Lecture 1 1/26/2024

Riemannian metrics

Let M^n be a smooth manifold. A Riem metric is a smoothly varying family of inner products

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

on the tangent spaces of M, i.e., a smooth section of the vector bundle $Sym^2(TM^*) \to M$ which is pointwise positive-definite. More concretely, $\forall p \in M$,

- $g_p(v, w) = g_p(w, v) \quad \forall v, w \in T_pM$
- $g_p(v, v) \geq 0$
  $= 0 \iff v = 0$
- $M \ni p \mapsto g_p(X_p, Y_p) \in \mathbb{R}$ is smooth $\forall X, Y \in \mathfrak{X}(M)$.

smooth vector fields on M, i.e., smooth sections of $TM \to M$

Endowed with g, we call (M^n, g) a Riem. manifold.

In a chart $(x_1, \dots, x_n)$, with $T_pM = \mathrm{span}\left\{\frac{\partial}{\partial x_1}(p), \dots, \frac{\partial}{\partial x_n}(p)\right\}$, we write $g_p = \sum_{i,j} g_{ij}(p) \cdot dx_i\, dx_j$

(strictly speaking there's a $\otimes$ here)

$g_{ij} = g_{ji} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ are smooth fct.
$\{dx_i\}$ is the dual basis on T_pM^*.
Usually abbreviate $dx_i^2 = dx_i\, dx_i$ and omit "$\otimes$" in $dx_i \otimes dx_j$

Examples: Let M be a 1-dim mfld.

If M is compact, then $M \underset{diff}{\cong} S^1$; if noncompact $M \underset{diff}{\cong} \mathbb{R}$.

In both cases, we can define $X \in \mathfrak{X}(M)$ such that $\forall p \in M$, $T_pM = \mathrm{span}\{X_p\}$. Thus, a Riem. metric on M is determined by a single smooth positive function $g_{11} : M \to \mathbb{R}$.

- The "usual way" to write "the" canonical metric on M is:

Can think of $\theta:(0,2\pi)\to S^1$ as a chart. Need 2 such charts to cover S^1.

on $S^1 := [0,2\pi]/\sim$, $\quad T_\theta S^1 = \operatorname{span}\left\{\frac{\partial}{\partial\theta}\right\}$, $\quad g = 1\, d\theta\otimes d\theta = d\theta^2$

$(T_\theta S^1)^* = \operatorname{span}\{d\theta\}$ dual space

on $\mathbb{R}$, $\quad T_x\mathbb{R} = \operatorname{span}\left\{\frac{\partial}{\partial x}\right\}$, $\quad g = 1\, dx\otimes dx = dx^2$

common slight abuse of notation...

1 global chart $x:\mathbb{R}\to\mathbb{R}$

$(T_x\mathbb{R})^* = \operatorname{span}\{dx\}$ dual space

- However, that's not the only metric there is; e.g., on S^1, define $h = f(\theta)^2 d\theta^2$ where $f: S^1\to\mathbb{R}$ is positive, e.g.,

$$h = (2+\cos\theta)^2\, d\theta^2.$$

Are the circles $(S^1, d\theta^2)$ and (S^1, h) the "same"?

Def: The Riem. manifolds (M^n, g) and (N^n, h) are <u>isometric</u> if there is a diffeomorphism $\phi: (M^n, g)\to(N^n,h)$ such that $\phi^* h = g$, i.e., $\forall p\in M$, $\forall v,w\in T_pM$,

$$h_{\phi(p)}\big(d\phi(p)v,\, d\phi(p)w\big) = g_p(v,w).$$

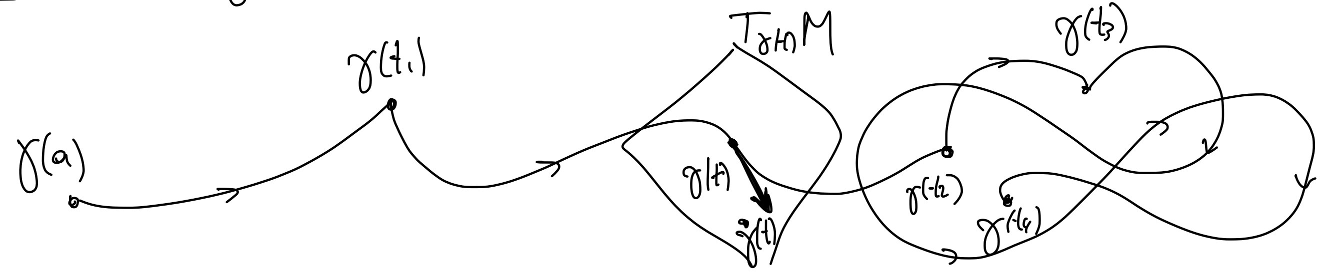
Such ϕ is called an <u>isometry</u>.

