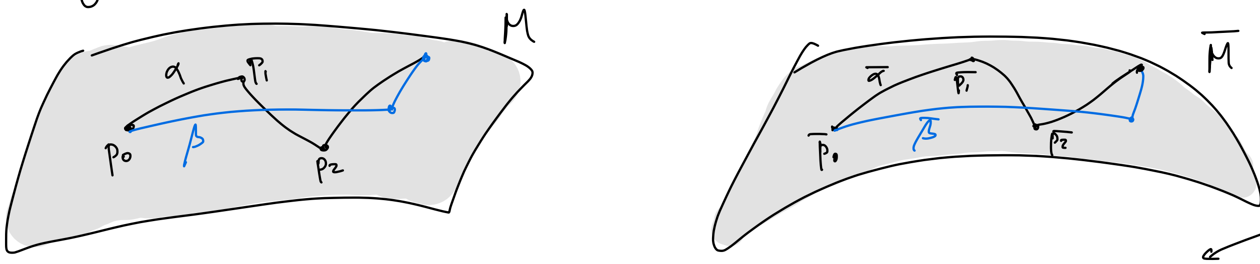
Lecture 12 3/13/2024

From now on, we almost always assume that all manifolds are complete to deal with global questions. "Global Riem. Geom." "in the large" (1960s)

Thm. (Cartan–Ambrose–Hicks). Suppose (M^n, g) and $(\overline{M}^n, \overline{g})$ are complete Riem. mflds, M simply-connected, fix $I : T_pM \to T_{\overline{p}}\overline{M}$ lin. isometry, and let $\varphi = \exp_{\overline{p}} \circ I \circ \exp_p^{-1} : B_\varepsilon(p) \to B_{\overline{\varepsilon}}(\overline{p})$. Suppose for all piecewise geodesic curves γ in M, the map $I_\gamma : T_{\gamma(t)}M \to T_{\varphi(\gamma(t))}\overline{M}$, given by $I_\gamma = P_{\overline{\gamma}} \circ I \circ P_\gamma$ (parallel transport), satisfies $I_\gamma(R(X,Y)Z) = \overline{R}(I_\gamma X, I_\gamma Y) I_\gamma Z$ for all $X, Y, Z \in T_{\gamma(t)}M$.
Then φ has a unique extension to a Riem. covering map $\varphi : M \to \overline{M}$. In particular, if $\overline{M}$ is simply-connected, then φ is a global isometry.

Pf. Using completeness, can "iterate" previous argument for Cartan's theorem:



hence α, β are homotopic b/c $\pi_1 M = \{1\}$.

Given $\alpha, \beta : [0, L] \to M$ piecewise geodesics in M with $\alpha(0) = \beta(0)$, $\alpha(L) = \beta(L)$, the piecewise geodesics $\overline{\alpha}, \overline{\beta}$ in $\overline{M}$ obtained repeating the construction at each break point have the same endpoints $\overline{\alpha}(L) = \overline{\beta}(L)$, so we can extend φ mapping geodesic endpoints in M to geodesic endpoints in $\overline{M}$. For details, see Cheeger–Ebin §12. $\square$

Using the above, and the (previously shown) fact that (M^n, g) has $\sec \equiv k$ iff

$$\langle R(X \wedge Y), Z \wedge W \rangle = k \cdot \langle X \wedge Y, Z \wedge W \rangle = k\big(\langle X, Z\rangle \langle Y, W\rangle - \langle X, W\rangle \langle Y, Z\rangle\big)$$

i.e., $$R(X,Y)Z = k\big(\langle Y, Z\rangle X - \langle X, Z\rangle Y\big)$$

So, if $\sec_M \equiv k$ and $\sec_{\overline{M}} \equiv k$, then clearly $I(R(X,Y)Z) = \overline{R}(IX, IY)IZ$.

We obtain:

Thm (Killing–Hopf). A complete connected Riem mfld (M^n, g) with $\sec \equiv k$ is isometric to a quotient of $S^n(1/\sqrt{k})$, $\mathbb{R}^n$, or $H^n(1/\sqrt{-k})$, according to $k > 0$, $k = 0$, $k < 0$, by a free properly discontinuous action of a subgroup of isometries.

<u>Recall</u>: The Riem. metrics of $S^n(1/\sqrt{k})$, $\mathbb{R}^n$, $H^n(1/\sqrt{-k})$ can be collectively written as the warped product metric $dr^2 + \mathrm{sn}_k(r)^2 g_{S^{n-1}}$, where sn_k solves $\begin{cases} \mathrm{sn}_k'' + k\,\mathrm{sn}_k = 0 \\ \mathrm{sn}_k(0) = 0, \ \mathrm{sn}_k'(0) = 1. \end{cases}$

<u>Ex</u>: Show that a closed manifold M^n, $n \geq 3$, with $\pi_2 M \neq \{0\}$ (e.g., $M = \mathbb{C}P^k$, $k \geq 2$) does not admit any Riem. metric with constant sectional curvature.

<u>Sol</u>: $\pi_2 M = \pi_2 \widetilde{M}$ and $\pi_2 S^n = \pi_2 \mathbb{R}^n = \pi_2 H^n = \{0\}$.

Basic Global Results

Ex: Give a counter-example if completeness is dropped.
Hint: Can $\mathbb{R}^n \setminus \{0\}$ be simply-connected?

> **Thm** (Cartan–Hadamard). If (M^n, g) is a complete connected Riem. mfld with $\sec \leq 0$, then $\widetilde{M} \underset{\text{diff.}}{\cong} \mathbb{R}^n$. In particular, if $\pi_1 M = \{1\}$, then $M \underset{\text{diff}}{\cong} \mathbb{R}^n$.

> **Lemma.** If $\sec \leq 0$, then Jacobi fields with $J(0)=0$ and $J'(0) \neq 0$ satisfy $J(t) \neq 0$, $\forall t > 0$.

i.e., there are no conjugate points on manifolds with $\sec \leq 0$.

Pf. Let $J(t)$ be a Jacobi field along $\gamma(t) = \exp_p tv$, with $J(0)=0$, and set $f(t) = \frac{1}{2}\|J(t)\|^2 = \frac{1}{2}\langle J(t), J(t)\rangle$. Then $f'(t) = \langle J, J'\rangle$, and

$$
\begin{aligned}
f''(t) &= \langle J', J'\rangle + \langle J, J''\rangle \\
&= \|J'\|^2 - \underbrace{\langle J, R(J,\dot\gamma)\dot\gamma\rangle}_{\leq 0 \text{ because } \sec \leq 0} \\
&\geq \|J'\|^2 .
\end{aligned}
$$

$J'' + R(J,\dot\gamma)\dot\gamma = 0$

Thus, $f'(t)$ is nondecreasing. As $f(0)=0$ and $f'(0)=0$, it follows that $f'(t) \geq 0$ for all $t \geq 0$, i.e., $f(t)$ is nondecreasing. Moreover, as $J'(0) \neq 0$, then

$$
f(t) = \frac{f''(0)}{2} t^2 + O(t^3) \geq \frac{1}{2}\|J'(0)\|^2 t^2 > 0
$$

for $t > 0$ sufficiently small, so $f(t) > 0$ for all $t > 0$ because f is nondecreasing. □

Rmk: Later on, we will prove that $\|J(t)\| \geq t\|J'(0)\|$ for all $t \geq 0$ (Rauch I).

> **Cor.** If $\sec \leq 0$, then $\exp_p : T_pM \longrightarrow M$ is a local diffeo.

Pf. By the Inverse Function Theorem, it suffices to show $d(\exp_p)_x : T_x T_pM \to T_{\exp_p x} M$ is invertible for all $x \in T_pM$. Given $w \neq 0 \in T_pM \cong T_{tv}T_pM$, let $J(t)$ be the Jacobi field along $\gamma(t) = \exp_p tv$ with $J(0)=0$ and $J'(0)=w$. Then ← cf. HW3
$J(t) = d(\exp_p)_{tv}\, tJ'(0)$, so for $t \neq 0$, $\|d(\exp_p)_{tv} w\| = \|\frac{1}{t} J(t)\| > 0$ by the Lemma and for $t=0$ we have shown before that $d(\exp_p)_0 = \mathrm{id}$. Thus $d(\exp_p)_x$ is invertible $\forall x \in T_pM$. □

Lemma. If $(\overline{M},\overline{g})$ and (M,g) are connected, $(\overline{M},\overline{g})$ complete, and $\pi:(\overline{M},\overline{g})\to(M,g)$ is a local isometry, then (M,g) is complete and π is a Riem. covering map.

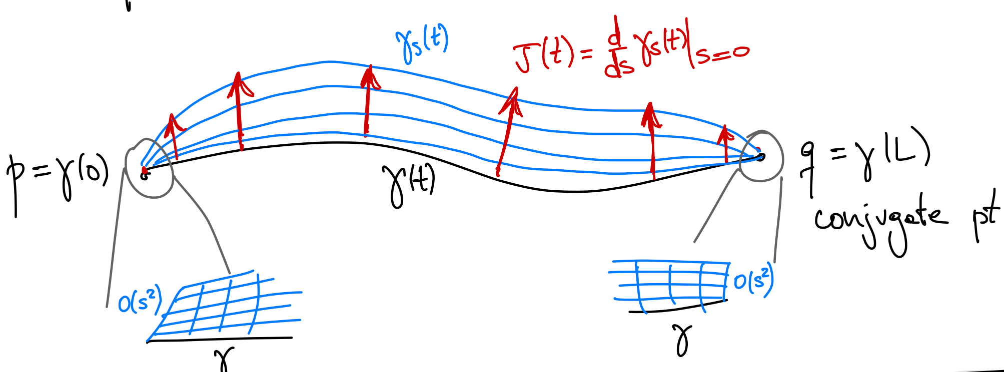
Pf. Basic topology: show that π has the path-lifting property (see [Lee, Thm 6.23] for details).

Pf of Cartan-Hadamard: Since M is complete, we have $\exp_p: T_pM\to M$ well-defined. By Cor. above, it is a local diffeo everywhere, so we can use it to pull back the metric g from M to a metric $\overline{g}=\exp_p^* g$ on T_pM. Thus, $\exp_p:(T_pM,\overline{g})\to(M,g)$ is a local isometry. The manifold $(T_pM,\overline{g})$ is complete by Hopf–Rinow, because the straight lines $t\mapsto tv$ through the origin of T_pM are geodesics w.r.t. $\overline{g}$, and extend to all $t\in\mathbb{R}$. Thus, by Lemma, $\exp_p: T_pM\to M$ is a covering map. $\square$

Cor: There does not exist a metric with $\sec\leq 0$ on S^n, $\mathbb{CP}^n$, ...

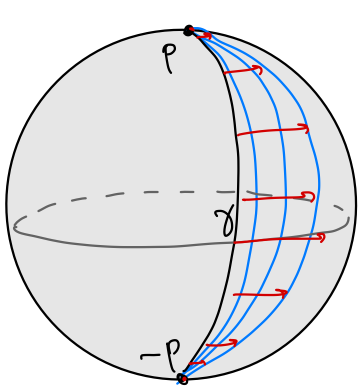
Def: A point $q\in M$ is conjugate to $p\in M$ along a geodesic $\gamma:[0,L]\to M$ if $\gamma(0)=p$, $\gamma(L)=q$ and there exists a Jacobi field $J:[0,L]\to M$ along γ with $J(0)=0$, $J(L)=0$.

Note: By the above, if $\sec\leq 0$, then there are no conjugate points. Moreover, $q=\exp_p Lv$ is conjugate to p along $\gamma(t)=\exp_p tv$ iff $d(\exp_p)_{Lv}: T_{Lv}T_pM\to T_qM$ is noninvertible. In other words, $\exists\,\gamma_s(t)$ a variation of γ by geodesic with endpoints that, to first order, coincide with p, q:

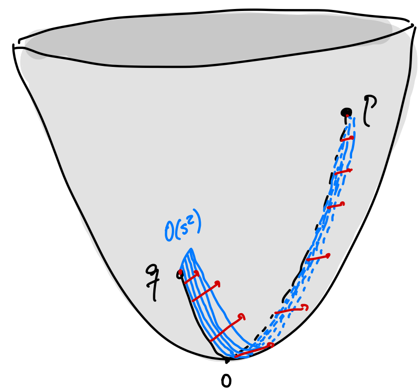


Examples:

On (S^n, g_{S^n}), antipodal pts are conjugate along any geodesic that joins them



On a paraboloid, conjugate points arise along all meridians but no geodesic other than the meridian joins them!



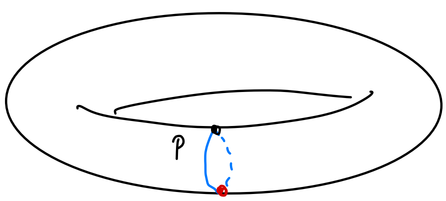
Recall the second variation of energy: $V = \frac{d}{ds}\gamma_s(t)\big|_{s=0}$

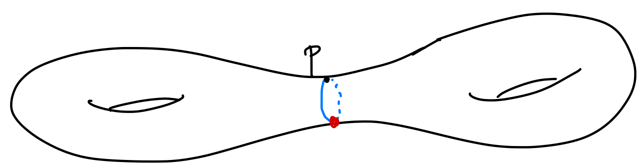
$$\frac{d^2}{ds^2}E(\gamma_s)\Big|_{s=0}(V,V) = g\left(\frac{DV}{ds},\dot\gamma\right)\Big|_a^b + \int_a^b \left\|\frac{DV}{dt}\right\|^2 - g\big(R(V,\dot\gamma)\dot\gamma, V\big)\, dt$$

If the variation has fixed endpoints ($\gamma_s(a)\equiv\gamma_0(a)$, $\gamma_s(b)\equiv\gamma_0(b)$), then $\frac{DV}{ds}(a)=0$, $\frac{DV}{ds}(b)=0$.
Moreover, if $\sec\leq 0$, then $-g\big(R(V,\dot\gamma)\dot\gamma,V\big)\geq 0$, so it follows that

$$\frac{d^2}{ds^2}E(\gamma_s)\Big|_{s=0}(V,V) = \int_a^b \left\|\frac{DV}{dt}\right\|^2 - g\big(R(V,\dot\gamma)\dot\gamma,V\big)\, dt \geq 0$$

i.e., if $\sec\leq 0$, then all geodesics are local minima for E among curves with the same endpoints.
However, they need not be global minima: think of closed geodesics on a torus, or on a hyperbolic manifold, which are minimizing up to half their length.

