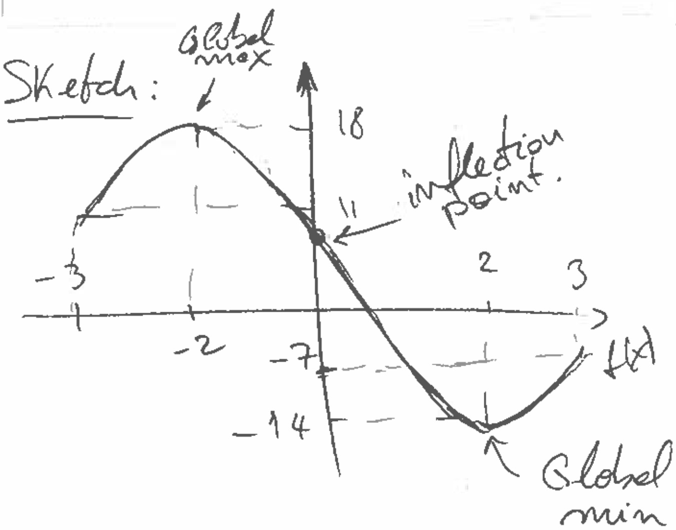
b. $f(-3) = -27 - 12.(-3) + 2 = -27 + 36 + 2 = 11$
$f(3) = 27 - 12.3 + 2 = -7$ $\Big\}$ endpoints

$f(-2) = (-2)^3 - 12(-2) + 2 = -8 + 24 + 2 = 18$
$f(2) = 2^3 - 12.2 + 2 = 8 - 24 + 2 = -14$ $\Big\}$ crit. pt. interior

Global maximum: $18 = f(-2)$

Global minimum: $-14 = f(2)$

Sketch:

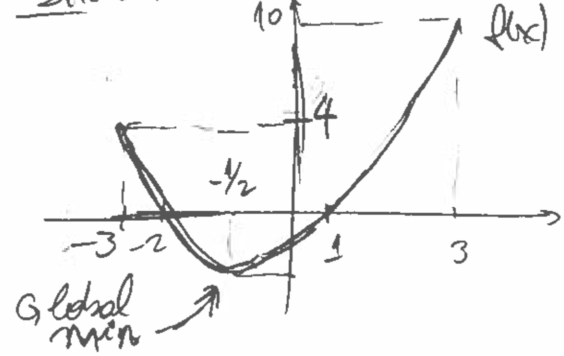


c. $f(-3) = (-3)^2 - 3 - 2 = 4$
$f(3) = 3^2 + 3 - 2 = 10$ $\Big\}$ endpoints

$f(-\frac{1}{2}) = (-\frac{1}{2})^2 - \frac{1}{2} - 2 = \frac{1}{4} - \frac{5}{2} = \frac{1-10}{4} = -\frac{9}{4}$ $\Big\}$ crit. pt. interior

Global min: $-\frac{9}{4} = f(-\frac{1}{2})$
Global max: $10 = f(3)$

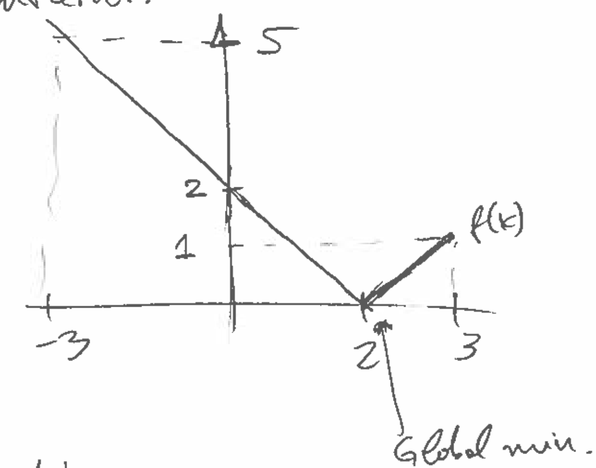
Sketch:



d. $f(-3) = |-3-2| = |-5| = 5$
$f(3) = |3-2| = |1| = 1.$ $\Big\}$ endpoints
$f(2) = |2-2| = 0$ $\Big\}$ crit. pt. interior.