- Two manifolds are "the same" if <u>diffeomorphic</u>
- Two Riem. mflds are "the same" if <u>isometric</u>.

To distinguish manifolds that are <u>not</u> the same, we can look for <u>invariants</u>:

- Smooth manifold invariants: dimension, Euler characteristic...
- Riem. manifold invariants: distances, volumes, curvature...

Length: Let $\gamma:[a,b]\to(M^n,g)$ be a piecewise C^1 curve



ie., $\gamma|_{[t_i,t_{i+1}]}$ is C^1. The length of γ in (M^n,g) is:

$$L_g(\gamma)=\int_a^b g(\dot\gamma(t),\dot\gamma(t))^{1/2}\,dt$$

Exercise: Show that L_g is invariant under reparametrization, ie, if $\eta:[a,b]\to[c,d]$ is a diffeomorphism, then $L_g(\gamma)=L_g(\gamma\circ\eta)$.

Exercise: Suppose $\phi:(M^n,g)\longrightarrow(M^n,h)$ is an isometry and $\gamma:[a,b]\to M$ is a piecewise C^1 curve. Show that $L_g(\gamma)=L_h(\phi\circ\gamma)$.

Let $\gamma:[0,2\pi]\to S^1$ be a parametrization of S^1; say $\gamma(\theta)=(\cos\theta,\sin\theta)$ if we take $S^1\subset\mathbb{R}^2$ to be the unit circle; and note $\dot\gamma(\theta)=\partial/\partial\theta$. Then, we have:

$$L_g(\gamma)=\int_0^{2\pi}\underbrace{g\left(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\right)}_{1}^{1/2}d\theta=2\pi.$$

($g=d\theta^2$ is called the "unit" metric on $S^1:=[0,2\pi]/\sim$.)

$$L_h(\gamma)=\int_0^{2\pi}h\left(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}\right)^{1/2}d\theta=\int_0^{2\pi}(2+\cos\theta)\,d\theta=4\pi.$$

Upshot: (S^1, g) and (S^1, h) are not isometric!

Claim: (S^1, h) is isometric to $\left([0,4\pi]/\sim, ds^2\right)$ ← "the" circle of length 4π

Pf: Let $\phi : \underset{\theta}{[0,2\pi]} \to \underset{s}{[0,4\pi]}$ be the increasing smooth function

$$\phi(\theta) = \int_0^\theta (2+\cos t)\,dt = 2\theta + \sin\theta. \qquad \left(\text{think } s = \phi(\theta), \text{ so } \phi^* ds = \phi'(\theta)\,d\theta\right)$$

ϕ induces a diffeo $\phi : \underbrace{[0,2\pi]/\sim}_{\cong S^1} \to \underbrace{[0,4\pi]/\sim}_{\cong S^1}$, such that

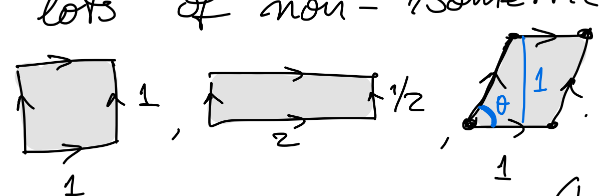
$\phi^* ds^2 = \phi'(\theta)^2 d\theta^2 = (2+\cos\theta)^2 d\theta^2 = h$, so we have an isometry

$\phi : (S^1, h) \to \left([0,4\pi]/\sim, ds^2\right)$. Moreover, $\left([0,4\pi]/\sim, ds^2\right) \underset{\text{isom}}{\cong} \left([0,2\pi]/\sim, 4\,d\theta^2\right)$. ← why?

In HW1!

Exercise: Show that circles (S^1, g) and (S^1, h) are isometric if and only if they have the same length.

Hint: Following the above, show that for any Riem. metric $g = f(\theta)^2 d\theta^2$ on S^1, there exists a constant $r > 0$ and a diffeom. $\phi : S^1 \to S^1$ s.t. $\phi^* g = r^2 d\theta^2$. If two metrics have the same $r > 0$, then compose the ϕ's to get an isometry.