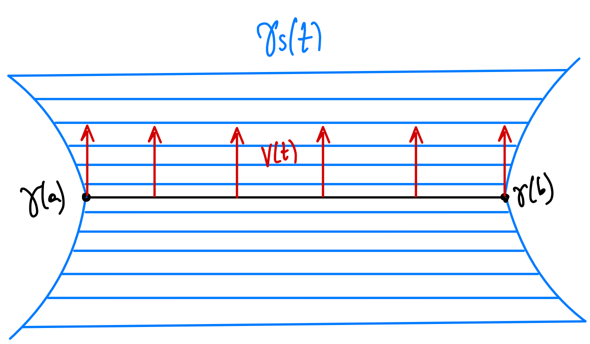




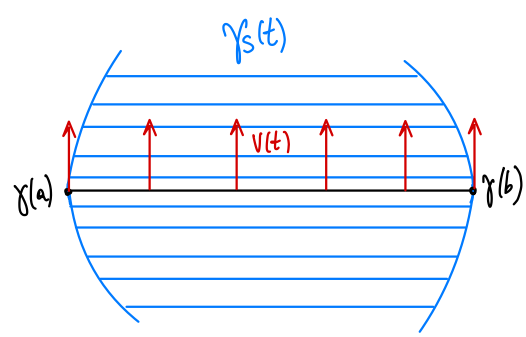
Parallel variations: Let $v\in T_{\gamma(a)}M$ and parallel transport it along the geodesic $\gamma:[a,b]\to M$ to obtain $V(t) = P_t\, v$ with $V(0)=v$ and $\frac{DV}{dt}\equiv 0$. Note that V is the variational vector field of $\gamma_s(t) = \exp_{\gamma(t)} s\cdot V(t)$. Moreover, $\frac{DV}{ds}\equiv 0$ because $s\mapsto\gamma_s(t)$ are geodesics.
Then:

$$\frac{d^2}{ds^2}E(\gamma_s)\Big|_{s=0}(V,V) = -\int_a^b g\big(R(V,\dot\gamma)\dot\gamma,V\big)\, dt$$

i.e., $\gamma_s(t)$ for $0<|s|<\varepsilon$ has $\begin{cases} E(\gamma_s) > E(\gamma) & \text{if } \sec<0 \\ E(\gamma_s) < E(\gamma) & \text{if } \sec>0 \end{cases}$.



$\sec<0$



$\sec>0$

Lecture 13 3/15/2024 $R(X,Y)Z = K$

Jacobi fields in constant curvature: if $\sec \equiv K$, then $R(X,Y)Z = K(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$

Let $\gamma:[0,L]\to M$ be a unit speed geodesic, set $e_1 = \dot\gamma(0)$ and complete it to an o.n.b. $\{e_i\}_{i=1}^n$ of $T_{\gamma(0)}M$, set $E_i(t) = P_t\, e_i$ to be their parallel transp along γ.

$$J(t) = \sum_{i=1}^n f_i(t)E_i(t) \text{ is a Jacobi field w/ } J(0), J'(0) \perp \dot\gamma(0) \quad {=}\mathrm{span}\{e_2,\dots,e_n\}$$

$$\Updownarrow$$

$$J'' + R(J,\dot\gamma)\dot\gamma = 0, \quad \begin{cases}\langle J(0),\dot\gamma(0)\rangle = 0\\ \langle J'(0),\dot\gamma(0)\rangle = 0\end{cases}$$

Recall: $\langle J(t),\dot\gamma(t)\rangle = \underbrace{\langle J'(0),\dot\gamma(0)\rangle}_{=0} t + \underbrace{\langle J(0),\dot\gamma(0)\rangle}_{=0}$ HW3 #2

so $f_1 = \langle J,\dot\gamma\rangle \equiv 0$

$$\Updownarrow \longleftarrow R(J,\dot\gamma)\dot\gamma = K(\cancel{\langle\dot\gamma,\dot\gamma\rangle}^{1} J - \cancel{\langle J,\dot\gamma\rangle}^{0 \text{ if } 2\le i\le n}\dot\gamma)$$

$$\sum_{i=1}^n f_i''(t)E_i(t) + K f_i E_i(t) = 0$$

$$\Updownarrow$$

for $i=1$, we have $f_1(t)\equiv 0$

$$f_i''(t) + K f_i(t) = 0 \quad \text{for all } i = 2,\dots,n.$$

Thus, if $J(0)=0$ and $J'(0) = e_j$, $j\neq 1$, then $\boxed{J(t) = sn_K(t)\cdot E_j(t)}$ for all $t\in[0,L]$,

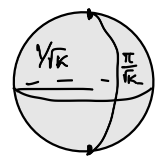
where $sn_K(t)$ is the solution to $\begin{cases} sn_K'' + K\, sn_K = 0\\ sn_K(0) = 0,\ sn_K'(0) = 1.\end{cases}$

$$sn_K(t) = \begin{cases}\frac{1}{\sqrt K}\sin t\sqrt K, & K>0\\ t, & K=0\\ \frac{1}{\sqrt{-K}}\sinh t\sqrt{-K}, & K<0\end{cases}$$

Note: if $K\le 0$, then $sn_K(t) > 0$ for all $t>0$. No conjugate points

if $K>0$, then $sn_K(t)>0$ for $t\in(0,\pi/\sqrt K)$, and $sn_K(\pi/\sqrt K) = 0$. Conjugate points at distance $\pi/\sqrt K$.

Note: A unit speed geodesic $\gamma:[0,L]\to S^n(1/\sqrt K)$ is not minimizing if $L > \pi/\sqrt K$. Indeed, geodesics cannot be minimizing after passing the first conjugate point:



> **Thm.** If $\gamma:[0,L]\to M$ is a unit speed geodesic and $0<c<L$ is s.t. $\gamma(c)$ is conjugate to $\gamma(0)$ along γ, then $\gamma:[0,L]\to M$ is not minimizing.

Pf. Let $J:[0,L]\to TM$ be a Jacobi field along γ with $J(0)=0$ and $J(c)=0$.

Let $W:[0,L]\to TM$ be a smooth vector field along γ with $W(0)=0$, $W(c) = -J'(c)$, $W(L)=0$.

$-J'(c) \neq 0$ b/c $J\not\equiv 0$.

Define $V:[0,L]\to TM$ as follows:

$$V(t)=\begin{cases} J(t)+\varepsilon\, W(t) & \text{if } t\in[0,c] \\ \varepsilon\, W(t) & \text{if } t\in[c,L] \end{cases}$$

Note. V' is not continuous at $t=c$, so V is differentiable but not C^1.

Then let $\gamma_s(t)=\exp_{\gamma(t)} sV(t)$ be a variation of γ with variational field V.

Note that $\frac{DV}{ds}\equiv 0$, and $\gamma_s(t)$ has fixed endpoints. So:

$$\frac{d^2}{ds^2}E(\gamma_s)\Big|_{s=0}(V,V)=\cancel{g\left(\frac{DV}{ds},\dot\gamma\right)\Big|_0^L}^{\,0}+\int_0^L g\left(\frac{DV}{dt},\frac{DV}{dt}\right)+g\big(R(V,\dot\gamma)V,\dot\gamma\big)\,dt$$

$$=\left(\int_0^c g(V',V')-g\big(R(V,\dot\gamma)\dot\gamma,V\big)\,dt\right)+\left(\int_c^L g(V',V')-g\big(R(V,\dot\gamma)\dot\gamma,V\big)\,dt\right)$$

within $(0,c)$ and (c,L) the vector field V is smooth, so can integrate by parts.

$$\overset{\text{int. by parts}}{=} g(V',V)\Big|_0^c-\int_0^c g\big(V''+R(V,\dot\gamma)\dot\gamma,V\big)\,dt$$

$$+\varepsilon^2\left(g(W',W)\Big|_c^L-\int_c^L g\big(W''+R(W,\dot\gamma)\dot\gamma,W\big)\,dt\right)$$

$$=g\big(J'(c)+\varepsilon W'(c),\underset{\substack{\| \\ -J'(c)}}{\varepsilon W(c)}\big)+O(\varepsilon^2)$$

$$=-\varepsilon\|J'(c)\|^2+O(\varepsilon^2)$$

so for $\varepsilon>0$ and $s\neq 0$ sufficiently small, we have $E(\gamma_s)<E(\gamma)$, and hence $\gamma:[0,L]\to M$ is not minimizing (for E hence for L). $\square$

Rmk: Passing a conjugate point is not the only way in which a geodesic stops being minimizing; the other possibility (non-exclusive) is that there exists another geodesic with same length and endpoints (e.g., think of closed geodesics on manifolds with $\sec\le 0$, see p. 4).

Rmk: Using ODE comparison theorems, one can show that if (M,g) has $\sec\geq K$, then conjugate points along geodesics in (M,g) arise faster than along geodesics in $S^n(1/\sqrt{K})$. This yields an alternative proof of the next result (Myers theorem)

Def: $\operatorname{diam}(M,g) = \sup\{d(p,q) : p,q \in M\}$ is the diameter of (M,g).

From basic topology, $\operatorname{diam}(M,g) < \infty \iff M$ is compact.

Later on, we will show that a weaker curvature bound ($\operatorname{Ric} \geq K > 0$) is enough.

> **Thm** (Myers, 1941). If (M^n, g) is a complete manifold with $\sec \geq K > 0$, then it has $\operatorname{diam}(M,g) \leq \pi/\sqrt{K}$. In particular, it is compact and $\pi_1 M$ is finite.

Pf: Let $p,q \in M$ and let $\gamma : [0,L] \to M$ be a unit speed minimizing geodesic with $\gamma(0) = p$ $\gamma(L) = q$. Since γ is minimizing, for all variations γ_s of γ with fixed endpoints, $\frac{d^2}{ds^2} E(\gamma_s)\big|_{s=0} \geq 0$. Let $v \in T_pM$ with $\|v\| = 1$ and $\langle \dot\gamma(0), v\rangle = 0$, set

$$V(t) = \sin\left(\frac{\pi t}{L}\right) \cdot P_t v$$

parallel transport $v \in T_{\gamma(0)}M$ along γ, so $\|P_t v\| = 1$ and $g(P_t v, \dot\gamma(t)) = 0$.

Clearly, $V(0) = 0$, $V(L) = 0$, and $V'(t) = \frac{\pi}{L}\cos\left(\frac{\pi t}{L}\right) P_t v$, $V''(t) = -\frac{\pi^2}{L^2}\sin\left(\frac{\pi t}{L}\right) P_t v$.

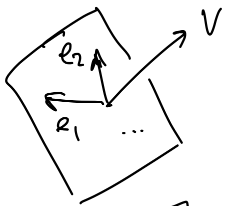
Then $\gamma_s(t) = \exp_{\gamma(t)} sV(t)$ is a variation of γ with fixed endpoints, hence:

$$0 \leq \frac{d^2}{ds^2} E(\gamma_s)\Big|_{s=0}(V,V) = \underbrace{\cancel{g\left(\frac{DV}{ds}, \dot\gamma\right)\Big|_0^L}}_{0} + \underbrace{\cancel{g\left(\frac{DV}{dt}, V\right)\Big|_0^L}}_{0}$$
$$- \int_0^L g\left(\frac{D^2V}{dt^2}, V\right) + g(R(V,\dot\gamma)\dot\gamma, V)\, dt$$
$$= -\int_0^L \left(-\frac{\pi^2}{L^2}\sin\left(\frac{\pi t}{L}\right)^2 \|P_t v\|^2 + \sin\left(\frac{\pi t}{L}\right)^2 g(R(P_t v, \dot\gamma)\dot\gamma, P_t v)\right) dt$$
$$= \int_0^L \sin\left(\frac{\pi t}{L}\right)^2 \left(\frac{\pi^2}{L^2} - \underbrace{\sec(P_t v \wedge \dot\gamma)}_{\geq K}\right) dt$$
$$\leq \left(\frac{\pi^2}{L^2} - K\right) \int_0^L \sin\left(\frac{\pi t}{L}\right)^2 dt$$

Thus, $\frac{\pi^2}{L^2} - K \geq 0$, i.e., $L \leq \frac{\pi}{\sqrt{K}}$. It follows that $d(p,q) \leq \frac{\pi}{\sqrt{K}}$ for all $p,q \in M$ hence $\operatorname{diam}(M,g) \leq \frac{\pi}{\sqrt{K}}$ and hence M is compact. If $\pi : \tilde M \to M$ is the universal covering, then by the same argument $(\tilde M, \pi^* g)$ is compact, so $\pi_1 M$ is finite. $\square$

Lecture 14 3/20/2024

Ricci curvature: Let $v \in T_pM$ and $\{e_1, \dots, e_{n-1}\}$ be an o.n.b. of $v^\perp$; so that $\{e_1, \dots, e_{n-1}, v\}$ is an o.n.b. of T_pM. Then



$$\mathrm{Ric}(v,v) = \sum_{i=1}^{n-1} \langle R(e_i, v)v, e_i \rangle$$

if $|v|=1$, then this is $\sec(v \wedge e_i)$

The above quadratic form defines, via polarization, a bilinear symmetric form

$$\mathrm{Ric}(v,w) = \sum_{i=1}^{n} \langle R(e_i, v)w, e_i \rangle = \mathrm{tr}\left(x \mapsto R(x,v)w\right)$$

which is represented by a symmetric endomorphism $\mathrm{Ric}_p : T_pM \to T_pM$, s.t.

$$\mathrm{Ric}(v,w) = g(\mathrm{Ric}(v), w)$$

All of the above are referred to as "Ricci curvature."

(For us, usually I mean $\mathrm{Ric} : TM \times TM \to \mathbb{R}$)

Scalar curvature: $\mathrm{scal} = \mathrm{tr}\,\mathrm{Ric} = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i)$ for $\{e_i\}$ o.n.b. of T_pM.

$$= \sum_{i,j=1}^{n} \langle R(e_i, e_j)e_j, e_i \rangle = 2\,\mathrm{tr}\left(R : \Lambda^2 TM \to \Lambda^2 TM\right)$$

$\mathrm{scal} : M \to \mathbb{R}$ is a function.

Both $\langle R(e_i \wedge e_j), e_i \wedge e_j \rangle$ and $\langle R(e_j \wedge e_i), e_j \wedge e_i \rangle$ appear in scal, but only one appears in $\mathrm{tr}\,R$.