Global max: $5 = f(-3)$

Global min $0 = f(2)$

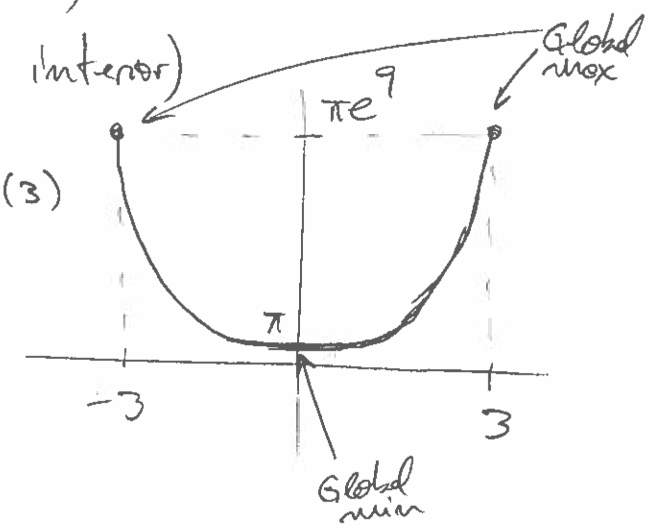


e. $f(-3) = \pi e^9 = f(3)$ (endpoints)

$f(0) = \pi e^0 = \pi$ (crit. pt. interior)

Global max: $\pi e^9 = f(-3) = f(3)$

Global min $\pi = f(0)$

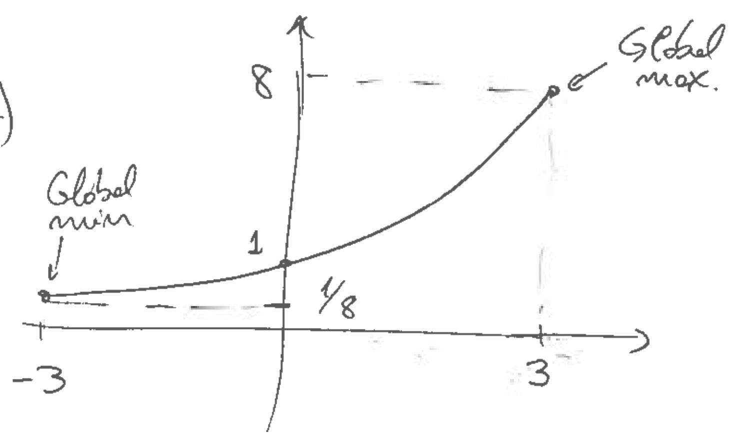


f. $f(-3) = 2^{-3} = \frac{1}{8}$
$f(3) = 2^3 = 8$ $\Big\}$ endpoints.

(No critical pts. in the interior)

Global min: $\frac{1}{8} = f(-3)$

Global max: $8 = f(3)$



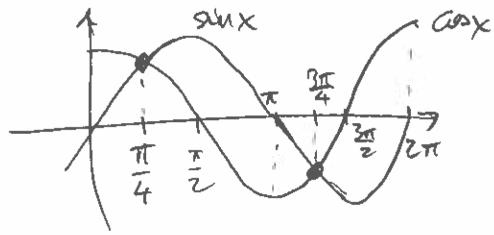
6. $f(x) = e^x \cos x, \quad x \in [0, 2\pi]$

