Name: ANSWERS Lehman ID: ______________

MAT 175
Midterm 1
October 24, 2018

Instructions:

Turn off and put away your cell phone.
Please write your Name and Lehman ID # on the top of this page.
Please sign and date the pledge below to comply with the Code of Academic Integrity.
No consultation material, calculators, or electronic devices are allowed during the exam.
If any question is unclear, raise your hand to ask for clarifications.
The regular amount of time you have to complete the exam is 100 minutes.
You must show all of your work! *No credit will be given for unsupported answers.*
Please try to be as organized, objective, and logical as possible in your answers.

#	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

My signature below certifies that I complied with the CUNY Academic Integrity Policy and the Lehman College Code of Academic Integrity in completing this examination.

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Problem 1 (10 pts): Compute the following limits, or explain why they do not exist:

a) **(5 pts)** $\lim_{x\to 1} \frac{x^3 - x}{x - 1}$

$$\lim_{x\to 1} \frac{x^3-x}{x-1} = \lim_{x\to 1} \frac{x(x+1)\cancel{(x-1)}}{\cancel{x-1}} = \lim_{x\to 1} x(x+1) = \boxed{2}$$

b) **(5 pts)** $\lim_{x\to 0} \frac{2\sin(\pi x)}{x}$

$$\lim_{x\to 0} \frac{2\sin(\pi x)}{x} = 2 \lim_{x\to 0} \pi \underbrace{\frac{\sin(\pi x)}{\pi x}}_{\to 1} = \boxed{2\pi}$$

Problem 2 (10 pts): Compute the following limits, or explain why they do not exist:

a) **(5 pts)** $\lim_{x\to -1} \frac{x^2+x+3}{x+1}$ $\boxed{\text{Does not exist}}$, because

$$\lim_{x\to -1_+} \frac{x^2+x+3 \;(\to 3_+)}{x+1 \;(\to 0_+)} = +\infty$$

$$\lim_{x\to -1_-} \frac{x^2+x+3 \;(\to 3_-)}{x+1 \;(\to 0_-)} = -\infty$$

lateral limits do not agree (vertical asymptote)

b) **(5 pts)** $\lim_{x\to 1} (x-1)^2 \cos\left(\frac{2\pi}{x-1}\right)$

$$\lim_{x\to 1} \underbrace{(x-1)^2}_{\text{goes to zero}} \underbrace{\cos\left(\frac{2\pi}{x-1}\right)}_{\text{bounded}} = \boxed{0}$$ by the Squeeze theorem

Problem 3 (10 pts): For what value of a is the function below continuous at all points?

$$f(x) = \begin{cases} ax - 1 & \text{if } x \leq 1 \\ \dfrac{\sqrt{x+8}-3}{x-1} & \text{if } x > 1 \end{cases}$$

In order for $f(x)$ to be continuous, we need the lateral limits at $x = 1$ to agree:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} ax - 1 = a - 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{\sqrt{x+8} - 3}{x - 1} = \lim_{x \to 1^+} \frac{(\sqrt{x+8} - 3)(\sqrt{x+8} + 3)}{(x-1)(\sqrt{x+8} + 3)}$$

$$= \lim_{x \to 1^+} \frac{x + 8 - 9}{(x-1)(\sqrt{x+8} + 3)} = \lim_{x \to 1^+} \frac{\cancel{x - 1}}{\cancel{(x-1)}(\sqrt{x+8} + 3)} = \frac{1}{6}.$$

Thus, we need

$$a - 1 = \frac{1}{6}$$

Hence $\boxed{a = \frac{7}{6}}$

Problem 4 (10 pts): Compute the following limits, or explain why they do not exist:

a) **(5 pts)** $\lim\limits_{x\to+\infty} \dfrac{x^3-2x+4}{x-1}$

$$\lim_{x\to+\infty} \frac{x^3-2x+4}{x-1} = \lim_{x\to+\infty} \frac{\cancel{x^3}^{\,x^2}\left(1-\overbrace{\frac{2}{x^2}}^{\to 0}+\overbrace{\frac{4}{x^3}}^{\to 0}\right)}{\cancel{x}\left(1-\underbrace{\frac{1}{x}}_{\to 0}\right)} = \boxed{+\infty}$$

b) **(5 pts)** $\lim\limits_{t\to-\infty} \dfrac{\sqrt{2}t-3}{t^2+8\pi}$

$$\lim_{t\to-\infty} \frac{\sqrt{2}\cdot t-3}{t^2+8\pi} = \lim_{t\to-\infty} \frac{\cancel{t}\left(\sqrt{2}-\overbrace{\frac{3}{t}}^{\to 0}\right)}{t\,\cancel{t^2}\left(1+\underbrace{\frac{8\pi}{t^2}}_{\to 0}\right)} = \boxed{0}$$