Def: The metric g is called Einstein, with Einstein constant λ if $\mathrm{Ric} = \lambda \cdot g$.

Note: If g is Einstein, then $\mathrm{scal}_g = \mathrm{tr}\,\mathrm{Ric} = \mathrm{tr}\,\lambda \cdot g = \lambda \cdot n$.

If (M,g) has $\sec \equiv K$, then $\mathrm{Ric} = K(n-1) \cdot g$, so it is Einstein, $\mathrm{scal} = n(n-1)K$.

Often write $A \geq b$ to mean $A \geq b \cdot \mathrm{Id}$ where $b \in \mathbb{R}$.

Note: Inequality between symmetric tensors $A \geq B$ means $g(Av, v) \geq g(Bv, v)$, $\forall v$, e.g., $\mathrm{Ric} \geq g$, often written $\mathrm{Ric} \geq 1$, means $\mathrm{Ric}(v,v) \geq g(v,v)$ for all v.

Clearly, $\boxed{\sec \geq K \implies \mathrm{Ric} \geq (n-1)K \cdot g \implies \mathrm{scal} \geq n(n-1) \cdot K}$, same for $\leq$.

This hypothesis is weaker than how Thm was stated earlier!

Thm (Myers, 1941). If (M^n, g) is a complete manifold with $\mathrm{Ric} \geq K(n-1) \cdot g$, then it has $\mathrm{diam}(M,g) \leq \pi/\sqrt{K}$. In particular, it is compact and $\pi_1 M$ is finite.

Pf: Let $p, q \in M$ and let $\gamma : [0,L] \to M$ be a unit speed minimizing geodesic with $\gamma(0) = p$, $\gamma(L) = q$. Since γ is minimizing, for all variations γ_s of γ with fixed endpoints, $\frac{d^2}{ds^2} E(\gamma_s)\big|_{s=0} \geq 0$. Let $\{e_i\}$ be an o.n.b. of $\dot\gamma(0)^\perp \subset T_{\gamma(0)}M$, set

$$V_i(t) = \sin\left(\frac{\pi t}{L}\right) \cdot P_t e_i$$

parallel transport $e_i \in T_{\gamma(0)}M$ along γ, so $g(P_t e_i, P_t e_j) = \delta_{ij}$ and $g(P_t e_i, \dot\gamma(t)) = 0$.

Clearly, $V_i(0) = 0$, $V_i(L) = 0$, and $V_i'(t) = \frac{\pi}{L} \cos\left(\frac{\pi t}{L}\right) P_t e_i$, $V_i''(t) = -\frac{\pi^2}{L^2} \underbrace{\sin\left(\frac{\pi t}{L}\right) P_t e_i}_{V_i(t)}$

Then $\gamma_s^i(t) = \exp_{\gamma(t)} s V_i(t)$ is a variation of γ with fixed endpoints, hence:

$$0 \leq \frac{d^2}{ds^2} E(\gamma_s^i)\Big|_{s=0} (V_i, V_i) = \cancelto{0}{g\left(\frac{DV_i}{ds}, \dot\gamma\right)\Big|_0^L} + \cancelto{0}{g\left(\frac{DV_i}{dt}, V_i\right)\Big|_0^L}$$

$$- \int_0^L g\left(\frac{D^2 V_i}{dt^2}, V_i\right) + g\left(R(V_i, \dot\gamma)\dot\gamma, V_i\right) \, dt$$

$$= - \int_0^L \left(- \frac{\pi^2}{L^2} \sin\left(\frac{\pi t}{L}\right)^2 \|P_t e_i\|^2 + \sin\left(\frac{\pi t}{L}\right)^2 g\left(R(P_t e_i, \dot\gamma)\dot\gamma, P_t e_i\right) \right) dt$$

$$= \int_0^L \sin\left(\frac{\pi t}{L}\right)^2 \left(\frac{\pi^2}{L^2} - \sec(P_t e_i \wedge \dot\gamma) \right) dt$$

Add over $i = 1, \dots, n-1$ to get:

$$0 \leq \sum_{i=1}^{n-1} \frac{d^2}{ds^2} E(\gamma_s^i)\Big|_{s=0} (V_i, V_i) = \sum_{i=1}^{n-1} \int_0^L \sin\left(\frac{\pi t}{L}\right)^2 \left(\frac{\pi^2}{L^2} - \sec(P_t e_i \wedge \dot\gamma) \right) dt$$

$$= \int_0^L \sin\left(\frac{\pi t}{L}\right)^2 \left((n-1)\frac{\pi^2}{L^2} - \underbrace{\sum_{i=1}^{n-1} \sec(P_t e_i \wedge \dot\gamma)}_{= Ric(\dot\gamma, \dot\gamma) \geq (n-1)K} \right) dt$$

$$\leq (n-1)\left(\frac{\pi^2}{L^2} - K\right) \int_0^L \sin\left(\frac{\pi t}{L}\right)^2 dt.$$

Thus, $\frac{\pi^2}{L^2} - K \geq 0$, i.e., $L \leq \frac{\pi}{\sqrt{K}}$. It follows that $d(p,q) \leq \frac{\pi}{\sqrt{K}}$ for all $p, q \in M$ hence $\operatorname{diam}(M, g) \leq \frac{\pi}{\sqrt{K}}$ and hence M is compact. If $\pi : \tilde{M} \to M$ is the universal covering, then by the same argument $(\tilde{M}, \pi^* g)$ is compact, so $\pi_1 M$ is finite. $\square$

Ex: $S^n \times S^1$ does not admit a metric with $Ric > 0$.
(If it did, then it would have $Ric \geq K > 0$ by compactness, contradicting $|\pi_1| = +\infty$).

Note: $S^n \times S^1$ admits metrics with $\mathrm{scal} > 0$. Indeed, for a product metric, $(M_1 \times M_2, g_1 \oplus g_2)$ has $\mathrm{scal}_{g_1 \oplus g_2} = \mathrm{scal}_{g_1} + \mathrm{scal}_{g_2}$, so the product metric $g_{S^n} \oplus d\theta^2$ has $\mathrm{scal} = \mathrm{scal}_{g_{S^n}} = n(n-1)$.

Upshot: $|\pi_1| = +\infty$ detects non existence of metrics with $\mathrm{Ric} > 0$, and can be used to distinguish the classes of closed manifolds that admit metrics with $\mathrm{scal} > 0$ and $\mathrm{Ric} > 0$. However, we have the following:

Open Problem: Does there exist a closed simply-connected manifold M that admits metrics with $\mathrm{scal} > 0$ but does not admit metrics with $\mathrm{Ric} > 0$?

Issue: Lack of known topological obstructions to $\mathrm{Ric} > 0$ besides $|\pi_1| = +\infty$.

On the other hand, using h-principle techniques, it is known that there are no topological obstructions to $\mathrm{Ric} < 0$:

Thm (Lohkamp, Annals of Math 1994). Every manifold M^n, $n \geq 3$, admits a complete metric with $\mathrm{Ric} < 0$.

either compact or noncompact!

Even more: for any metric g on M, and $C > 0$, $\varepsilon > 0$, there is a metric g' on M with $\mathrm{Ric}_{g'} \leq -C$ and $\|g - g'\|_{C^0} < \varepsilon$...

In particular, also $\mathrm{scal} < 0$ is topologically unobstructed. In fact, using PDE one can show that every manifold admits a complete metric with $\mathrm{scal} \equiv -1$. However, $\mathrm{scal} > 0$ is topologically obstructed (this is a very active research area!)

Thm (Lichnerowicz, 1963). If (M^n, g) is a closed Riem. spin mfld with $\mathrm{scal} > 0$, then $\hat{A}(M) = 0$.

Ex: $M^4 := \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{CP}^3 \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$ is a smooth closed spin 4-mfld with $\hat{A}(M) \neq 0$, therefore it does not admit Riem. metrics with $\mathrm{scal} > 0$.

The results above are best suited for a second course in Riem. geometry...

Lecture 15 3/22/2024

Back to the (relatively classical) world of sectional curvature:

Lemma. If $P : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is a linear isometry, i.e. $P \in O(n-1)$, and $\det P = (-1)^n$, then 1 is an eigenvalue of P, so $\exists\, v \in \mathbb{R}^{n-1}$, $v \neq 0$, $Pv = v$.

Pf. If n is even, then the characteristic polynomial $\det(P - \lambda \mathrm{Id})$ has odd degree $n-1$, and real coeff., so P has some real eigenvalues. Since P is orthogonal they are ± 1. The product of complex

eigenvalues is ≥ 0 since they come in conjugate pairs: $(\alpha+\beta i)(\alpha-\beta i)=\alpha^2+\beta^2$; and $\det P = 1$, so at least one of the real eigenvalues is $+1$.
If n is odd, then $\det P = -1$, and as the product of cplx eigenvalues is ≥ 0, there's a positive even number of real eigenvalues, so at least one is $+1$. $\square$

<u>Rmk</u>. By the Max. Torus Thm, $\forall P \in O(n-1)$, $\exists U \in O(n-1)$ s.t.
$$UPU^{-1} = \begin{pmatrix} \boxed{R_{\theta_1}} & & & & \\ & \ddots & & & \\ & & \boxed{R_{\theta_k}} & & \\ & & & \pm 1 & \\ & & & & \ddots \\ & & & & & \pm 1 \end{pmatrix}$$

Rotation 2×2 blocks (eigenvalues $\pm i\theta$)

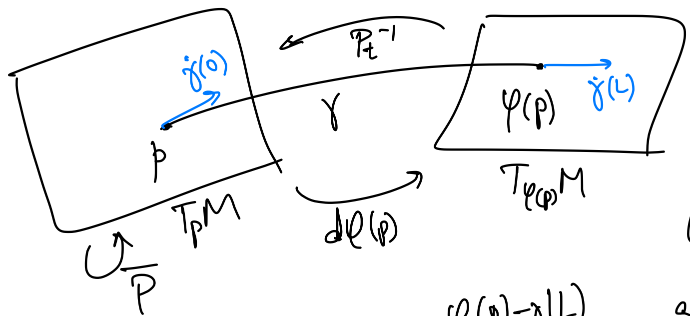
eigenvalues ± 1

<u>Thm</u>. (Weinstein). Let (M^n, g) be an oriented closed Riem. mfld with $\sec > 0$, and $\varphi : M \to M$ an isometry that preserves orientation if n is even, reverses if n is odd. Then φ has a fixed point.

<u>Pf</u>. Suppose $\varphi(x) \neq x$, $\forall x \in M$ and let $p \in M$ be s.t. $\mathrm{dist}_g(p, \varphi(p)) = \min_{x \in M} \{\mathrm{dist}_g(x, \varphi(x))\}$.
Let $\gamma : [0, L] \to M$ be a minimizing geod. from $p = \gamma(0)$ to $\varphi(p) = \gamma(L)$. Let

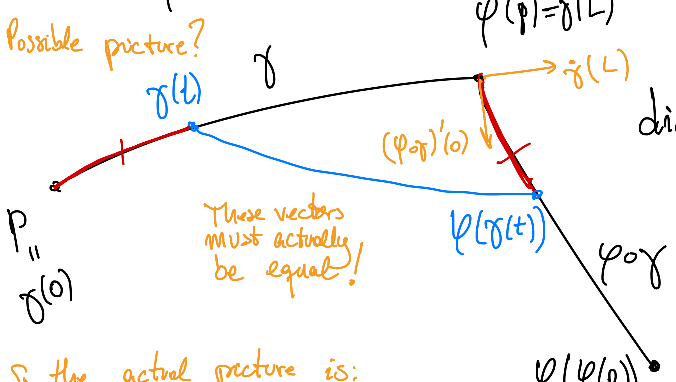
$$\overline{P} = P_t^{-1} \circ d\varphi(p) : T_pM \to T_pM$$

where P_t^{-1} is the parallel transp. along γ from $\varphi(p) = \gamma(L)$ back to $p = \gamma(0)$.



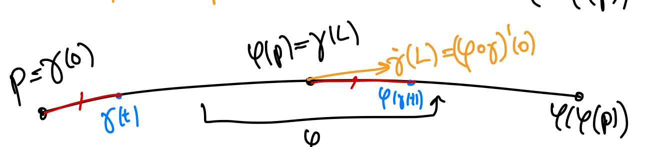
<u>Claim</u>. $(\varphi \circ \gamma)'(0) = \gamma'(L)$.

Consider $\varphi \circ \gamma : [0, L] \to M$, which is a geodesic from $\varphi(p)$ to $\varphi(\varphi(p))$. Given $t \in (0, L)$,



$$\mathrm{dist}_g(\gamma(t), \varphi(\gamma(t))) \overset{(*)}{\leq} \mathrm{dist}_g(\gamma(t), \varphi(p)) + \mathrm{dist}_g(\varphi(p), \varphi(\gamma(t)))$$
φ is isom.
$$= \mathrm{dist}(\gamma(t), \varphi(p)) + \mathrm{dist}(p, \gamma(t))$$
γ is minimizing.
$$= \mathrm{dist}(p, \varphi(p)).$$

So the <u>actual</u> picture is:



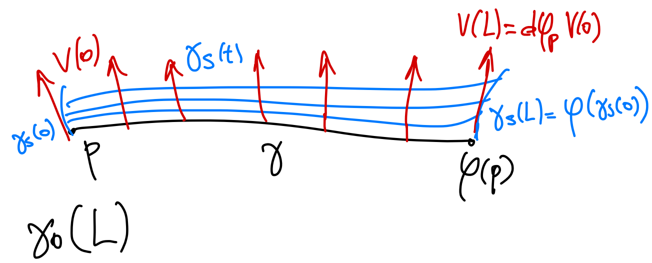
As p was chosen to <u>minimize</u> displacement by φ,
$$\mathrm{dist}_g(\gamma(t), \varphi(\gamma(t))) \geq \mathrm{dist}_g(p, \varphi(p))$$
hence equality holds. From equality in $(*)$, it follows that $\gamma([0,L]) \cup \varphi(\gamma([0,L]))$ is a minimizing curve (i.e., distances are achieved along that curve), so it is a geodesic. Thus it is smooth at $\varphi(p)$, proving the claim.