$$f'(x) = e^x \cos x + e^x(-\sin x) = e^x(\cos x - \sin x)$$

$$f'(x) = 0 \iff \cos x = \sin x$$

$$\iff \tan x = 1$$

$$\iff x = \frac{\pi}{4}, \text{ or } x = \frac{3\pi}{4}$$



Critical points: $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$

Values at endpoints: $f(0) = e^0 \cos 0 = 1$

$$f(2\pi) = e^{2\pi} \cos 2\pi = e^{2\pi}$$

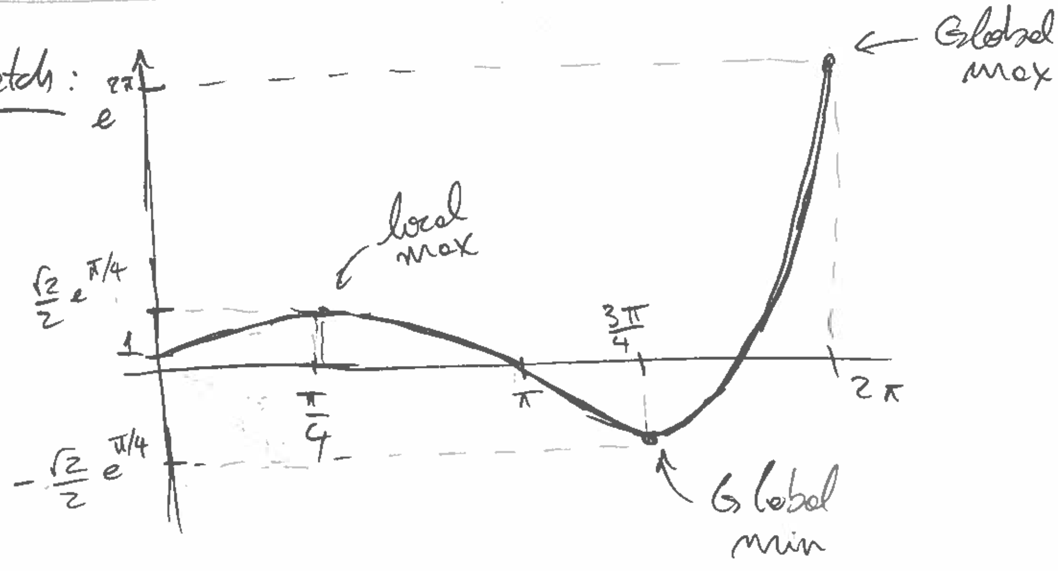
Values at crit. pts. in the interior: $f\left(\frac{\pi}{4}\right) = e^{\pi/4} \cdot \frac{\sqrt{2}}{2}$

$$f\left(\frac{3\pi}{4}\right) = e^{\frac{3\pi}{4}} \cdot \left(-\frac{\sqrt{2}}{2}\right)$$

Global minimum: $-\frac{\sqrt{2}}{2} \cdot e^{\frac{3\pi}{4}} = f\left(\frac{3\pi}{4}\right)$

Global maximum: $e^{2\pi} = f(2\pi)$

Sketch:



7. (HARD) $f(x) = 5x^{4/5} - x^2$

$$f'(x) = 5 \cdot \frac{4}{5} x^{-1/5} - 2x = \frac{4}{x^{1/5}} - 2x$$

Thus, $x = 0$ is a critical pt (derivative does not exist at $x = 0$!)

If $x \neq 0$, then

$$f'(x) = 0 \iff \frac{4}{x^{1/5}} = 2x \overset{x \neq 0}{\iff} 4 = 2x^{6/5}$$

$$\iff x^{6/5} = 2$$

$$\iff x^6 = 2^5 = 32$$

$$\iff (x^3)^2 = 32 = (4\sqrt{2})^2$$

$$\iff x^3 = \pm 4\sqrt{2} = \pm 2^{5/2}$$

$$\iff x = \pm 2^{5/6}.$$

or also directly from here, noting that $x^{6/5} = 2 \iff x = \pm 2^{5/6}$ where the $\pm$ comes from extracting 6th root (6 is even)

Critical points: $x = -2^{5/6}$, $x = 0$, $x = 2^{5/6}$

$$f''(x) = \frac{d}{dx}\left(4x^{-1/5} - 2x\right) = -\frac{4}{5} x^{-6/5} - 2 = -\frac{4}{5x^{6/5}} - 2$$

- $f''(\pm 2^{5/6}) = -\dfrac{4}{5 \cdot 2} - 2 < 0 \Rightarrow$ $x = \pm 2^{5/6}$ are local maxima (2nd derivative test)

$f(\pm 2^{5/6}) = 5 \cdot 2^{2/3} - 2^{5/3}$ (>5)