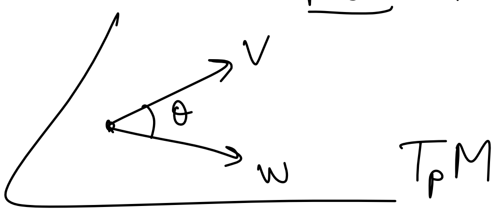
Remark: Any two Riem. metrics on a 1-dim Riem. manifold are locally isometric. However, loc. isom. is not enough to conclude that mflds can only differ by a homothety; e.g., any two flat tori are locally isometric, but there exist lots of non-isometric flat tori of the same volume; e.g. 

Rmk: If (M^n, g) is a Riem. manifold, then $\alpha \cdot g$ is a Riem. metric on M^n for any $\alpha > 0$, called homothetic to g.

Def. If $f: M \to \mathbb{R}$ is a smooth positive function, then $f \cdot g$ is a metric on M^n called conformal to g.

i.e., same angles...

Note: Riemannian angle between v, w is:



$$\theta = \arccos \frac{g_p(v,w)}{\sqrt{g_p(v,v)\, g_p(w,w)}}$$

does not change if we replace g with $f \cdot g$.

Ex: On $B_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, consider the Riem. metrics

$$g_{Eucl} = dx^2 + dy^2, \quad \text{i.e.,} \quad (g_{Eucl}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ on each } T_{(x,y)}\mathbb{R}^2 \cong \mathbb{R}^2 \text{ w.r.t. } \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}.$$

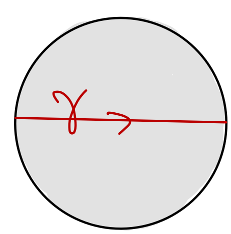
$$\left.\begin{aligned} g_{hyp} &= \frac{4}{(1-x^2-y^2)^2}(dx^2+dy^2) \\ g_{sph} &= \frac{4}{(1+x^2+y^2)^2}(dx^2+dy^2) \end{aligned}\right\}$$

both are conformal but not isometric to g_{Eucl}!

E.g., let $\gamma: (-1,1) \to B_1$, $\gamma(t) = (t, 0)$, be the "diameter".

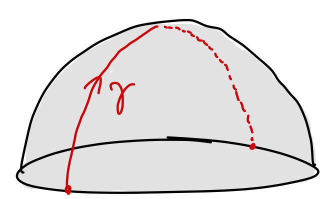
i.e. $\begin{cases} x(t) = t \\ y(t) = 0 \end{cases}$

Clearly, $\dot\gamma(t) = (1, 0)$, so



$$L_{g_{Eucl}}(\gamma) = \int_{-1}^{1} 1 \cdot dt = 2$$

$$L_{g_{hyp}}(\gamma) = \int_{-1}^{1} \frac{2}{1-t^2}\, dt = \ln\left(\frac{1+t}{1-t}\right)\Big|_{-1}^{1} = +\infty$$



$$L_{g_{sph}}(\gamma) = \int_{-1}^{1} \frac{2}{1+t^2}\, dt = 2 \arctan t \,\Big|_{-1}^{1} = \pi$$

Q: How can we show that $g_{Eucl}, g_{hyp}, g_{sph}$ are not isometric?

- Just having different lengths for γ is not enough! (there could be some diffeo ϕ "lurking" so that $L_{g_{Eucl}}(\gamma) = L_g(\phi\circ\gamma)$.)
- One way would be to compute the Area of B_1 with each metric: $\text{Area}(B_1, g_{Eucl}) = \pi$, $\text{Area}(B_1, g_{hyp}) = \infty$, $\text{Area}(B_1, g_{sph}) = 2\pi$.
- Another way would be to compute their curvature: More on how to do this later!

$$\sec_{g_{Eucl}} \equiv 0, \quad \sec_{g_{hyp}} \equiv -1, \quad \sec_{g_{sph}} \equiv 1.$$

For now, the important upshot is that just by "looking at" the metric tensor, one usually cannot distinguish metrics or recognize a given "canonical" or "best" metric, because the expression depends on a choice of coordinates and there are lots of such choices (lots of diffeomorphisms...)

E.g., on S^1, the metrics $4d\theta^2$ and $(2+\cos\theta)^2 d\theta^2$ are isometric.

Perhaps some more interesting examples can be found on $\mathbb{R}^2$:

Ex: Let $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism. Compute $\phi^* g_{Eucl}$.