Thus $\overline{P}\dot\gamma(0) = P_t^{-1}\, d\varphi(p)\, \gamma'(0) = P_t^{-1}(\varphi\circ\gamma)'(0) = P_t^{-1}\dot\gamma(L) = \dot\gamma(0)$; so $\overline{P}$ fixes $\dot\gamma(0) \in T_pM$. Moreover, $P := \left(\overline{P}|_{\dot\gamma(0)^\perp}\right) : \dot\gamma(0)^\perp \to \dot\gamma(0)^\perp$ is an isometry of $\dot\gamma(0)^\perp \subset T_pM$, and since $\overline{P}\dot\gamma(0) = \dot\gamma(0)$, we have

$$\det P = \det \overline{P} = \det\big(\underset{\substack{\uparrow\\ \text{preserves}\\ \text{orientation!}}}{P_t^{-1}} \circ \underset{\substack{\uparrow\\ \det d\varphi(p) = (-1)^n\\ \text{by hypothesis.}}}{d\varphi(p)}\big) = (-1)^n.$$

By the Lemma, $\exists\, v \in \dot\gamma(0)^\perp$, $v \neq 0$, s.t. $Pv = v$.

Let $V(t) = P_t v$ be the parallel transport of $v \in T_pM$ along $\gamma : [0,L] \to M$, note that $g(V(t), \dot\gamma(t)) = 0$ and $d\varphi(p)\, V(0) = V(L)$.



Then $\gamma_s(t) = \exp_{\gamma(t)} sV(t)$ is a variation by geodesics s.t.

$$\gamma_s(L) = \exp_{\varphi(p)} sV(L) \overset{\substack{\text{ODE}\\ \text{uniqueness}\\ \downarrow}}{=} \varphi(\gamma_s(0))$$

since $s \mapsto \varphi(\gamma_s(0))$ is a geod. with $\begin{cases} \varphi(\gamma_0(0)) = \varphi(p) \\ \frac{d}{ds}\varphi(\gamma_s(0)) = d\varphi_p V(0) = V(L). \end{cases}$

and,

$$\frac{d^2}{ds^2}E(\gamma_s)\Big|_{s=0}(V,V) = g\Big(\underbrace{\cancel{\tfrac{DV}{ds}}}_{0}, \dot\gamma\Big)\Big|_a^b + \int_a^b \Big\|\underbrace{\cancel{\tfrac{DV}{dt}}}_{0}\Big\|^2 - \underbrace{g(R(V,\dot\gamma)\dot\gamma, V)}_{>0 \text{ b/c } \sec>0}\, dt < 0,$$

contradicting the choice of p, which yields $s=0$ is a minimum for $s \mapsto \mathrm{dist}_g(\gamma_s(0), \varphi(\gamma_s(0)))$. $\square$

As a corollary, we recover:

> **Thm** (Synge, 1936). Let (M^n, g) be a closed manifold with $\sec > 0$.
>
> (i) If n is even and M is orientable, then $\pi_1 M = \{1\}$,
> M is non-orientable, then $\pi_1 M \cong \mathbb{Z}_2$.
>
> (ii) If n is odd, then M is orientable. ← Note: $\pi_1 M$ can be arbitrarily large, e.g., lens space $S^3/\mathbb{Z}_p$ has $\sec > 0$.

Pf. Since M^n is closed, $\sec \geq K > 0$. Let $\pi : \widetilde{M}^n \to M^n$ be the universal cover, and $\tilde g = \pi^* g$. Then $(\widetilde{M}^n, \tilde g)$ also has $\sec \geq K > 0$.

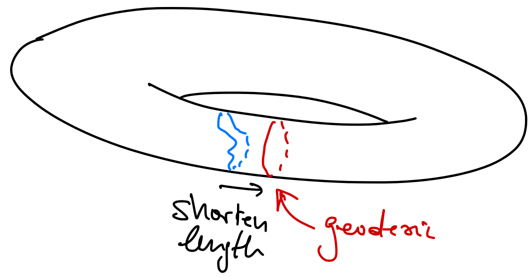
(i) Assume M^n is orientable, and endow $\widetilde{M}^n$ with a compatible orientation. Then any deck transformation $\varphi : \widetilde{M}^n \to \widetilde{M}^n$ preserves orientation and hence has a fixed point by Weinstein's Thm, hence $\varphi = \mathrm{id}$ and thus $\widetilde{M} = M$ is simply-connected. If M is non-orientable, apply previous argument to its orientable double-cover to conclude it is $\widetilde{M}$ and hence $\pi_1 M \cong \mathbb{Z}_2$.

(ii) If M^n is non-orientable, then $\exists \varphi : \overline{M} \to \overline{M}$ an orientation-reversing isometry of the orientable double-cover $\overline{M} \to M$. By Weinstein's Theorem, φ has a fixed point, hence $\varphi = id$, contradicting that φ is orientation-reversing. $\square$

Alternatively, the above can be proven with the second variation of energy and the following result:

$$\boxed{\text{Prop. If } (M^n, g) \text{ is a closed Riem. mfld, then every nontrivial free homotopy class in } M \text{ is represented by a closed geodesic that has least length among curves in its free homotopy class.}}$$

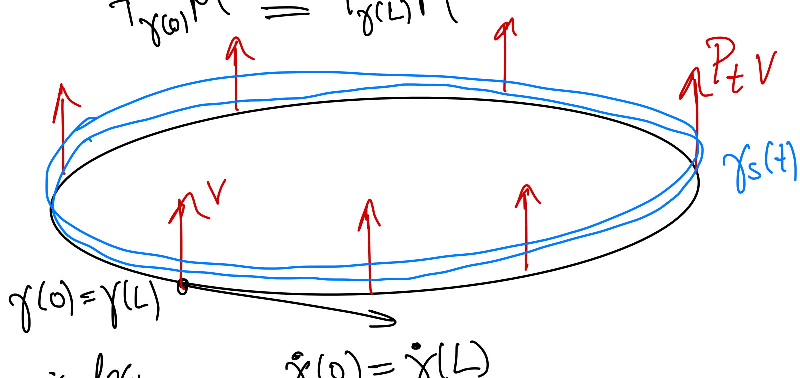
Pf of Synge: (i) Suppose M is oriented, $\dim M$ is even.



Given a nontrivial element in $\pi_1 M$, let γ be a closed geodesic with least length that represents that free homotopy class. The parallel transport along γ gives an orientation-preserving linear isometry

$P_t : \dot\gamma(0)^\perp \to \dot\gamma(L)^\perp$, which (by Lemma) has a fixed vector $v \in \dot\gamma(0)^\perp$

$\underset{T_{\gamma(0)}M \;\cong\; T_{\gamma(L)}M}{}$



Let $\gamma_s(t) = \exp_{\gamma(t)} sP_t v$

Then, as $V = \frac{d}{ds}\gamma_s(t)\big|_{s=0} = P_t v$ has $\frac{DV}{dt}\big|_{s=0} \equiv 0$, and $\frac{DV}{ds} \equiv 0$,

we obtain a contradiction:

$$\overset{\gamma \text{ is loc. minimizing}}{0 \le} \frac{d^2}{ds^2} E(\gamma_s)\Big|_{s=0} = \cancelto{0}{g\left(\frac{DV}{ds}, \dot\gamma\right)\Big|_0^L} + \int_0^L \cancelto{0}{g\left(\frac{DV}{dt}, \frac{DV}{dt}\right)} - \underbrace{g\big(R(V,\dot\gamma)\dot\gamma, V\big)}_{>0 \text{ b/c sec}>0}\, dt < 0$$

So $\pi_1 M = \{1\}$. If M is non-orientable, apply the above to its oriented double-cover to conclude that is its universal cover hence $\pi_1 M \cong \mathbb{Z}_2$.

(ii) Exercise.

Lecture 16 3/27/2024

Submanifold Geometry

Let $i : M \longrightarrow (\overline{M}, \overline{g})$ be an immersion, endow M with $g = i^* \overline{g}$.

$\overline{\nabla}$: Levi-Civita connection of $(\overline{M}, \overline{g})$

∇ : Levi-Civita connection of (M, g)

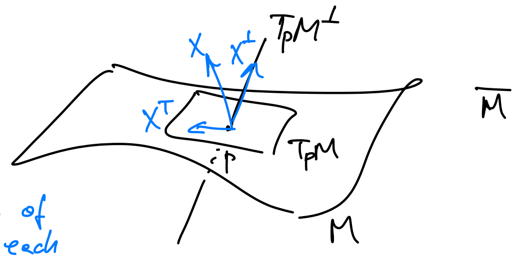
Clearly these need not be the same: e.g. geodesics on $S^n \subset \mathbb{R}^{n+1}$ are not straight lines in $\mathbb{R}^{n+1}$!

Henceforth, we often treat i as an inclusion $M \subset \overline{M}$ and write

$$T_p \overline{M} = T_p M \oplus T_p M^{\perp}$$

$$X = X^{T} + X^{\perp}$$

components of $X \in T_p\overline{M}$ in each summand



Let $U \ni p$ be a small neighborhood of $p \in M$. Given vector fields $X, Y \in \mathfrak{X}(U)$, there exist (many) extensions $\overline{X}, \overline{Y}$ to vector fields on $\overline{U} \ni p$, where $\overline{U} \subset \overline{M}$ is a neighborhood of $p \in \overline{M}$ s.t. $U = \overline{U} \cap M$.

These is old-fashioned terminology that stuck; the "first fundamental form" is just $g = i^* \overline{g}$

Def. The second fundamental form of $M \hookrightarrow \overline{M}$ is $\mathrm{II} : TM \times TM \to TM^{\perp}$, given by

$$\mathrm{II}(X,Y) = \left(\overline{\nabla}_{\overline{X}} \overline{Y}\right)^{\perp}.$$

Note. II is well-defined, i.e., independent of choice of extension of X, Y, tensorial, and symmetric. Indeed,

$$\mathrm{II}(X,Y) - \mathrm{II}(Y,X) = \left(\overline{\nabla}_{\overline{X}} \overline{Y}\right)^{\perp} - \left(\overline{\nabla}_{\overline{Y}} \overline{X}\right)^{\perp}$$

$$= \left(\overline{\nabla}_{\overline{X}} \overline{Y} - \overline{\nabla}_{\overline{Y}} \overline{X}\right)^{\perp} = [\overline{X}, \overline{Y}]^{\perp}$$

but along $M \subset \overline{M}$, the vector fields $\overline{X}, \overline{Y}$ are tangent to M, hence so is their bracket, so the above vanishes (along M), i.e.

$$\mathrm{II}(X,Y) = \mathrm{II}(Y,X) \quad \forall X, Y \in TM$$

Since $\left(\overline{\nabla}_{\overline{X}} \overline{Y}\right)_p$ only depends on $\overline{X}_p = X_p$, it is independent of the extension chosen for X, and $C^\infty(M)$-linear in X. By symmetry, same for Y.

Prop. $\left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)\Big|_M = \nabla_X Y + \mathrm{II}(X,Y)$, for any extensions $\overline{X}, \overline{Y}$ of X, Y.

Pf. Since $\mathrm{II}(X,Y) = \left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)^{\perp}$, it suffices to show $\nabla_X Y = \left(\overline{\nabla}_{\overline{X}}\overline{Y}\right)^{T}$. Both are torsion-free connections on M compatible with g, hence agree by the uniqueness of the Levi-Civita connection on (M,g). $\square$

Def: Given $\vec{n} \in TM^{\perp}$ a normal vector field, the symmetric linear map $S_{\vec{n}}: TM \to TM$ s.t. $\overline{g}(S_{\vec{n}}X, Y) = \overline{g}(\mathrm{II}(X,Y), \vec{n})$ for all $X, Y \in TM$ is called the Shape operator (or Weingarten operator) of M in direction $\vec{n}$.

Prop. $S_{\vec{n}}(X) = -\left(\overline{\nabla}_X \vec{n}\right)^{T}$

Pf.

$$\begin{aligned}
0 = \overline{X}\left(\overline{g}(\vec{n}, \overline{Y})\right) &= \overline{g}\left(\overline{\nabla}_{\overline{X}}\vec{n}, \overline{Y}\right) + \overline{g}\left(\vec{n}, \overline{\nabla}_{\overline{X}}\overline{Y}\right) \\
&= \overline{g}\left(\overline{\nabla}_{\overline{X}}\vec{n}, \overline{Y}\right) + \overline{g}\Big(\vec{n}, \underset{\substack{\cap \\ TM}}{\nabla_X Y} + \underset{\substack{\cap \\ TM^{\perp}}}{\mathrm{II}(X,Y)}\Big) \\
&= \overline{g}\left(\overline{\nabla}_{\overline{X}}\vec{n}, \overline{Y}\right) + \overline{g}\left(\vec{n}, \mathrm{II}(X,Y)\right) \\
&= \overline{g}\left(\overline{\nabla}_{\overline{X}}\vec{n}, \overline{Y}\right) + \overline{g}\left(S_{\vec{n}}X, Y\right)
\end{aligned}$$

for all $Y \in TM$, so, along M, we have $\left(\overline{\nabla}_{\overline{X}}\vec{n} + S_{\vec{n}}X\right)^{T} \equiv 0$, i.e. $S_{\vec{n}}X = -\left(\overline{\nabla}_{\overline{X}}\vec{n}\right)^{T}$. $\square$

Thm. (Gauss Equation). The difference between ambient and intrinsic curvature is:

$$\boxed{\overline{g}\left(\overline{R}(\overline{X},\overline{Y})\overline{Z}, \overline{W}\right) - g\left(R(X,Y)Z, W\right) = g\left(\mathrm{II}(X,Z), \mathrm{II}(Y,W)\right) - g\left(\mathrm{II}(X,W), \mathrm{II}(Y,Z)\right)}$$

Pf:

$$\begin{aligned}
\overline{g}\left(\overline{R}(\overline{X},\overline{Y})\overline{Z}, \overline{W}\right) &= \overline{g}\left(\overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z} - \overline{\nabla}_{\overline{Y}}\overline{\nabla}_{\overline{X}}\overline{Z} - \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z}, \overline{W}\right) \\
&= \overline{g}\left(\overline{\nabla}_{\overline{X}}\left(\overline{\nabla_Y Z} + \overline{\mathrm{II}(Y,Z)}\right) - \overline{\nabla}_{\overline{Y}}\left(\overline{\nabla_X Z} + \overline{\mathrm{II}(X,Z)}\right) - \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z}, \overline{W}\right) \\
&= \overline{g}\left(\overline{\nabla}_{\overline{X}}\overline{\nabla_Y Z} - S_{\mathrm{II}(Y,Z)}X - \overline{\nabla}_{\overline{Y}}\overline{\nabla_X Z} + S_{\mathrm{II}(X,Z)}Y - \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z}, \overline{W}\right)
\end{aligned}$$