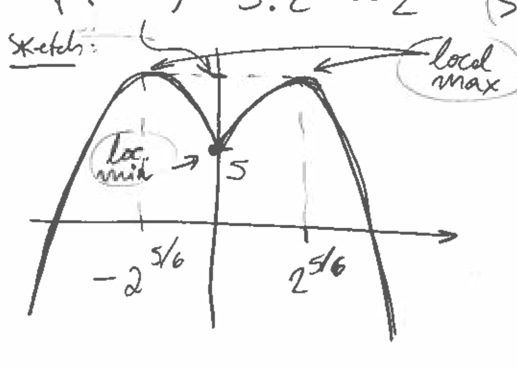
- $\boxed{x = 0 \text{ is a local minimum}}$ by the 1st derivative test, since

$f'(x) < 0$ if $x < 0$ near $x = 0$

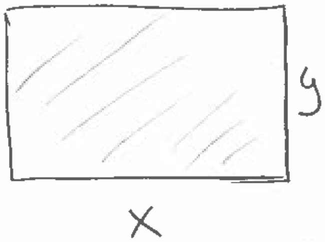
$f'(x) > 0$ if $x > 0$ near $x = 0$.

$f(0) = 5$

Sketch:



8.



x = width of rectangle
y = height of rectangle
$A = xy$ Area of rectangle
$P = 2x + 2y$ Perimeter of rectangle.

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 8. \quad [\text{cm/s}]$$

$$\frac{dA}{dt} = \frac{d}{dt}\big(x(t)\,y(t)\big) = \frac{dx}{dt}\cdot y + x \cdot \frac{dy}{dt} = 3y + 8x$$

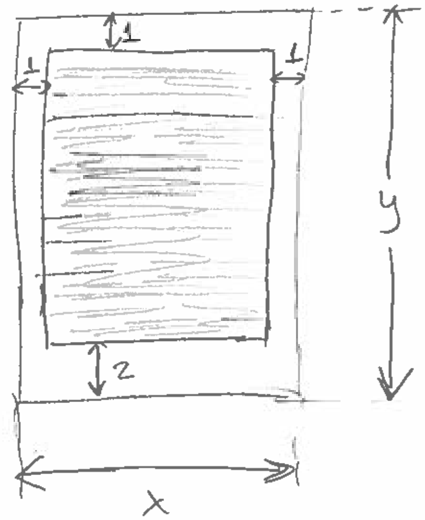
a). when $x = 10$, $y = 20$, the area is increasing at a rate of:

$$\frac{dA}{dt} = 3 \cdot 20 + 8 \cdot 10 = 60 + 80 = \boxed{\underline{\underline{140}} \quad [\text{cm}^2/\text{s}]}$$

b) when $x = 10$, $y = 20$, the perimeter is increasing at a rate of

$$\frac{dP}{dt} = \frac{d}{dt}\big(2x(t) + 2y(t)\big) = 2\frac{dx}{dt} + 2\frac{dy}{dt} = 2 \cdot 3 + 2 \cdot 8$$
$$= 6 + 16$$
$$\boxed{= \underline{\underline{22}} \quad [\text{cm/s}]}$$

(Note: This rate is constant, independent of x & y ...)

9. 

$x =$ width of page

$y =$ height of page

$A =$ total area of page

$$A = xy$$

$$\left.\begin{aligned} P &= \text{printed area} \\ P &= (x-2)(y-3) \end{aligned}\right\} \text{(Discounting margins!)}$$

Constraint: $A = 150 \Rightarrow xy = 150$

$$\Rightarrow y = \frac{150}{x}$$

$$P = (x-2)(y-3) = (x-2)\left(\frac{150}{x} - 3\right) \leftarrow \text{target function}$$

$$\Rightarrow P(x) = (x-2)\left(\frac{150}{x} - 3\right)$$

$$\Rightarrow P'(x) = \frac{150}{x} - 3 + (x-2)\left(-\frac{150}{x^2}\right) = \cancel{\frac{150}{x}} - 3 - \cancel{\frac{150}{x}} + \frac{300}{x^2}$$

$$= \frac{300}{x^2} - 3$$

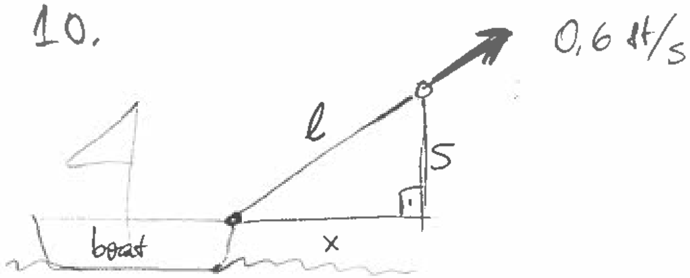
$$P'(x) = 0 \iff \frac{300}{x^2} = 3 \iff x^2 = 100 \iff x = 10.$$