Take a chart (u,v) on $\mathbb{R}^2$ and suppose $\phi(u,v) = (x,y)$, say $x = \phi_1(u,v)$, $y = \phi_2(u,v)$, and let $h = \phi^* g_{Eucl}$.

$$d\phi_{(u,v)} = \begin{bmatrix} \frac{\partial\phi_1}{\partial u} & \frac{\partial\phi_1}{\partial v} \\ \frac{\partial\phi_2}{\partial u} & \frac{\partial\phi_2}{\partial v} \end{bmatrix}, \text{ i.e., } \begin{cases} d\phi_{(u,v)} \frac{\partial}{\partial u} = \frac{\partial\phi_1}{\partial u}\frac{\partial}{\partial x} + \frac{\partial\phi_2}{\partial u}\frac{\partial}{\partial y} \\ d\phi_{(u,v)} \frac{\partial}{\partial v} = \frac{\partial\phi_1}{\partial v}\frac{\partial}{\partial x} + \frac{\partial\phi_2}{\partial v}\frac{\partial}{\partial y} \end{cases}$$

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = g_{Eucl}\left(d\phi\frac{\partial}{\partial u}, d\phi\frac{\partial}{\partial u}\right) = \left(\frac{\partial\phi_1}{\partial u}\right)^2 + \left(\frac{\partial\phi_2}{\partial u}\right)^2 = h_{11}$$

$$h\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = g_{Eucl}\left(d\phi\frac{\partial}{\partial u}, d\phi\frac{\partial}{\partial v}\right) = \frac{\partial\phi_1}{\partial u}\cdot\frac{\partial\phi_1}{\partial v} + \frac{\partial\phi_2}{\partial u}\cdot\frac{\partial\phi_2}{\partial v} = h_{12} = h_{21}$$

$$h\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = g_{Eucl}\left(d\phi\frac{\partial}{\partial v}, d\phi\frac{\partial}{\partial v}\right) = \left(\frac{\partial\phi_1}{\partial v}\right)^2 + \left(\frac{\partial\phi_2}{\partial v}\right)^2 = h_{22}$$

so $$h = \left[\left(\frac{\partial \phi_1}{\partial u}\right)^2 + \left(\frac{\partial \phi_2}{\partial u}\right)^2\right] du^2$$

$$+ 2\left[\frac{\partial \phi_1}{\partial u}\cdot\frac{\partial \phi_1}{\partial v} + \frac{\partial \phi_2}{\partial u}\cdot\frac{\partial \phi_2}{\partial v}\right] du\,dv$$

$$+ \left[\left(\frac{\partial \phi_1}{\partial v}\right)^2 + \left(\frac{\partial \phi_2}{\partial v}\right)^2\right] dv^2 \qquad = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

Again, careful: $du\,dv = \frac{1}{2}\left(du\otimes dv + dv \otimes du\right)$

(More generally, $h = \phi^* g$ is given by $h_{ij} = \sum_{a,b} \frac{\partial \phi_a}{\partial x_i}\cdot\frac{\partial \phi_b}{\partial x_j} g_{ab}$, so, if $g_{ab} = \delta_{ab}$ is the Euclidean metric, then $h_{ij} = \sum_a \frac{\partial \phi_a}{\partial x_i}\frac{\partial \phi_a}{\partial x_j}$.)

E.g., with a linear $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\phi(u,v) = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{d\phi}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, $ad \neq bc$, we get

$$h = (a^2+c^2)\,du^2 + 2(ab+cd)\,du\,dv + (b^2+d^2)\,dv^2$$

$$h = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} \text{ in the basis } \left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right\}.$$

HW1

Ex: Show that any constant metric on $\mathbb{R}^2$, e.g., $h = 2\,du^2 - du\,dv + 5\,dv^2$, is isometric to $g_{Eucl} = dx^2 + dy^2$. (For the above h, use $\phi(u,v) = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$.)

But there are also nonlinear diffeos of $\mathbb{R}^2$, such as

$$\phi(u,v) = \begin{pmatrix} \cos(u^2+v^2) & -\sin(u^2+v^2) \\ \sin(u^2+v^2) & \cos(u^2+v^2) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

Ex: check this is a diffeo indeed...