Since $\overline{W} \in TM$, we can get rid of any normal components:

$$\left(\overline{\nabla}_{\overline{X}}\, \overline{\nabla}_Y Z\right)^T = \nabla_X \nabla_Y Z, \quad \left(\overline{\nabla}_{[\overline{X},\overline{Y}]} \overline{Z}\right)^T = \nabla_{[X,Y]} Z, \text{ etc., so:}$$

$$\begin{aligned} \dots &= g\left(\nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, W\right) + g\left(S_{\mathrm{II}(X,Z)} Y, W\right) - g\left(S_{\mathrm{II}(Y,Z)} X, W\right) \\ &= g\left(R(X,Y)Z,W\right) + g\left(\mathrm{II}(X,Z), \mathrm{II}(Y,W)\right) - g\left(\mathrm{II}(X,W), \mathrm{II}(Y,Z)\right). \end{aligned}$$

$\square$

__Cor:__ If X, Y are orthonormal, then

$$\overline{\sec}(X\wedge Y) - \sec(X \wedge Y) = \|\mathrm{II}(X,Y)\|^2 - g\left(\mathrm{II}(X,X), \mathrm{II}(Y,Y)\right).$$

__Def.__ $M \hookrightarrow \overline{M}$ is __totally geodesic__ if every geodesic in M is geodesic in $\overline{M}$.

__Prop:__ $M \hookrightarrow \overline{M}$ is totally geodesic if and only if $\mathrm{II} \equiv 0$.

__Pf.__ If $\mathrm{II} \equiv 0$, then Levi-Civita connections of $\overline{M}$ and M agree hence so do their geodesics. Conversely, if M is tot. geod., then let $p \in M$, $v \in T_pM$, and $\gamma : (-\varepsilon, \varepsilon) \to M$ be the geodesic in M (and $\overline{M}$) s.t. $\gamma(0) = p$, $\dot\gamma(0) = v$. Then since $\underbrace{\overline{\nabla}_{\dot\gamma}\dot\gamma}_{=0} = \underbrace{\nabla_{\dot\gamma}\dot\gamma}_{=0} + \mathrm{II}(\dot\gamma, \dot\gamma)$, we have $\mathrm{II}(v,v) = 0$. As v is arbitrary, $\mathrm{II} \equiv 0$. $\square$

__Cor:__ If $M \hookrightarrow \overline{M}$ is totally geodesic, then ambient and intrinsic curvatures agree.

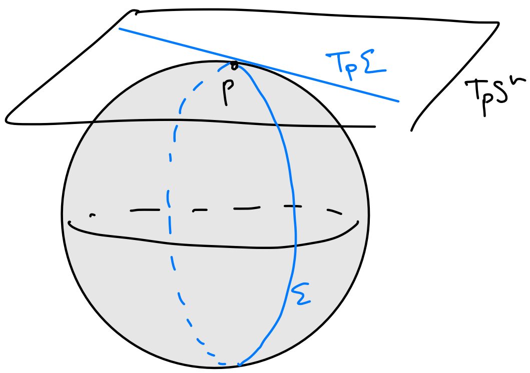
__Ex:__ If $M \subset \overline{M}$ is a hypersurface, i.e. $\dim \overline{M} = \dim M + 1$, and two-sided (i.e., transversely oriented), i.e. $TM^\perp$ is trivial, then let $\vec{n} \in TM^\perp$ be a unit normal to M and note $\mathrm{II}(X,Y) = \underbrace{h(X,Y)}_{\text{scalar}} \cdot \vec{n}$, so $\overline{g}(S_{\vec{n}} X, Y) = h(X,Y)$.

__Ex:__ Round sphere $i : S^n(r) \hookrightarrow \mathbb{R}^{n+1}$ of radius $r > 0$, with $\vec{n}(x) = -\frac{x}{r}$. Then $S_{\vec{n}}(X) = -\overline{\nabla}_{\overline{X}} \vec{n}^{\,T} = \frac{1}{r} X$ so $h(X,Y) = \frac{1}{r}\langle X, Y\rangle$ and we recover:

$$\begin{aligned} g\left(R(X,Y)Z,W\right) &= -g\left(\mathrm{II}(X,Z), \mathrm{II}(Y,W)\right) + g\left(\mathrm{II}(X,W), \mathrm{II}(Y,Z)\right) \\ &= \frac{1}{r^2}\left(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle\right), \quad \text{i.e. } \sec \equiv \frac{1}{r^2}. \end{aligned}$$

Ex: What are the totally geodesic submanifolds of S^n?

If $\Sigma^k \subset S^n$ is tot. geod; let $p \in \Sigma$ and $v \in T_p\Sigma$. Then $\exp_p tv \in \Sigma$ (this is a great circle in S^n)



for all $t \in \mathbb{R}$, so Σ contains a tot. geod. subsphere $S^k = \exp_p(T_p\Sigma) \subset S^n$.

There can't be any $x \in \Sigma \setminus S^k$, otherwise the minimizing geodesic γ from p to x in Σ would have $\dot\gamma(0) \in T_p\Sigma$ so $x \in S^k$; as such γ is also a geodesic in S^n. $\square$

Exercise: Given $k \geq 2$ distinct points $x_1, \dots, x_k \in S^n$, there is a unique (up to congruence) totally geodesic $S^{k-1} \subset S^n$ with $x_j \in S^{k-1}$, $\forall j$.

Note: Same is true on $\mathbb{R}^n$ and $\mathbb{H}^n$.

By a Theorem of Cartan, if (M^n, g) is such that $\forall p \in M$, $\forall \sigma \subset T_pM$ 2-dim $\exists \Sigma \subset M$ tot. geod with $T_p\Sigma = \sigma$, then (M^n, g) has $\sec_g \equiv$ const. On a generic Riem mfld, the only tot. geod. submanifolds are 1-dimensional...

Lecture 17 — 4/3/2024

Some basic definitions in Geometric Analysis: (we need those to begin our discussion of minimal hypersurfaces, since the relevant second variation formula has Δ_Σ instead of $\frac{D^2}{dt^2}$ if $\dim\Sigma > 1$...)

Let $f: M \to \mathbb{R}$ be a smooth function on a Riem. mfld (M^n, g).

- Gradient vector field:
  $\nabla f \in \mathfrak{X}(M)$ is the only vector field such that $g(\nabla f(p), v) = df_p v$, $\forall v \in T_pM$, $\forall p \in M$.

- Hessian:
  $\operatorname{Hess} f \in \operatorname{Sym}^2(TM)$ is defined by $(\operatorname{Hess} f)(X, Y) = g(\nabla_X \nabla f, Y)$ for all $X, Y \in \mathfrak{X}(M)$

  Note: $\operatorname{Hess} f$ is symmetric, since: (Some authors write $\nabla^2 f = \operatorname{Hess} f$)

$$
\begin{aligned}
(\operatorname{Hess} f)(Y, X) - \operatorname{Hess} f(X, Y) &= g(\nabla_Y \nabla f, X) - g(\nabla_X \nabla f, Y) \\
&= Y g(\nabla f, X) - g(\nabla f, \nabla_Y X) - X g(\nabla f, Y) + g(\nabla f, \nabla_X Y) \\
&= Y\, df(X) - X\, df(Y) + df([X, Y]) = YXf - XYf + [X, Y]f = 0.
\end{aligned}
$$

Note: $f: M \to \mathbb{R}$ is convex iff $\operatorname{Hess} f \geq 0$, and concave iff $\operatorname{Hess} f \leq 0$. Equivalently, f is convex iff $\forall \gamma: \mathbb{R} \to M$ geodesic, the function $\mathbb{R} \ni t \mapsto f(\gamma(t)) \in \mathbb{R}$ is convex; similarly for concave. (see HW4)

- Laplacian:

$\Delta f \in C^\infty(M)$ is defined as the trace of the Hessian: $\Delta f = \mathrm{tr}(\mathrm{Hess}\, f)$

i.e., $\Delta f(p) = \sum_{i=1}^{n} \mathrm{Hess}\, f(e_i, e_i)$, where $\{e_i\}$ is an o.n.b. of T_pM.

Note: If $X \in \mathfrak{X}(M)$, the divergence of X is $\mathrm{div}\, X = \mathrm{tr}\, \nabla X \in C^\infty(M)$, so at $p \in M$,

$(\mathrm{div}\, X)(p) = \sum_{i=1}^{n} g(\nabla_{e_i} X, e_i)$, where $\{e_i\}$ is an o.n.b. of T_pM. In particular, $\Delta f = \mathrm{div}\, \nabla f$.

By Stokes Thm, if M is a Riem. mfld w/ boundary ∂M, then $\int_M \mathrm{div}\, X = \int_{\partial M} g(X, \vec{n})$

where $\vec{n}$ is outward unit normal to ∂M. In particular, $\int_M \Delta f = \int_{\partial M} \frac{\partial f}{\partial \vec{n}}$, and if M is closed ($\partial M = \emptyset$), then $\int_M \mathrm{div}\, X = 0$; $\int_M \Delta f = 0$. (integrals of functions are always w.r.t. vol_g!)

Exercise: Using the above, show that if M is closed, then $\int_M |\nabla f|^2 = -\int_M f \Delta f$. [Hint: Compute $\mathrm{div}(f \nabla f)$]

HW4

Some facts about $(-\Delta)$: often called Laplacian (on functions) or Laplace-Beltrami operator.

In what follows, we assume (M^n, g) is connected and closed, i.e., compact and $\partial M = \emptyset$.

Since $\langle (-\Delta) f_1, f_2 \rangle_{L^2} = -\int_M f_1 \Delta f_2 = \int_M g(\nabla f_1, \nabla f_2) = -\int_M f_2 \Delta f_1 = \langle (-\Delta) f_2, f_1 \rangle_{L^2}$

the operator $(-\Delta): C^\infty(M) \to C^\infty(M)$ is essentially self-adjoint in $L^2(M)$ and nonnegative. We also denote by $(-\Delta)$ its self-adjoint extension to $L^2(M)$, which has compact resolvent, and hence the following hold:

1. The spectrum of $-\Delta$ consists of a sequence $\underbrace{0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots}_{= \mathrm{Spec}(-\Delta) \text{ or } \mathrm{Spec}(M,g)} \nearrow +\infty$ of eigenvalues, each with finite multiplicity, which accumulates only at $+\infty$.
2. For each eigenvalue λ_k, the corresponding eigenspace $E_k = \mathrm{Ker}(\Delta + \lambda_k \mathrm{Id}) \subset W^{1,2}(M)$ consists of smooth functions and is finite-dimensional: $m_k = \dim E_k < +\infty$.
3. Eigenfunctions with different eigenvalues are L^2-orthogonal, since $\begin{cases} -\Delta f_i = \lambda_i f_i \\ -\Delta f_j = \lambda_j f_j \end{cases}$

$$\lambda_i \int_M f_i f_j = -\int_M f_j \Delta f_i = -\int_M f_i \Delta f_j = \lambda_j \int_M f_i f_j \Rightarrow (\lambda_i - \lambda_j) \langle f_i, f_j \rangle_{L^2} = 0.$$

4. The eigenfunctions of $-\Delta$ form a complete orthogonal set, so $L^2(M) = \overline{\bigoplus_{k \geq 0} E_k}$
5. Eigenvalues have a variational (min-max) characterization using Rayleigh quotients:

$$\lambda_0 = \inf_{f \in W^{1,2}(M)} \frac{\int_M |\nabla f|^2}{\int_M f^2} \overset{(HW4)}{=} 0, \quad \lambda_k = \inf_{\substack{V \subset W^{1,2}(M) \\ \dim V = k+1}} \sup_{f \in V \setminus \{0\}} \frac{\int_M |\nabla f|^2}{\int_M f^2} = \inf_{\substack{f \in W^{1,2}(M) \setminus \{0\} \\ \langle f, h \rangle = 0,\, \forall h \in E_j,\, j < k}} \frac{\int_M |\nabla f|^2}{\int_M f^2}$$

(These inf and sup are actually min and max...)

6. Weyl's Law: $N(\lambda) = \sum_{\{k:\lambda_k\leq\lambda\}} m_k = \#\left(\mathrm{Spec}(-\Delta)\cap[0,\lambda]\right) \approx \frac{\omega_n}{(2\pi)^n}\mathrm{Vol}(M,g)\cdot\lambda^{\frac{n}{2}} + O(\lambda^{\frac{n-1}{2}})$,

counting with multiplicity

So: $\left(\begin{matrix}\text{k-th eigenvalue}\\ \text{listed according to}\\ \text{multiplicity}\end{matrix}\right)^{\frac{n}{2}} \approx \frac{(2\pi)^n}{\omega_n}\cdot\frac{k}{\mathrm{Vol}(M,g)}$.

volume of unit ball in $\mathbb{R}^n$, so $\frac{\omega_n}{(2\pi)^n} = \frac{1}{(4\pi)^{n/2}\,\Gamma(\frac{n}{2}+1)}$

Some foundational questions about higher order asymptotics remain unanswered in general, see e.g., Polya's conjecture from 1954.

7. Courant's Nodal Domain Theorem: if $f\in E_k$, then

$M\setminus\{f=0\}$ has $\leq k+1$ connected components. How large/small can they be? cf. Yau's conjecture and Logunov's work.

<u>Example</u>: The Laplace spectrum on $(S^n, g_{\mathrm{round}})$ consists of:

$\lambda_k(S^n, g_{\mathrm{round}}) = k(k+n-1)$, with multiplicity $m_k = \binom{n+k}{k} - \binom{n+k-2}{k}$, and the corresponding eigenfunctions are the restriction to $S^n\subset\mathbb{R}^{n+1}$ of harmonic homogeneous polynomials on $\mathbb{R}^{n+1}$ of degree k. Since $(-\Delta)_{\alpha g} = \frac{1}{\alpha}(-\Delta_g)$, the eigenvalues on $S^n(r)$ are $\frac{1}{r^2}k(k+n-1)$.