$(x = \text{width} > 0)$

$$P''(x) = -\frac{600}{x^3} < 0$$ So every critical point is a local max.

$\boxed{\text{A: The dimensions that maximize the printed area are } x = 10\text{ cm},\ y = 15\text{ cm.}}$

10.



l = length of rope out

x = distance from boat to dock.

Pythagoras: $x^2 + 5^2 = l^2$ $\qquad \frac{dl}{dt} = -0.6 \frac{ft}{s}$

$\implies x^2 + 25 = l^2$

Differentiate w.r.t. t: $2x \frac{dx}{dt} = 2l \frac{dl}{dt}$

Want: $\frac{dx}{dt}$ when $l = 13$:

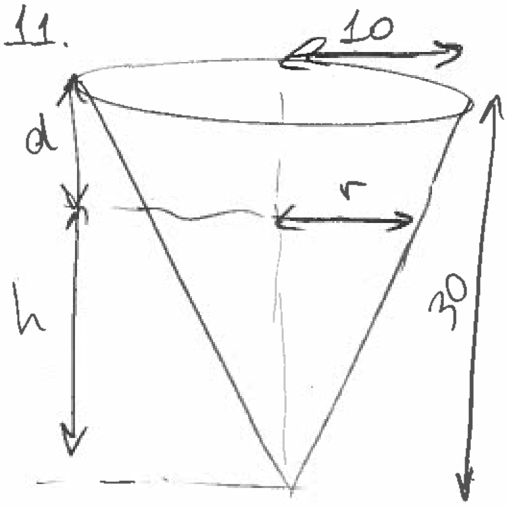
$l = 13 \implies x^2 + 25 = 13^2 = 169 \implies x^2 = 144 \implies x = 12$ $(x > 0)$

$\cancel{2} \cdot 12 \cdot \frac{dx}{dt} = \cancel{2} \cdot 13 \cdot (-0.6) \implies \frac{dx}{dt} = -\frac{6}{10} \cdot \frac{13}{12}$

$\frac{dx}{dt} = -\frac{13}{20}$.

A: The boat is approaching the dock at speed $\frac{13}{20} \frac{ft}{s}$.

11.



h = height of water in tank

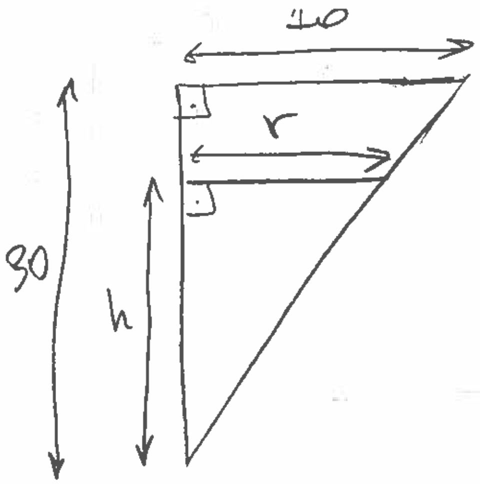
d = depth $= 30 - d$

V = Volume of water in tank.

r = radius of water level

$V = \frac{1}{3} \pi r^2 \cdot h$ $\qquad$ Want: $\frac{dh}{dt} = ?$ when $d = 3$.