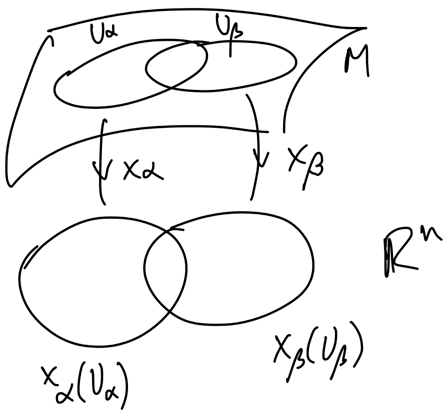
And
$$\phi^* g_{Eucl} = \left(1 + 4u^4 - 4uv + 4u^2v^2\right) du^2$$
$$+ 4\left(u^2 - v^2 + 2u^3 v + 2uv^3\right) du\,dv$$
$$+ \left(1 + 4v^4 + 4uv + 4u^2v^2\right) dv^2.$$

Of course, this "ugly" polynomial metric is isometric to g_{Eucl} (we found it this way) but it would not be obvious at all how to see this... until we learn about curvature!

Some routine stuff:

Prop: Every smooth manifold can be endowed with a Riem. metric.

Pf: Choose an atlas $\{x_\alpha : U_\alpha \to x_\alpha(U_\alpha)\}$ and a subordinate partition of unity $\rho_\alpha : U_\alpha \to [0,1]$, i.e. $\sum_\alpha \rho_\alpha \equiv 1$.
On each $x_\alpha(U_\alpha) \subset \mathbb{R}^n$ take, e.g., the Euclidean metric



and let $g^{(\alpha)} := x_\alpha^*(g_{Eucl})$ on U_α;
then set $g = \sum_\alpha \rho_\alpha g^{(\alpha)}$. $\square$

Ex: Is g locally isometric to g_{Eucl}? Why not? (If so, every manifold could be made flat!)

Q: How to "construct" Riem. metrics?
E.g., recall another result proven with partitions of unity:

Thm (Whitney Embedding). If M^n is a smooth closed mfld, then there exists a smooth embedding $\phi : M^n \hookrightarrow \mathbb{R}^{2n+1}$.

M compact, $\partial M = \phi$.

sometimes can reduce this dim...

Using the above, we can endow M^n with the metric $\phi^* g_{Eucl}$, so that ϕ becomes an <u>isometric</u> <u>embedding</u>.

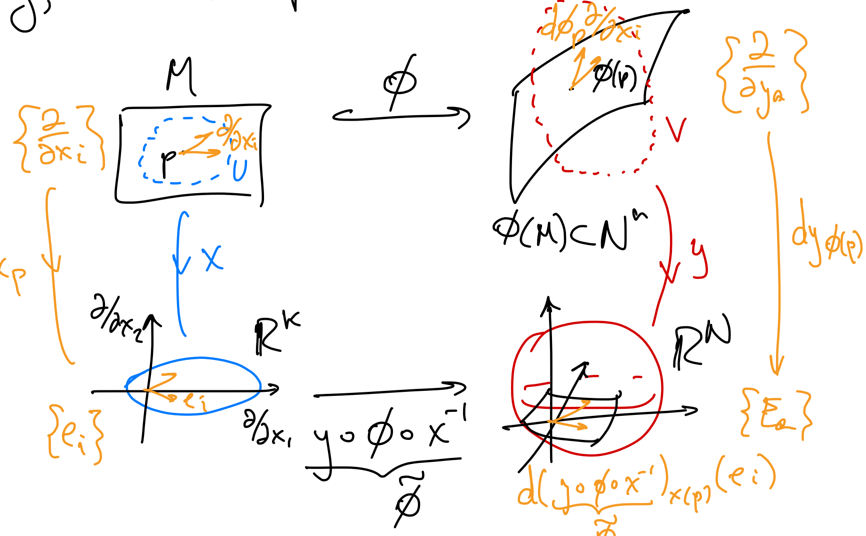
Recall from computations above that, in coordinates $(x_1, \dots, x_n)$ in M,

$$\phi^*(g_{Eucl}) = \underbrace{\left(\sum_a \frac{\partial \phi_a}{\partial x_i} \frac{\partial \phi_a}{\partial x_j} \right)}_{g_{ij}} dx_i\, dx_j, \quad \text{where } \begin{array}{l} \phi : M^n \to \mathbb{R}^N \\ \phi = (\phi_1, \dots, \phi_N) \end{array}$$