If (M^n,g) is complete but not compact, then $(-\Delta_g)$ may have continuous spectrum, e.g., $\begin{cases}\mathrm{Spec}(\mathbb{R}^n) = [0,+\infty).\\ \mathrm{Spec}(\mathbb{H}^n) = [\frac{(n-1)^2}{4},+\infty).\end{cases}$

<u>Note</u>: "Spectral Geometry" is an active research area, investigating how $\mathrm{Spec}(-\Delta)$ is related to the geometry of (M^n,g), see e.g., recent AMS Book by Levitin, Mangoubi, Polterovich, the classic book by I. Chavel (CUNY prof!) "Eigenvalues in Riemannian Geometry" and Kac's "Can you hear the shape of a drum?"...

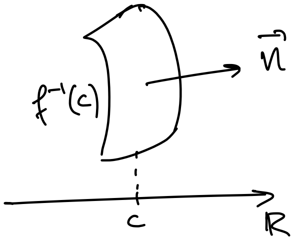
Back to submanifold geometry:

<u>Recall</u>: The second fundamental form of a submfld $M\hookrightarrow\overline{M}$ is $\mathrm{II}(X,Y) = \overline{\nabla}_X Y - \nabla_X Y = (\nabla_X Y)^\perp$

Suppose $f: M\to\mathbb{R}$ is smooth and $c\in\mathbb{R}$ is a <u>regular value</u>.

i.e., $df_p: T_pM\to\mathbb{R}$ is surjective; equivalently, $\nabla f(p)\neq 0$ for all $p\in f^{-1}(c)$.

Then $f^{-1}(c)\subset M$ is a <u>submanifold of codimension 1</u>, two-sided, "hypersurface"



$\vec{n} = \frac{\nabla f}{|\nabla f|}$

$\mathrm{II}(X,Y) = h(X,Y)\,\vec{n}$, where

Careful: Unless $|\nabla f|\equiv 1$, this is <u>not</u> $-\mathrm{Hess}\, f$.

$h(X,Y) = -g(\nabla_X\vec{n}, Y) = -g\left(\nabla_X\frac{\nabla f}{|\nabla f|}, Y\right)$

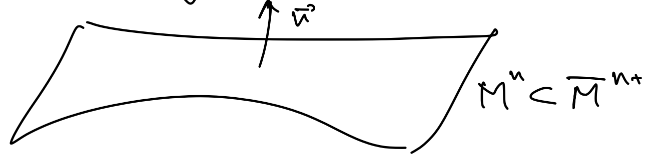
<u>Example</u>: Suppose $f: M\to\mathbb{R}$ is the distance function to a point, or submanifold, or more generally, a solution to the Eikonal equation $|\nabla f| = 1$. Then, if $c\in\mathbb{R}$ is a regular value of f, the hypersurface $f^{-1}(c)$ has unit normal $\vec{n} = \nabla f$, second fundamental form $\mathrm{II}(X,Y) = -(\mathrm{Hess}\, f)(X,Y)\,\vec{n}$, for all $X,Y\in T_p(f^{-1}(c)) = n^\perp$ and shape operator $S_{\vec{n}}X = -(\nabla_X\vec{n})^T = -(\nabla_X\nabla f)^T$, for all $X\in T_p(f^{-1}(c))$. Note also $\mathrm{II} = \mathcal{L}_{\vec{n}}\, g$.

<u>Ex</u>: $f:\mathbb{R}^{n+1}\to\mathbb{R}$, $f(x) = |x|$ is the Euclidean distance to $x=0$. Then:

$\nabla f(x) = \left(-\frac{x_1}{f(x)},\ldots,-\frac{x_n}{f(x)}\right) = -\frac{x}{f(x)}$ satisfies $|\nabla f| = \frac{|x|}{f(x)} = 1$; and all $r>0$ are regular values, so $f^{-1}(r) = S^n(r)$ has shape operator $S_{\vec{n}}X = -(\nabla_X\nabla f)^T = -\left(\nabla_X\left(-\frac{\mathrm{id}}{r}\right)\right)^T = \frac{1}{r}X$ for all $X\in T_xS^n(r) = x^\perp\subset\mathbb{R}^{n+1}$. cf. last lecture!

In what follows, assume $M \subset \overline{M}$ is a two-sided hypersurface, with unit normal $\vec{n}$,

$$\mathrm{II}(X,Y) = h(X,Y) \cdot \vec{n}$$



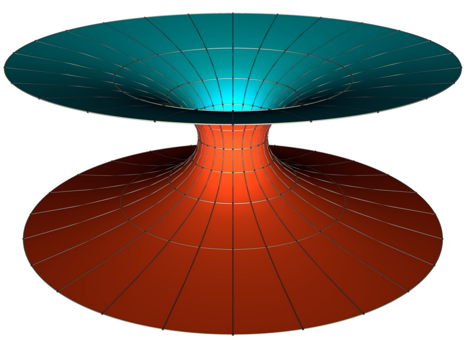
Since $h : T_pM \times T_pM \to \mathbb{R}$ is symmetric,
there is an o.n.b. $\{e_i\}$ of eigenvectors with eigenvalues K_i; that is,
$h(e_i, e_j) = K_i \delta_{ij}$, or, in terms of the shape operator, $S_{\vec{n}}\, e_i = K_i e_i$.

Def. K_i are the principal curvatures of $M \subset \overline{M}$, and e_i are the principal directions. The mean curvature of M is $H = \operatorname{tr} h = \sum_{i=1}^{n} K_i$.

Def. $M^n \subset \overline{M}^{n+1}$ is a minimal hypersurface if it has $H \equiv 0$. Similarly, a submanifold $M^k \subset \overline{M}^{n+1}$ of codimension > 1, is minimal if $\operatorname{tr} S_N = 0$ for all normal vectors N, or, equivalently, $\operatorname{tr} \mathrm{II} = 0$. vector valued $\mathrm{II} : TM \times TM \to TM^{\perp}$.

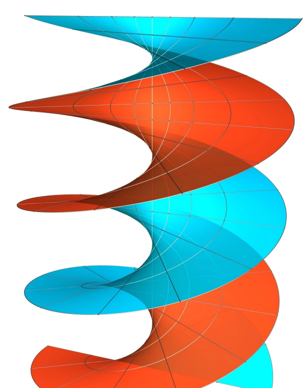
Ex: Minimal hypersurfaces M^n in $\mathbb{R}^{n+1}$:
$n=1$: affine subspaces (note $H=0$ for a 1-dim submfld iff it is a geodesic)
$n=2$: this is very classical; going back to Lagrange 1762. Besides affine subspaces, lots of examples are now known (see e.g., minimalsurfaces.blog, by M. Weber)

Catenoid (Euler 1744)

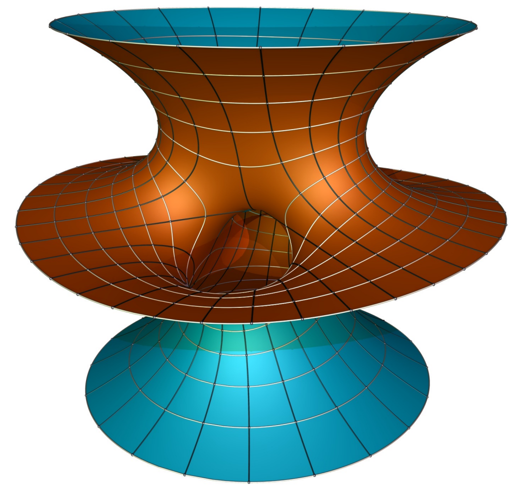


obtained rotating a catenary $y = \alpha \cosh\left(\frac{x}{\alpha}\right)$

Helicoid (Meusnier 1776)



Costa surface (1982)



Thm. A hypersurface $M^n \subset \mathbb{R}^{n+1}$ is minimal if and only if its coordinate functions in $\mathbb{R}^{n+1}$ restrict to harmonic functions on M^n; i.e., $\Delta_M \langle e_i, X \rangle = 0$, $i = 1, \dots, n+1$.

Pf: Given $v \in \mathbb{R}^{n+1}$, let $\overline{f} : \mathbb{R}^{n+1} \to \mathbb{R}$ be the function $\overline{f}(x) = \langle x, v \rangle$, and let $f = \overline{f}|_M : M \to \mathbb{R}$. Then $\overline{\nabla} \overline{f} = v$, so $\nabla f = (\overline{\nabla} \overline{f})^T = v - \langle v, \vec{n} \rangle \vec{n}$, where

$\vec{n}$ is a unit normal for $M \subset \mathbb{R}^{n+1}$. The Laplacian of f on M is:

$$\Delta_M f = \operatorname{div}_M \nabla f = \operatorname{div}_M \left(v - \langle v, \vec{n} \rangle \vec{n} \right) = - \langle \underbrace{\nabla \langle v, \vec{n} \rangle}_{\text{tangent to } M}, \vec{n} \rangle - \langle v, \vec{n} \rangle \operatorname{div}_M \vec{n}$$

0

and $\displaystyle \operatorname{div}_M \vec{n} = \sum_{i=1}^{n} \langle \nabla_{e_i} \vec{n}, e_i \rangle = - \sum_{i=1}^{n} (S_{\vec{n}} e_i, e_i) = - \operatorname{tr} S_{\vec{n}} = - H$;

so it follows: $\Delta_M f = H \langle v, \vec{n} \rangle$ on M

Setting v to be a coordinate vector in $\mathbb{R}^{n+1}$, it follows that $H \equiv 0$ implies all coordinate functions restrict to harmonic functions on M.
Conversely, if all coordinate functions restrict to harmonic functions on M, then $0 = H \langle v, \vec{n} \rangle$ for a linearly independent set of $v \in \mathbb{R}^{n+1}$, so $H \equiv 0$. $\square$

<u>Cor</u>: Complete minimal hypersurfaces in $\mathbb{R}^{n+1}$ are either <u>noncompact</u> or have <u>boundary</u>.
(cf. HW4: if M closed, then only harmonic functions are constant.)

Some important research questions regarding minimal hypersurfaces in $\mathbb{R}^3$:

<u>Q</u>: How many ends does it have? Does it have finite total curvature $\left| \int_M K \right| < +\infty$?

	Plane	Catenoid	Helicoid	Costa of genus K	Riemann
$\int_M K \, \mathrm{vol}_g$	0	-4π	$-\infty$	$-4\pi(K+2)$	$-\infty$
# ends	1	2	1	3	$+\infty$
genus	0	0	0	K	$+\infty$

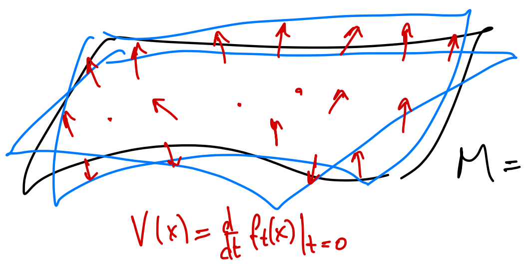
Classification results for <u>embedded</u> min. surfaces $M^2 \subset \mathbb{R}^3$ with $\left| \int_M K \right| < \infty$:

- # ends $= 1 \implies M^2$ is isometric to a plane
- # ends $= 2 \implies M^2$ is isometric to catenoid [Schoen, 1983]
- genus $(M) = 0 \implies M^2$ is isom. to plane or catenoid [López-Ros, 1991]
- genus $(M) = 1$, # ends $= 3 \implies M^2$ is isom. to Costa surface (or Hoffman-Meeks deformation) [Costa, 1991]

Open questions:
- Are there embedded genus 1 minimal surfaces in $\mathbb{R}^3$ w/ $\left| \int_M K \right| < \infty$ and > 3 ends? (Conjecturally <u>NO</u> by Hoffman-Meeks)
- genus$(M) \geq$ #ends$(M) - 2$?
- ...

Lecture 18 4/5/2024

First variation of Area. Given $M \subset \overline{M}$ a submfld, consider a variation $f_t : M^m \to \overline{M}^n$, ie, $f_0(x) = x$, $\forall x \in M$ and $f_t(M) \subset \overline{M}$ are nearby submflds



$$df_0(x) = id : T_xM \to T_xM$$

$$\text{Area}(f_t(M)) = \int_M \sqrt{\det (df_t)^T(df_t)}\, dx$$

$dx = vol_g$ is the volume form of $M \subset \overline{M}$; $g = f_0^* \bar{g}$.

Recall from calculus: if $A_t \in \text{Sym}(\mathbb{R}^n)$ with $A_0 = Id$,

$$\frac{d}{dt} \det(A_t)|_{t=0} = \text{tr}\left(\frac{d}{dt} A_t|_{t=0}\right). \quad \left(\text{e.g., use } \underset{\det(I + tX + \dots)}{\det e^{tX}} = e^{\text{tr}(tX)} \dots\right)$$

So:

$$\frac{d}{dt} \text{Area}(f_t(M))|_{t=0} = \int_M \frac{d}{dt} \sqrt{\det (df_t)^T(df_t)}\Big|_{t=0} dx$$

$$= \int_M \frac{1}{2\sqrt{\det (df_0)^T(df_0)}} \cdot \frac{d}{dt} \det\left((df_t)^T(df_t)\right)|_{t=0}\, dx$$

$$= \frac{1}{2} \int_M \text{tr}\left(\frac{d}{dt}\left((df_t)^T(df_t)\right)|_{t=0}\right) dx$$

Let $V(x) = \frac{d}{dt} f_t(x)|_{t=0}$ be the corresponding variational field, so, in normal coord. $\{x_i\}$ around a point, $df_t(x) = \left(\frac{\partial (f_t)^j}{\partial x_i}\right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}}$ so

$$\text{tr}\left(\frac{d}{dt}\left((df_t)^T(df_t)\right)|_{t=0}\right) = \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{d}{dt}\left(\frac{\partial (f_t)^k}{\partial x_i}\right)^2\Big|_{t=0}$$

$$= 2 \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{\partial (f_0)^k}{\partial x_i} \frac{\partial V^k}{\partial x_i}$$

$$= 2 \sum_{i=1}^{m} g(e_i, \nabla_{e_i} V) = 2 \, \text{div}_M V.$$

So: $$\frac{d}{dt} \mathrm{Area}(f_t(M))\Big|_{t=0} = \int_M \mathrm{div}_M V \, dx$$

It is useful to decompose $V = V^T + V^\perp$ along M, to disregard tangential variations, which are not geometric (just change coordinates on M...)

$$\mathrm{div}_M V = \mathrm{div}_M V^T + \mathrm{div}_M V^\perp$$

Stokes: $\displaystyle\int_M \mathrm{div}_M V^T = 0$ b/c M is closed $\left(\displaystyle\int_{\partial M} g(V, \vec{n}) \text{ if } \partial M \neq \emptyset \dots \right)$

$$\mathrm{div}_M V^\perp = \sum_{i=1}^n g(e_i, \nabla_{e_i} V^\perp) = \sum_{i=1}^n e_i\Big(\underbrace{g(e_i, V^\perp)}_{=0} - g\big((\nabla_{e_i} e_i), V^\perp\big)\Big)$$

$$= -\sum_{i=1}^n g\big(\mathrm{II}(e_i,e_i), V^\perp\big) = -g\Big(\underbrace{\sum_{i=1}^n \mathrm{II}(e_i,e_i)}_{\vec{H}}, V^\perp\Big) = -g(\vec{H}, V).$$

$\mathrm{II}(X,Y) = (\overline{\nabla}_X Y)^\perp$

So, if $M \subset \overline{M}$ is closed, we obtain:

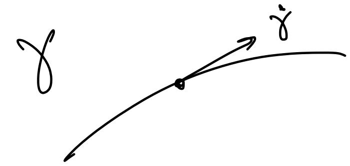
$$\frac{d}{dt} \mathrm{Area}(f_t(M))\Big|_{t=0} = -\int_M g(\vec{H}, V)\, dx.$$

Thus, minimal submanifolds are <u>critical points</u> of Area.