$h + d = 30$



By similar triangles:

$$\frac{r}{h} = \frac{10}{30} = \frac{1}{3} \Rightarrow \boxed{r = \frac{h}{3}}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{3}\frac{dh}{dt} \quad (*)$$

$$\frac{dV}{dt} = \frac{1}{3}\pi\, 2r\frac{dr}{dt}h + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

$$\overset{(*)}{=} \frac{2\pi}{3} r \frac{1}{3}\frac{dh}{dt}\cdot h + \frac{\pi}{3} r^2 \frac{dh}{dt}$$

$$= \frac{2\pi r h}{9}\frac{dh}{dt} + \frac{\pi r^2}{3}\frac{dh}{dt}$$

When $d=3$, $h = 30-3 = 27$, $r = \frac{h}{3} = \frac{27}{3} = 9$

$$\Rightarrow \frac{dV}{dt} = \frac{2\pi\cdot\cancel{9}\cdot 27}{\cancel{9}}\cdot\frac{dh}{dt} + \frac{\pi\cdot 9^2}{3}\frac{dh}{dt}$$

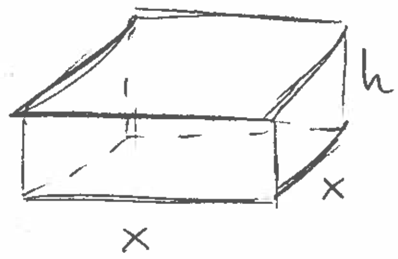
From problem $\frac{dV}{dt} = 10$

$$10 = 54\pi\frac{dh}{dt} + 27\pi\frac{dh}{dt} = 81\pi\frac{dh}{dt}$$

$$\Rightarrow \boxed{\frac{dh}{dt} = \frac{10}{81\pi}\ \frac{cm}{s}}$$

A: The water level is rising at a rate $\frac{10}{81\pi}$ cm/s.

12.



$(x>0,\ h>0)$

x = length of sides of base (square)

h = depth of swimming pool

$V = x^2 h$ volume of pool

$A = x^2 + 4xh$ surface area to be covered with tiles.

x^2: bottom (floor); $4xh$: 4 sides (walls)

$V = 32{,}000 \text{ m}^3 \leftarrow$ (constraint)

$x^2 h = 32\,000 \implies h = \dfrac{32000}{x^2}$

Target function: $A = x^2 + 4xh = x^2 + 4x \cdot \left(\dfrac{32000}{x^2}\right)$

$$A(x) = x^2 + \frac{128{,}000}{x}$$

$$A'(x) = 2x - \frac{128000}{x^2} = 0 \iff x = \frac{64{,}000}{x^2}$$

$$\iff x^3 = 64{,}000$$

$$\iff \boxed{x = 40}$$

$$A''(x) = 2 + \frac{256{,}000}{x^3} > 0$$ so all critical points are minima.

A: The dimensions that minimize the surface area to be tiled are $x = 40$ m, $h = \dfrac{32000}{1600} = 20$ m, that is, the optimal swimming pool w/ $V = 32{,}000 \text{ m}^3$ is $40 \text{ m} \times 40 \text{ m} \times 20 \text{ m}$
(length) (width) (depth)

13. a. $\int x^3 - 3x^2 + x + 1 \, dx = \boxed{\frac{x^4}{4} - x^3 + \frac{x^2}{2} + x + C}$

b. $\int 2\sin x - 3\cos x + 5 \, dx = \boxed{-2\cos x - 3\sin x + 5x + C}$

c. $\int \frac{t^4 + t^2 + 1}{\sqrt{t}} \, dt = \int t^{7/2} + t^{3/2} + t^{-1/2} \, dt$

$$= \boxed{\frac{t^{9/2}}{9/2} + \frac{t^{5/2}}{5/2} + \frac{t^{1/2}}{1/2} + C.}$$

d. $\int e^x - x^e \, dx = \boxed{e^x - \frac{x^{e+1}}{e+1} + C}$

e. $\int \frac{3}{y} - \frac{y}{3} \, dy = 3 \ln y - \frac{1}{3}\frac{y^2}{2} + C$