Lecture 2 $\quad$ 1/30/2024

Generalizing "induced metric" from embedding into $\mathbb{R}^n$, if $\phi: M^k \hookrightarrow (N^n, g)$ is an embedding, then the pullback metric $h = \phi^* g$ is

$$h_p(v,w) = g(d\phi_p v, d\phi_p w)$$



In charts,

$$d\phi_p \frac{\partial}{\partial x_i} = \sum_a f_i^a(p) \frac{\partial}{\partial y_a}$$

$$h_p\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = g\left(d\phi_p \frac{\partial}{\partial x_i}, d\phi_p \frac{\partial}{\partial x_j}\right)$$

$$= \sum_{a,b} f_i^a(p) f_j^b(p) \underbrace{g\left(\frac{\partial}{\partial y_a}, \frac{\partial}{\partial y_b}\right)}_{g_{ab}}$$

$$= \sum_{a,b} \underline{\frac{\partial \phi_a}{\partial x_i}} \frac{\partial \phi_b}{\partial x_j} g_{ab}$$

short-hand notation for $\left[\frac{\partial}{\partial x_i}(y\circ\phi\circ x^{-1})\right]_a$

see side discussion

$f_i^a(p) = \frac{\partial \tilde\phi_a}{\partial x_i}$, where $\tilde\phi = y \circ \phi \circ x^{-1}$, $\tilde\phi = (\tilde\phi_1, \dots, \tilde\phi_a, \dots)$

b/c $d\phi_p\left(\frac{\partial}{\partial x_i}\right) = \sum_a f_i^a(p) \underbrace{dy^{-1}_{\phi(p)} E_a}_{\partial/\partial y_a} \in T_{\phi(p)}N$

$dy_{\phi(p)} d\phi_p \underbrace{(dx_p)^{-1} e_i}_{\partial/\partial x_i} = \sum_a f_i^a(p) E_a \in \mathbb{R}^N$

$d\tilde\phi_{x(p)} e_i = d(y\circ\phi\circ x^{-1})_{x(p)} e_i = \sum_a f_i^a(p) E_a \in \mathbb{R}^N$

hence $= \frac{\partial \tilde\phi_a}{\partial x_i}$ $\square$

In short,

$$\boxed{h_{ij} = \sum_{a,b} \frac{\partial \phi_a}{\partial x_i} \frac{\partial \phi_b}{\partial x_j} g_{ab}}$$

HW 1.

Ex: If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is a smooth function, find the metric on U s.t. the embedding $\phi: U \to \mathbb{R}^{n+1}$ given by $\phi(x) = (x, f(x))$ is isometric.

$\mathbb{R}^{n+1}$ endowed with $g_{Eucl.}$

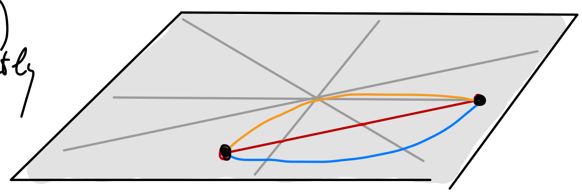
E.g., say $n=2$ and $f(x_1,x_2) = x_1^2 + x_2^2$. Then $\phi(x_1,x_2) = (\underset{\phi_1}{x_1}, \underset{\phi_2}{x_2}, \underset{\phi_3}{x_1^2+x_2^2})$

$$h_{11} = \sum_{a,b} \frac{\partial\phi_a}{\partial x_1}\frac{\partial\phi_b}{\partial x_1}\delta_{ab} = \sum_a \left(\frac{\partial \phi_a}{\partial x_1}\right)^2 = 1 + (2x_1)^2 = 1 + 4x_1^2,$$

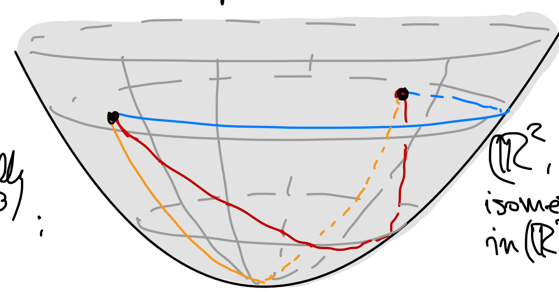
$$h_{12} = \sum_{a,b} \frac{\partial\phi_a}{\partial x_1}\frac{\partial\phi_b}{\partial x_2}\delta_{ab} = \sum_a \frac{\partial\phi_a}{\partial x_1}\frac{\partial\phi_a}{\partial x_2} = 4x_1x_2,$$

$h_{22} = \cdots = 1 + 4x_2^2$ so $h = (1+4x_1^2)\,dx_1^2 + 8x_1x_2\,dx_1dx_2 + (1+4x_2^2)\,dx_2^2$ is the induced metric on $\mathbb{R}^2$ seen as a paraboloid in $\mathbb{R}^3$.