Note: If $\vec{H} \neq 0$ at $p \in M$, then we can find V s.t. $g(\vec{H}, V) > 0$ near p and $g(\vec{H}, V) = 0$ away from p, so for $\varepsilon > 0$ small, we have $\mathrm{Area}(f_\varepsilon(M)) < \mathrm{Area}(f_0(M))$. Thus, if $M \subset \overline{M}$ <u>minimizes</u> Area, then it is <u>minimal</u>. However, the converse does not hold!

"Area-minimizing submanifold" v. "minimal submanifold"

Note: A 1-dim. submanifold is minimal iff it is geodesic:



$$\vec{H} = \operatorname{tr} \mathrm{II} = \mathrm{II}(\dot\gamma, \dot\gamma) = \left(\nabla_{\dot\gamma} \dot\gamma\right)^{\perp}$$

(recall $\left(\nabla_{\dot\gamma}\dot\gamma\right)^{T} = 0$ if γ has constant speed.)

Second Variation of Area: Assume $\vec{H} = 0$ and, for simplicity, $V^{T} \equiv 0$.

$$\frac{d^2}{dt^2} \operatorname{Area}(f_t(M))\Big|_{t=0} = \int_M \underbrace{\|(\nabla V)^{\perp}\|^2}_{\sum_{i=1}^{m} \|(\nabla_{e_i} V)^{\perp}\|^2} - \sum_{i=1}^{m} \bar{g}\left(\bar{R}(e_i, V)V, e_i\right) - \underbrace{\|\langle \mathrm{II}, V\rangle\|^2}_{\left(\sum_{i,j=1}^{m} \langle \mathrm{II}(e_i, e_j), V\rangle\right)^2}$$

If $M \subset \bar{M}$ is a two-sided hypersurface, let $\vec{n}$ be a unit normal, and $\phi : M \to \mathbb{R}$ be s.t. $V = \phi \cdot \vec{n}$; variational field of $f_t(x) = \exp_x(\phi(x)\,\vec{n}_x)$,

$$\frac{d^2}{dt^2} \operatorname{Area}(f_t(M))\Big|_{t=0} = \int_M \|\nabla\phi\|^2 - \operatorname{Ric}(\vec{n}, \vec{n})\,\phi^2 - \|h\|^2 \phi^2 \, dx$$

$$= \int_M \left((-\Delta_g)\phi - \left(\operatorname{Ric}(\vec{n}) + \|h\|^2\right)\phi \right) \phi \, dx$$

$$= \langle J\phi, \phi \rangle_{L^2(M)}$$

where $J\phi := (-\Delta_g)\phi - \left(\operatorname{Ric}(\vec{n}) + \|h\|^2\right)\phi$ is the Jacobi operator (also called "stability operator")

Def. The two-sided minimal hypersurface $M \subset \bar{M}$ is stable if for all normal variations $f_t(M)$, we have $\frac{d^2}{dt^2}\operatorname{Area}(f_t(M))\big|_{t=0} \geq 0$; equivalently, $\forall \phi \in C^{\infty}(M)$,

$\int_M \|\nabla\phi\|^2 \geq \int_M \left(\operatorname{Ric}(\vec{n}) + \|h\|^2\right)\phi^2$; equivalently, J is a $\underbrace{\text{nonnegative operator}}_{\operatorname{Spec}(J) \subset (0, +\infty)}$.

<u>Note:</u> Area-minimizing (minimal) hypersurfaces are <u>stable</u>.

<u>Thm</u> (Simons '68). If $\overline{M}$ has $Ric>0$, then it has no two-sided stable minimal hypersurfaces. If $\overline{M}$ has $Ric \geq 0$ and $M \subset \overline{M}$ is a two-sided stable minimal hypersurface, then M is totally geodesic and $Ric(\vec{n}) \equiv 0$.

<u>Pf</u>. Set $\phi \equiv 1$ on the stability inequality: $0 \geq \int_M Ric(\vec{n}) + \|h\|^2$.

> <u>Prop.</u> (Schoen-Yau '79). Suppose $(\overline{M}^3, \bar{g})$ has $scal > 0$ and $M^2 \hookrightarrow \overline{M}$ is a connected closed two-sided stable min. hypersurface. Then $M^2 \cong S^2$.

<u>Pf.</u> Choose $\{e_1, e_2\}$ that diagonalize h, so $\|h\|^2 = h(e_1,e_1)^2 + h(e_2,e_2)^2 = K_1^2 + K_2^2$.
Using the Gauss equation; as $\{e_1, e_2\}$ is o.n.b. of T_xM, setting $e_3 = \vec{n}$,

$$\begin{aligned} sec_{\overline{M}}(e_1 \wedge e_2) &= sec_M(e_1 \wedge e_2) + h(e_1,e_2)^2 - h(e_1,e_1)h(e_2,e_2) \\ &= sec_M(e_1 \wedge e_2) - K_1 K_2 \end{aligned}$$

so:
$$\begin{aligned} Ric(\vec{n}) &= sec_{\overline{M}}(e_1 \wedge \vec{n}) + sec_{\overline{M}}(e_2 \wedge \vec{n}) \\ &= \left(\sum_{i=1}^{3} sec_{\overline{M}}(e_i \wedge \vec{n}) \right) - sec_{\overline{M}}(e_1 \wedge e_2) \\ &= \frac{1}{2} scal_{\overline{M}} - sec_M(e_1 \wedge e_2) + K_1 K_2 \end{aligned}$$

$$Ric(\vec{n}) + \|h\|^2 = \frac{1}{2} scal_{\overline{M}} - sec_M + K_1 K_2 + K_1^2 + K_2^2$$

$$H = K_1 + K_2 = 0$$
so: $$K_1^2 + K_2^2 = -2K_1K_2$$
$$K_1 K_2 = -\frac{K_1^2 + K_2^2}{2}$$

$$= \frac{1}{2} scal_{\overline{M}} - sec_M + \frac{1}{2}\|h\|^2 \geq \frac{1}{2} scal_{\overline{M}} - sec_M$$

Set $\phi \equiv 1$ in the stability inequality and use Gauss-Bonnet:

$$0 \geq \int_M Ric(\vec{n}) + \|h\|^2 \geq \underbrace{\frac{1}{2}\int_M scal_{\overline{M}}}_{>0} - 2\pi \chi(M) \Rightarrow \chi(M) > 0$$

M orient., connected $\Rightarrow M \cong S^2$. $\square$

Thm (Federer, Fleming, De Giorgi, Almgren, Allard). If $(\overline{M}^n, \overline{g})$ is a closed oriented Riem. mfld, $n \leq 7$, and $\alpha \in H_{n-1}(\overline{M}, \mathbb{Z})$, there exist (embedded) two-sided stable minimal hypersurfaces $M_1, \dots, M_k$ so that $\alpha = [M_1] + \dots + [M_k]$, obtained by minimizing area in α.

also Gromov–Lawson 83, for all $n \geq 2$, w/ spinors instead of minimal surfaces.

$$\boxed{\text{Thm (Schoen–Yau 79). } T^n,\ 2 \leq n \leq 7, \text{ does not admit metrics with scal} > 0.}$$

Pf. $(n=3)$. Suppose $(T^3, \overline{g})$ has $\mathrm{scal} > 0$, and let $\alpha \in H_2(T^3, \mathbb{Z})$ be the class $\alpha = [\{x_3 = 0\}]$, so that any representative $M \in \alpha$ has $\int_M \omega = 1$ where $\omega = dx_1 \wedge dx_2 \in H^2_{dR}(M, \mathbb{R})$. Minimize area in α to find $M_1, \dots, M_k$ stable min. hyp. s.t. $\alpha = [M_1] + \dots + [M_k]$. Then $\sum_{j=1}^{k} \int_{M_j} \omega = 1$ so $\int_{M_j} \omega \neq 0$ for some $1 \leq j \leq k$. This implies $[dx_1|_{M_j}]$ $[dx_2|_{M_j}] \in H^1_{dR}(M_j, \mathbb{R})$ are nonzero. Indeed, if $dx_1|_{M_j}$ is exact in M_j, then let $f: M_j \to \mathbb{R}$ be s.t. $df = dx_1|_{M_j}$, and compute:

$$0 \neq \int_{M_j} \underset{\substack{\shortparallel \\ df}}{dx_1} \wedge dx_2 = \underbrace{\int_{M_j} d(f\, dx_2)}_{\substack{=0 \text{ by Stokes} \\ \text{b/c } \partial M_j = \emptyset}} - \int_{M_j} f\, \underbrace{d(dx_2)}_{=0} = 0$$

so $[dx_1|_{M_j}] \neq 0$ in $H^1_{dR}(M_j, \mathbb{R})$. This contradicts $M_j \cong S^2$. $\square$

b/c $M_j^2 \subset \overline{M}^3$ is connected 2-sided stable min. surf. in mfld w/ scal > 0.

For $3 < n \leq 7$, there is a dimension-reduction scheme that reduces the problem to the case $n=3$. The above proved a conjecture of Geroch.

Rmk. The above proof is adapted from notes of Otis Chodosh.

The original proof by Schoen-Yau uses a different area-minimization technique, showing that if $\Gamma_g < \pi_1(\overline{M}^3)$ is a subgroup isom. to the fund. group of a surface of genus $g \geq 1$, then there is a two-sided stable min. surface $M^2 \subset \overline{M}^3$ of genus g. For the case $\overline{M}^3 = T^3$, take $\Gamma_1 = \pi_1(T^2) = \mathbb{Z}^2$ and get a contradiction.

Lecture 19 4/10/2024

Comparison theory for Jacobi fields

> **Prop:** If $\gamma : [0,L] \to M$ is a geodesic with $\gamma(0) = p$, $\dot\gamma(0) = v$, $w \in T_v T_p M$ has $\|w\| = 1$ and $J(t)$ is the Jacobi field along $\gamma(t)$ with $J(0) = 0$ and $J'(0) = w$, i.e., $J(t) = d(\exp_p)_{tv}\, tw$, then:
>
> $$\|J(t)\|^2 = t^2 - \frac{1}{3}\langle R(v,w)w, v\rangle t^4 + O(t^5)$$

Pf:

$$\langle \underset{\overset{\shortparallel}{0}}{J}, J\rangle(0) = 0$$

$$\langle J, J\rangle'(0) = 2\langle \underset{\overset{\shortparallel}{0}}{J}, J'\rangle(0) = 0$$

$$\langle J, J\rangle''(0) = \underbrace{2\langle J', J'\rangle(0)}_{\|w\|^2 = 1} + 2\langle J'', \underset{\overset{\shortparallel}{0}}{J}\rangle(0) = 2$$

Also, $J''(0) = -R(\underset{\overset{\shortparallel}{0}}{J}, \dot\gamma)\dot\gamma\,(0) = 0$ so

$$\langle J, J\rangle'''(0) = \underset{\overset{\shortparallel}{\binom{3}{1}+\binom{3}{2}}}{6}\langle J', \underset{\overset{\shortparallel}{0}}{J''}\rangle(0) + \underset{\overset{\shortparallel}{\binom{3}{0}+\binom{3}{3}}}{2}\langle J''', \underset{\overset{\shortparallel}{0}}{J}\rangle(0) = 0$$

Moreover, for any vector field W along γ,

$$\left\langle \frac{D}{dt} R(J(t),\dot\gamma(t))\dot\gamma(t), W\right\rangle = \frac{d}{dt}\underbrace{\langle R(J,\dot\gamma)\dot\gamma, W\rangle}_{=\langle R(W,\dot\gamma)\dot\gamma, J\rangle} - \langle R(J,\dot\gamma)\dot\gamma, W'\rangle$$

$$= \left\langle \frac{D}{dt} R(W,\dot\gamma)\dot\gamma, J\right\rangle + \underbrace{\langle R(W,\dot\gamma)\dot\gamma, J'\rangle}_{=\langle R(J',\dot\gamma)\dot\gamma, W\rangle} - \langle R(J,\dot\gamma)\dot\gamma, W'\rangle$$

So at $t=0$:

$$\frac{D}{dt} R(J,\dot\gamma)\dot\gamma = R(J',\dot\gamma)\dot\gamma$$

(all other terms are zero at $t=0$, b/c $J(0)=0$.)