14. a. $\int_0^1 x^3 - 3x^2 + x + 1 \, dx = \left(\frac{x^4}{4} - x^3 + \frac{x^2}{2} + x\right)\Big|_0^1$

$$= \frac{1}{4} - \cancel{1} + \frac{1}{2} + \cancel{1} - 0 = \boxed{\frac{3}{4}}$$

b. $\int_0^\pi 2\sin x - 3\cos x + 5 \, dx = \left(-2\cos x - 3\sin x + 5x\right)\Big|_0^\pi$

$$= \left(-2\cos\pi - 3\underbrace{\cancel{\sin\pi}}_{0} + 5\pi\right) - \left(-2\cos 0 - 0\right)$$

$$= +2 + 5\pi + 2 = \boxed{4 + 5\pi}$$

c. $\int_1^4 \frac{t^4+t^2+1}{\sqrt{t}}\,dt = \left(\frac{2}{9}t^{9/2}+\frac{2}{5}t^{5/2}+2t^{1/2}\right)\Big|_1^4$

$$= \left(\frac{2}{9}2^9+\frac{2}{5}\cdot 2^5+2.2\right)-\left(\frac{2}{9}+\frac{2}{5}+2\right)$$

$$= \frac{1024}{9}+\frac{64}{5}+4-\frac{10+18+90}{45}$$

$$= \frac{5120+576+180-118}{45} = \frac{5758}{45}$$

(sorry for the "ugly" numbers here!)

d. $\int_0^1 e^x - x^e\,dx = \left(e^x - \frac{x^{e+1}}{e+1}\right)\Big|_0^1$

$$= \boxed{e-\frac{1}{e+1}-1}$$

e. $\int_1^e \frac{3}{y}-\frac{y}{3}\,dy = \left(3\cdot\ln y - \frac{y^2}{6}\right)\Big|_1^e = \left(3\cancel{\ln e}^{1} - \frac{e^2}{6}\right)-\left(0-\frac{1}{6}\right)$

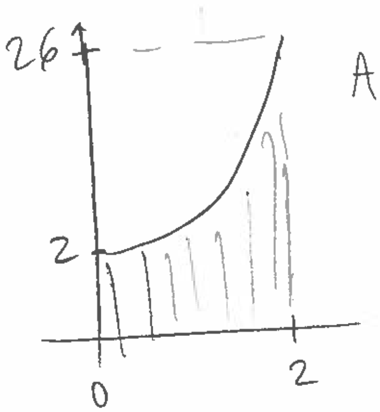
$$= 3-\frac{e^2}{6}+\frac{1}{6} = \boxed{\frac{19-e^2}{6}}$$

15. $\int_0^5 f(x)\,dx = \int_0^8 f(x)dx - \int_5^8 f(x)dx$

$$= \int_0^7 f(x)dx + \int_7^8 f(x)dx - \int_5^8 f(x)dx$$

$$= 8-2-1 = \boxed{5}$$

16.



$$A = \int_0^2 6x^2 + 2\,dx = \left(6\frac{x^3}{3} + 2x\right)\Big|_0^2 = \left(2x^3 + 2x\right)\Big|_0^2$$

$$= 2\cdot 8 + 4 = \boxed{20.}$$

17. (Left) Riemann Sum on interval $[a,b] = [0,1]$

$$\Delta x = \frac{b-a}{n} = \frac{1}{n}, \qquad x_i = a + i\Delta x = 0 + i\cdot\frac{1}{n} = \frac{i}{n}$$

$$\int_0^1 x^4\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} x_i^4\,\Delta x = \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{i}{n}\right)^4 \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{i^4}{n^5} = \lim_{n\to\infty} \frac{1}{n^5} \sum_{i=0}^{n-1} i^4 =$$

$$= \lim_{n\to\infty} \frac{1}{n^5}\Big(\underbrace{\sum_{i=0}^{n} i^4}_{\text{use formula}} - n^4\Big) = \lim_{n\to\infty} \frac{1}{n^5}\left(\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} - n^4\right)$$

$$= \lim_{n\to\infty} \frac{1}{5} + \underbrace{\frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{30n^4} - \frac{1}{n}}_{\to 0} = \boxed{\frac{1}{5}}$$

18. (left) Riemann Sum on $[a,b] = [1,2]$ with $n = 5$

$$\Delta x = \frac{b-a}{5} = \frac{1}{5}, \qquad x_i = a + i\Delta x = 1 + \frac{i}{5}.$$

$$RS = \sum_{i=0}^{4} \frac{\sin(x_i)}{x_i}\,\Delta x = \sum_{i=0}^{4} \frac{\sin\left(1 + \frac{i}{5}\right)}{\left(1 + \frac{i}{5}\right)}\cdot\frac{1}{5} = \boxed{\sum_{i=0}^{4} \frac{\sin\left(1 + \frac{i}{5}\right)}{5 + i}}$$