$(\mathbb{R}^2, h)$ abstractly



or, seen isometrically inside $\mathbb{R}^3$:



$(\mathbb{R}^2, h)$ sits isometrically in $(\mathbb{R}^3, g_{Eucl})$

Note: "Most" metrics on surfaces cannot be realized as induced metric by some isometric embedding into $\mathbb{R}^3$. E.g., any flat torus cannot be isometrically embedded in $\mathbb{R}^3$... (Why?)

> Thm (Nash Embedding, 1956). Every C^k Riemannian manifold (M^n, g) can be C^k isometrically embedded in Euclidean space $\mathbb{R}^N$ for some N.

$3 \leq k \leq +\infty$

$\left(\begin{array}{l} M \text{ compact: } N \leq n(3n+11)/2 \\ M \text{ noncompact: } N \leq n(n+1)(3n+11)/2 \end{array} \right)$

Stranger things happen for C^1 Riem. manifolds, see "Nash–Kuiper Embedding Theorem".

Isometries $\leftarrow$ Postponed to Lecture 3

The isometry group of (M^n, g) is $\mathrm{Iso}(M^n, g) = \{\phi : M \xrightarrow[\text{diff}]{} M,\ \phi^* g = g\}$.

> Theorem (Myers–Steenrod, 1939). $\mathrm{Iso}(M^n, g)$ is a Lie group.

Ex. $\mathrm{Iso}(\mathbb{R}^n, g_{Eucl}) = O(n) \ltimes \mathbb{R}^n = \left\{ \phi : \mathbb{R}^n \to \mathbb{R}^n,\ \phi(x) = Ax + b,\ A \in O(n),\ b \in \mathbb{R}^n \right\}$

$\mathrm{Iso}(S^n, g_{round}) = O(n+1)$

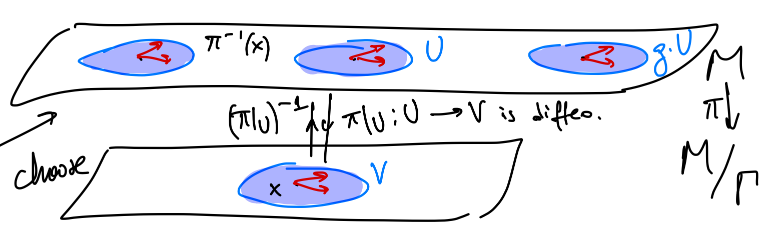
$\forall p \in M, \exists U \ni p$ neighbd. s.t. $g \cdot U \cap U \neq \emptyset \Leftrightarrow g = e$.

Prop. If $\Gamma < \mathrm{Iso}(M^n, g)$ acts properly discontinuously on (M^n, g), then M/Γ is a smooth manifold, and it inherits a Riemannian metric $\check{g}$ so that the covering map $(M^n, g) \xrightarrow{\pi} (M/\Gamma, \check{g})$ is a local isometry.

i.e., there exist neighborhoods $U \ni p$ in M and $V \ni \pi(p)$ in M/Γ s.t. $\pi|_U : U \to V$ is an isometry, i.e., $(\pi|_U)^*(\check{g}|_V) = g|_U$. $\forall p \in M$ and $\pi(p) \in M/\Gamma$,

Pf:

All isometric so can just choose one...



If $x \in M/\Gamma$, $\exists V \ni x$ open neighbd and $U \subset M$ s.t. $\pi^{-1}(V) = \bigcup_{g \in \Gamma} g \cdot U$. Define $\check{g}|_V := ((\pi|_U)^{-1})^* g|_U$.

Cor: Can endow S^n/Γ, $\mathbb{R}^n/\Gamma$, $\mathbb{H}^n/\Gamma$ with Riem. metrics that are locally isometric to (S^n, g_{round}), $(\mathbb{R}^n, g_{Eucl})$, $(\mathbb{H}^n/\Gamma, g_{hyp})$, e.g., $\mathbb{RP}^n$, T^n, ...

n-torus

Volume form.

Def: The volume form $vol_g \in \Omega^n(M^n)$ induced by a Riemannian metric on an orientable mfld is given in local coordinates $(x_1, \dots, x_n)$ by

$$vol_g = \sqrt{\det(g_{ij})}\, dx_1 \wedge \dots \wedge dx_n.$$

The Volume of (M^n, g) is $Vol(M, g) = \int_M vol_g$.