Thus:

$$\langle J,J\rangle''''(0) = \overset{\binom{4}{1}+\binom{4}{3}}{\overset{\shortparallel}{8}}\langle J', J'''\rangle(0) + \overset{\binom{4}{2}}{\overset{\shortparallel}{6}}\langle \underset{0}{\underset{\shortparallel}{J''}}, J''\rangle(0) + \overset{\binom{4}{0}+\binom{4}{4}}{\overset{\shortparallel}{2}}\langle J'''', \underset{0}{\underset{\shortparallel}{J}}\rangle(0)$$

$$= -8\langle J', R(J',\dot\gamma)\gamma\rangle(0) = -8\langle R(w,v)v,w\rangle$$

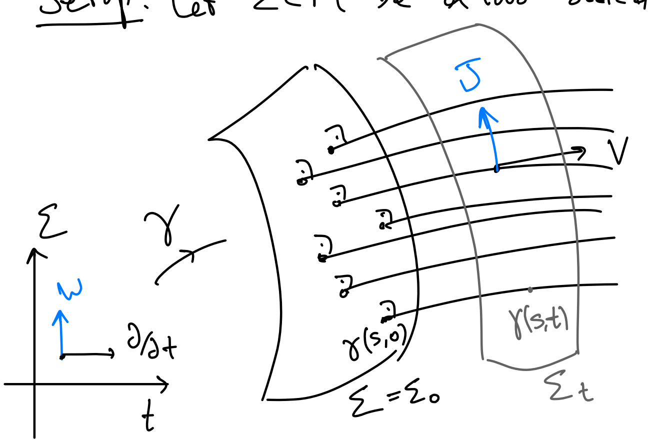
$$= -8\langle R(v,w)w,v\rangle.$$

$J'' = -R(J,\dot\gamma)\dot\gamma$
so $J''' = -R(J',\dot\gamma)\dot\gamma$

$\square$

The goal is to extend the above comparison result beyond just $t\approx 0$, and up to $0\le t\le T$ where $\gamma(T)$ is the first conjugate point to $\gamma(0)$ along γ (Rauch comparison thm).

<u>Setup</u>: Let $\Sigma\subset M$ be a two-sided hypersurface, and consider unit speed geodesics



$$\gamma: \underset{s}{\underset{\psi}{\Sigma}} \times \underset{t}{\underset{\psi}{(-\varepsilon,\varepsilon)}} \longrightarrow M$$

with $\gamma(s,0)=s$, $\frac{d}{dt}\gamma(s,t)\big|_{t=0}\in T\Sigma^\perp$, $\forall s\in\Sigma$,

$V = d\gamma(s,t)\left(\frac{\partial}{\partial t}\right)$ tangent field to geodesics (so $V|_\Sigma\in T\Sigma^\perp$ and $\|V\|=1$)

$J = d\gamma(s,t)(w)$, $w\in T_s\Sigma$ Jacobi field.

In other words, $\gamma(s,t) = \exp_s t\vec{n}_s$, where $\vec{n}$ is unit normal to Σ.

We can choose $\varepsilon>0$ suff. small so that $\Sigma_t=\{\gamma(s,t): s\in\Sigma\}\subset M$ are smooth hypersurfaces for each $t\in(-\varepsilon,\varepsilon)$; cf. "focal radius" of Σ_0.

Let $S=\nabla V$ i.e. $S:\mathfrak{X}(M)\to\mathfrak{X}(M)$, $X\longmapsto\nabla_X V$, and $R_V:\mathfrak{X}(M)\to\mathfrak{X}(M)$, $X\longmapsto R(X,V)V$

This is the shape operator of Σ, with opposite sign, i.e., with the opposite unit normal $-\vec{n}$.

Since $[J,V]=0$, we have $\nabla_V J=\nabla_J V=S(J)$, so the

$$\boxed{\text{Jacobi eqn.: } J''+\underbrace{R(J,V)V}_{R_V J}=0 \iff \begin{cases} J'=SJ \\ \underbrace{S'+S^2+R_V=0} \end{cases}}$$

(2nd order ODE) (System of 1st order ODEs)

"Riccati equation"

Indeed:

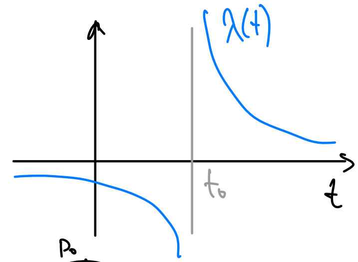
$$\begin{aligned}(\nabla_V S)X &= \nabla_V(SX)-S(\nabla_V X)\\ &= \nabla_V\nabla_X V - S(\nabla_X V+[V,X])\\ &= R(V,X)V+\nabla_X\underbrace{\nabla_V V}_{=0}+\nabla_{[V,X]}V-\nabla_{(\nabla_X V+[V,X])}V\\ &= -R_V(X)-S(S(X)),\quad\forall X\end{aligned}$$

i.e. $S'+S^2+R_V=0$. (This equation can be solved independently!)

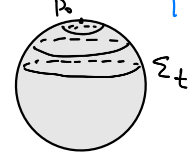
Note S is <u>self-adjoint</u> for each t, i.e., $\langle SX,Y\rangle=\langle X,SY\rangle$, $\forall X,Y\in T\Sigma_t$, since it is (the opposite of) the shape operator of the hypersurface $\Sigma_t=\{\gamma(s,t): s\in\Sigma_0\}$. Eigenvalues of S are <u>principal curvatures</u> (with normal $-\vec{n}$) and $H_{\Sigma_t}=|\operatorname{tr} S|$.

<u>Example:</u> If $\sec\equiv k$, then $R_V=k\,\mathrm{Id}$, and the Riccati equation becomes a scalar equation for umbilical surfaces (with $S=\lambda\cdot\mathrm{Id}$).

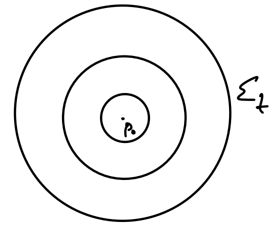
$$S'+S^2+R_V=0\iff\lambda'+\lambda^2+k=0$$



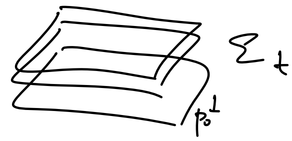
- If $k>0$, the solutions are $\lambda(t)=\sqrt{k}\cot\left(\sqrt{k}(t-t_0)\right)$ corresponding to $\Sigma_t=\{p\in S^n(1/\sqrt{k}): \operatorname{dist}(p,p_0)=|t-t_0|\}$, which are <u>concentric spheres</u> (latitude circles).

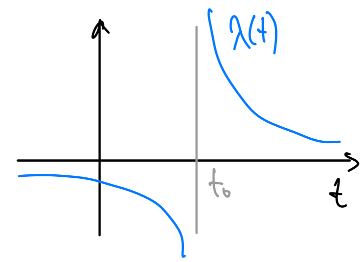


• If $K=0$, the solutions are $\lambda(t) = \dfrac{1}{t-t_0}$, corresponding to <u>concentric spheres</u> $\Sigma_t = \{p \in \mathbb{R}^n : \operatorname{dist}(p, p_0) = |t-t_0|\}$,



and $\lambda \equiv 0$, corresponding to <u>parallel hyperplanes</u>
$\Sigma_t = \{p \in \mathbb{R}^n : \operatorname{dist}(p, p_0^\perp) = |t-t_0|\}$





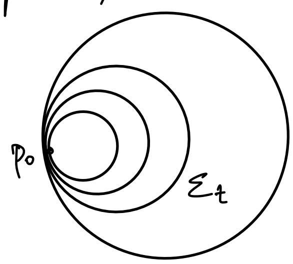
• If $K<0$, the solutions are

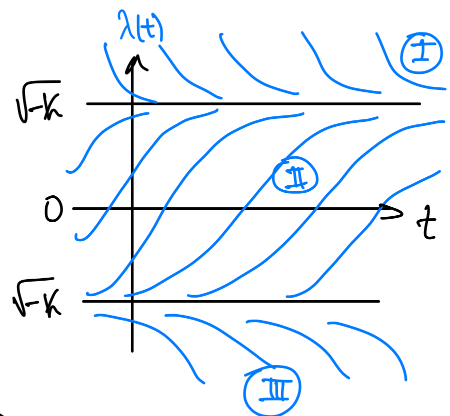
$\lambda(t) = \sqrt{-K}\coth\left(\sqrt{-K}(t-t_0)\right)$, I, III

corresponding to Σ_t being concentric spheres,

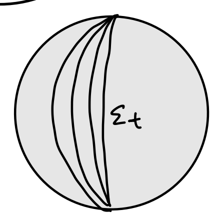
$\lambda(t) = \sqrt{-K}\tanh\left(\sqrt{-K}(t-t_0)\right)$, II

corresponding to Σ_t being horospheres,





and $\lambda(t) \equiv \pm\sqrt{-K}$, corresponding to Σ_t being hypersurfaces parallel to $\mathbb{H}^{n-1}(1/\sqrt{-K}) \subset \mathbb{H}^n(1/\sqrt{-K})$.



<u>Note</u>: The above are <u>all</u> the umbilic hypersurfaces of space forms! Their principal curvatures are given by $\lambda(t)$, and mean curvature by $H = (n-1)|\lambda(t)|$.

To facilitate comparison, identify $T_{\gamma(t)}M \cong T_{\gamma(0)}M$ via parallel transport along γ; so that $S_t : T_{\gamma(t)}M \to T_{\gamma(t)}M$ can be written as $S_t : E \to E$, where $E = T_{\gamma(0)}M$, i.e., $S_t \in \operatorname{Sym}^2 E$ is a curve of self-adjoint operators on a fixed vector space. We prove the following ODE comparison results:

<u>Thm.</u> Let $R_1, R_2 : \mathbb{R} \to \operatorname{Sym}^2 E$ be smooth curves with $R_1(t) \geq R_2(t)$, $\forall t$. Let $S_i : [t_0, t_i) \to \operatorname{Sym}^2 E$ be the maximal solutions to $S_i' + S_i^2 + R_i = 0$. If $S_1(t_0) \leq S_2(t_0)$, then $t_1 \leq t_2$ and $S_1(t) \leq S_2(t)$ for all $t \in [t_0, t_1)$.

Pf. Let $U = S_2 - S_1$, so $U(t_0) \geq 0$.

$$U' = S_2' - S_1' = S_1^2 - S_2^2 + \underbrace{R_1 - R_2}_{\Delta}$$

Define $\Delta = R_1 - R_2$ and $X = -\frac{1}{2}(S_1 + S_2)$, so that

$$XU + UX = -\tfrac{1}{2}(S_1+S_2)(S_2-S_1) - \tfrac{1}{2}(S_2-S_1)(S_1+S_2) = S_1^2 - S_2^2$$

So $U' = XU + UX + \Delta$, an inhomogeneous linear ODE we can solve by "variation of constants". Namely, let $g : (t_0, t') \to \mathrm{Sym}^2 E$ be the solution to the homogeneous linear ODE $g' = Xg$, where $t' = \min\{t_1, t_2\}$. Then $U = g V g^T$ is the desired solution, where V satisfies $V' = g^{-1} \Delta (g^{-1})^T$.

Indeed:
$$\begin{aligned} U' &= g' V g^T + g V' g^T + g V (g^T)' \\ &= X g V g^T + g\cancel{g^{-1}} \Delta \cancel{(g^{-1})^T g^T} + g V g^T X^T \qquad (X^T = X) \\ &= XU + \Delta + UX. \end{aligned}$$

Since $\Delta = R_1 - R_2 \geq 0$, we have $V' = g^{-1} \Delta (g^{-1})^T \geq 0$.

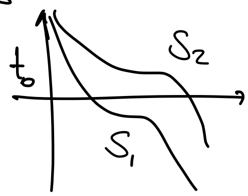
Since $U(t_0) = g(t_0) V(t_0) g(t_0)^T = S_2(t_0) - S_1(t_0) \geq 0$, we have $V(t_0) \geq 0$.

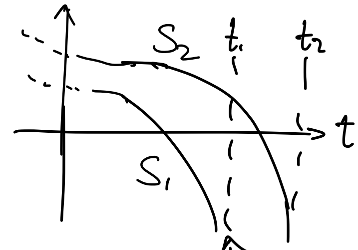
Thus $V(t) \geq 0$ for all $t \in (t_0, t')$ and hence also

$S_2(t) - S_1(t) = U(t) = g(t) V(t) g(t)^T \geq 0$ for all $t \in (t_0, t')$; i.e. $S_1(t) \leq S_2(t)$ for $t \in (t_0, t')$. Since S_i' is bounded from above $\left(S_i' \leq -S_i^2 - R_i \leq -R_i\right)$ the only singularity possibile is $-\infty$, so $S_1 \leq S_2$ implies $t' = t_1 \leq t_2$.

$\square$

Rmk: The above still holds if S_1, S_2 are singular at t_0, but $U = S_2 - S_1$ has a continuous extension to t_0, with $U(t_0) \geq 0$.





first singularity will be $t' = t_1$!

Geometric interpretation: "Principal curvatures of equidistant hypersurfaces grow (in absolute value) faster on the space of larger curvature."

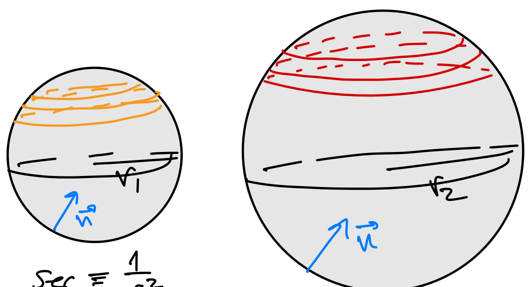
$\mathrm{Spec}(-\nabla\vec{n}) = S_{\vec{n}}$ (shape operator)	$\mathrm{Spec}(\nabla\vec{n}) = S$ (as above)
$-S_{\vec{n}}' + S_{\vec{n}}^2 + R_V = 0$	$S' + S^2 + R_V = 0$

if $\vec{n}$ is inward-pointing

(could also just replace w/ $-\vec{n}$, which is outward pointing, and would flip $S = \nabla\vec{n}$ and $S_{\vec{n}} = -\nabla\vec{n}$)

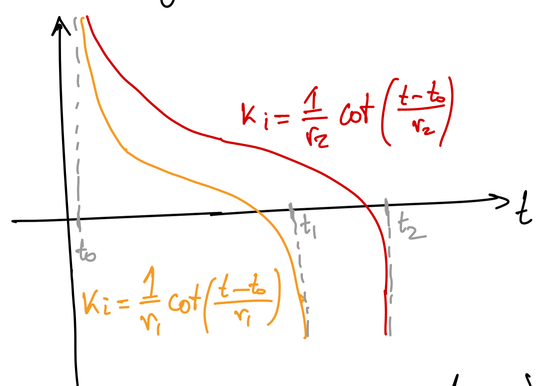
Ex: In the umbilic case with $\sec \equiv K > 0$:

$$0 < r_1 < r_2 \Rightarrow \frac{1}{r_1^2} > \frac{1}{r_2^2}$$



$\sec \equiv \frac{1}{r_1^2}$ $\qquad$ $\sec \equiv \frac{1}{r_2^2}$

The (opposite of) principal curvatures are eigenvalues of $S = \nabla V$



(for actual principal curvatures, flip $\updownarrow$, i.e., change sign!)