19. $\begin{cases} \dfrac{dy}{dx} = x^2 - 6\cos x \\ y(0) = 0 \end{cases}$ $\Rightarrow$ $y(x) = \dfrac{x^3}{3} - 6\sin x + C$ (General sol'n)

$0 = y(0) = 0 + C \Rightarrow C = 0$

$$\boxed{y(x) = \frac{x^3}{3} - 6\sin x}$$ (Particular sol'n)

20. $\begin{cases} \dfrac{ds}{dt} = t^{1/3} + e^t \\ s(0) = \sqrt{3} \end{cases}$ $\Rightarrow$ $s(t) = \dfrac{t^{4/3}}{4/3} + e^t + C$ (General sol'n)

$\sqrt{3} = s(0) = 0 + e^0 + C = 1 + C$

$\Rightarrow C = \sqrt{3} - 1$

$$\boxed{y(x) = \frac{3}{4} t^{4/3} + e^t + \sqrt{3} - 1}$$ (Particular sol'n)

21 a) $\dfrac{d}{dx}\displaystyle\int_0^x \sqrt{1+t^4}\,dt = \boxed{\sqrt{1+x^4}}$

b) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \sqrt{1+t^4}\,dt = \left(\dfrac{d}{du}\displaystyle\int_0^u \sqrt{1+t^4}\,dt\right) \cdot \dfrac{du}{dx}$

$u = x^2$

$= \sqrt{1+u^4} \cdot 2x = \boxed{\sqrt{1+x^8} \cdot 2x}$

c) $\dfrac{d}{dx}\displaystyle\int_{-\pi}^{\cos x} 4e^s\,ds = \left(\dfrac{d}{du}\displaystyle\int_{-\pi}^{u} 4e^s\,ds\right) \dfrac{du}{dx} = 4e^u \cdot (-\sin x)$

$u = \cos x$

$= \boxed{-4e^{\cos x} \sin x}$

d) $\frac{d}{dz}\int_1^{1/z^2} \ln(y)\, dy = \left(\frac{d}{du}\int_1^{u} \ln y\, dy\right)\cdot \frac{du}{dz}$

$u = \frac{1}{z^2}$

$= \ln u \cdot \left(-\frac{2}{z^3}\right) = \ln \frac{1}{z^2}\cdot\left(-\frac{2}{z^3}\right)$

$= -\frac{2}{z^3}\left(\ln 1 - \ln z^2\right) = \frac{2}{z^3}\ln z^2$

$= \boxed{\frac{4}{z^3}\ln z}$

22. a) $\int 4x\cos(x^2)\, dx = \int 2\cos(u)\, du = 2\sin(u) + C$

$u = x^2$
$du = 2x\,dx$

$= \boxed{2\sin(x^2) + C}$

b) $\int x e^{-x^2} dx = \int e^{u}\frac{du}{-2} = -\frac{1}{2}e^{u} + C = \boxed{-\frac{1}{2}e^{-x^2} + C}$

$u = -x^2$
$du = -2x\,dx$

c) $\int z\sqrt{4-z^2}\, dz = \int \sqrt{u}\,\frac{du}{-2} = -\frac{1}{2}\frac{u^{3/2}}{3/2} + C = -\frac{u^{3/2}}{3} + C$

$u = 4 - z^2$
$du = -2z\,dz$

$= \boxed{-\frac{1}{3}(4-z^2)^{3/2} + C}$

d) $\int e^x\sqrt{1-e^x}\, dx = \int \sqrt{1-u}\, du = -\int v^{1/2} dv = -\frac{v^{3/2}}{3/2} + C$

$u = e^x$
$du = e^x dx$

$V = 1-u$
$dV = -du$

$= -\frac{2}{3}(1-u)^{3/2} + C = \boxed{-\frac{2}{3}(1-e^x)^{3/2} + C}$