Problem 5 (10 pts): Show that the derivative of $f(x) = 6x - 2$ is equal to $f'(x) = 6$ **using the definition as a limit** of a difference quotient.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{6(x+h) - 2 - (6x - 2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{6x} + 6h - \cancel{2} - \cancel{6x} + \cancel{2}}{h}$$

$$= \lim_{h \to 0} \frac{6h}{h} = \boxed{6.}$$

Problem 6 (10 pts): Compute the first derivative of the following functions:

a) **(5 pts)** $f(x) = 3\cos(x) + 2e^x + 9$

$f'(x) = -3\sin(x) + 2e^x$

b) **(5 pts)** $g(r) = \sqrt{r} + \sqrt{2}r + \sqrt{2}$

$g'(r) = \dfrac{1}{2\sqrt{r}} + \sqrt{2}$

Problem 7 (10 pts): Find the tangent line to the graph of $f(x) = 4(x+1)^{3/2}$ at the point $(0, 4)$.

Slope of the tangent line is given by

$$f'(x) = 4 \cdot \frac{3}{2}(x+1)^{1/2} = 6(x+1)^{1/2}$$

At the point (0,4), that is

$$f'(0) = 6$$

Hence the tangent line to the graph is:

$$y - 4 = 6(x - 0)$$

$$\boxed{y = 6x + 4}$$

Problem 8 (20 pts): A Knicks player is 6 ft tall and successfully scores 3 points for his team after shooting the basketball in such way that its height from the ground after t seconds is given by $h(t) = -16t^2 + 20t + 6$.

a) **(5 pts)** What is the (vertical) velocity of the basketball after t seconds?

Velocity is given by <u>first derivative</u> of position

$h'(t) = -32t + 20.$ $[ft/s]$

b) **(5 pts)** What is the (vertical) acceleration of the basketball after t seconds?

Acceleration is given by <u>second derivative</u> of position

$h''(t) = -32$ $[ft/s^2]$

c) **(5 pts)** According to NBA regulations, the basketball rim is exactly 10 ft above ground. How many seconds after being shot does the basketball go through the rim?

The height of the basketball is equal to 10 ft if

$h(t) = 10$

$\Leftrightarrow -16t^2 + 20t + 6 = 10$

$\Leftrightarrow -16t^2 + 20t - 4 = 0$

$\Leftrightarrow -4(4t^2 - 5t + 1) = 0$

$\Leftrightarrow 4t^2 - 5t + 1 = 0$

$\Leftrightarrow t = \dfrac{5 \pm \sqrt{25-16}}{8} = \dfrac{5 \pm 3}{8} = \begin{cases} 1 & \leftarrow \text{going through the rim} \\ \frac{1}{4} & \leftarrow \text{still on the way up.} \end{cases}$

A: The basketball goes through the rim $\boxed{\text{after } 1 \text{ s.}}$

d) **(5 pts)** What is the (vertical) velocity of the basketball when it goes through the rim?

$h'(1) = -32 \cdot 1 + 20 = \boxed{-12 \text{ ft/s}}$

Problem 9 (10 pts): The equation $y^5 + 2y + 3x^4 = 6$ implicitly defines a function $y = y(x)$ near the point $(1, 1)$. What is the equation of the tangent line to the curve $y^5 + 2y + 3x^4 = 6$ at the point $(1, 1)$?

Implicit Differentiation gives:

$$5y^4 \cdot y' + 2y' + 12x^3 = 0$$

$$(5y^4 + 2)\, y' = -12x^3$$

$$\Rightarrow y' = -\frac{12x^3}{5y^4 + 2}.$$

At the point (1,1), that gives a slope of the tangent line of

$$m = y'(1) = -\frac{12}{5+2} = -\frac{12}{7}$$

Thus, the equation of the tangent line is:

$$y - 1 = -\frac{12}{7}(x - 1)$$

$$y = -\frac{12}{7}x + \frac{12}{7} + 1$$

$$\boxed{y = -\frac{12}{7}x + \frac{19}{7}}$$

Problem 10 (10 pts): Find the first derivative $z'(t)$ of the function

$$z(t) = \frac{e^{\sqrt{t^3-1}} + 2t^2\tan(t^3)}{t^4+1}$$

$$z'(t) = \left[\left(e^{\sqrt{t^3-1}} \cdot \frac{1}{2\sqrt{t^3-1}} \cdot 3t^2 + 4t\tan(t^3) + 2t^2\sec^2(t^3) \cdot 3t^2\right)(t^4+1) - \left(e^{\sqrt{t^3-1}} + 2t^2\tan(t^3)\right)(4t^3)\right] \Big/ (t^4+1)^2$$