E.g., on $(\mathbb{R}^2, h)$ where h is the "paraboloid" metric above, there's a global chart and

$$\det(h_{ij}) = \det\begin{pmatrix} 1+4x_1^2 & 4x_1x_2 \\ 4x_1x_2 & 1+4x_2^2 \end{pmatrix} = 1 + 4x_1^2 + 4x_2^2 + 16x_1^2x_2^2 - 16x_1^2x_2^2$$

$$= 1 + \|\nabla f\|^2 \quad \text{where } f(x_1, x_2) = x_1^2 + x_2^2.$$

so $vol_h = \sqrt{1 + 4x_1^2 + 4x_2^2}\, dx_1 dx_2$; and, e.g., $Vol(U, h) = \int_U \sqrt{1+4x_1^2+4x_2^2}\, dx_1 dx_2$.

Distances.

Def: The distance between $p, q \in M$ with respect to a Riem. metric g is

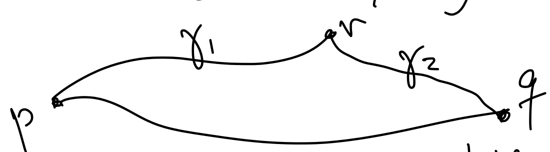
$$dist_g(p,q) = \inf\left\{ L_g(\gamma) : \gamma \text{ piecewise } C^1 \text{ curve on } M \text{ joining } p \text{ and } q. \right\}$$

Prop: $(M^n, dist_g)$ is a metric space, with the same topology as M.

Use a chart and fact ⊛ below...

Pf: Clearly, $dist_g$ is symmetric, $dist_g(p,q) \geq 0$ and $dist_g(p,q) = 0 \iff p = q$.

The triangle inequality, use concatenation of paths: if given $\varepsilon > 0$ we have



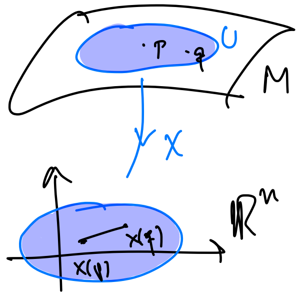
$$L_g(\gamma_1) < dist_g(p,r) + \varepsilon$$

$$L_g(\gamma_2) < dist_g(r,q) + \varepsilon$$

then $dist(p,q) \leq L_g(\gamma_1 * \gamma_2) = L_g(\gamma_1) + L_g(\gamma_2) = dist_g(p,r) + dist(r,q) + 2\varepsilon$

then let $\varepsilon \to 0$. To see the topologies agree, $\forall p \in M, \exists U \ni p$ chart and $C > 0$ s.t.

$$\frac{1}{C^2} x^*(g_{Eucl}) \leq (g|_U) \leq C^2 x^*(g_{Eucl})$$



e.g., compare the eigenvalues of the positive-def. matrices (g_{ij}) and $(x^* g_{Eucl})_{ij}$. Thus,

$$\frac{1}{C}\|x(p)-x(q)\| \leq \operatorname{dist}_g(p,q) \leq C\,\|x(p)-x(q)\| \quad \circledast$$

so the topologies agree $\square$

Thm (Myers-Steenrod, 1939). If $\phi : (M, \operatorname{dist}_g) \to (N, \operatorname{dist}_h)$ is a metric isom., i.e., $\operatorname{dist}_g(x,y) = \operatorname{dist}_h(\phi(x), \phi(y))$ for all $x, y \in M$ then ϕ is a Riem. isom., i.e., a smooth diffeomorphism st. $h_{\phi(p)}(d\phi_p v, d\phi_p w) = g_p(v,w)$, $\forall p \in M, \forall v,w \in T_pM$.

Simple exercise: prove the converse!

Product manifolds $M = M_1 \times M_2$, $TM = TM_1 \oplus TM_2$

If (M_i, g_i) are Riem. manifolds, then $M = M_1 \times M_2$ can be endowed with the "product metric" $g_1 \oplus g_2$, i.e., for all $p = (p_1, p_2) \in M$,

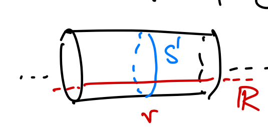
$$g_p(v,w) = (g_1)_{p_1}(v_1, w_1) + (g_2)_{p_2}(v_2, w_2), \quad \forall v = (v_1, v_2),\ w = (w_1, w_2) \in T_pM$$

Given a positive function $f : M_1 \to \mathbb{R}$, one may also endow M with a "warped product" metric

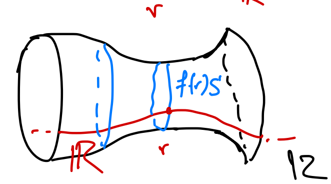
"warping function"

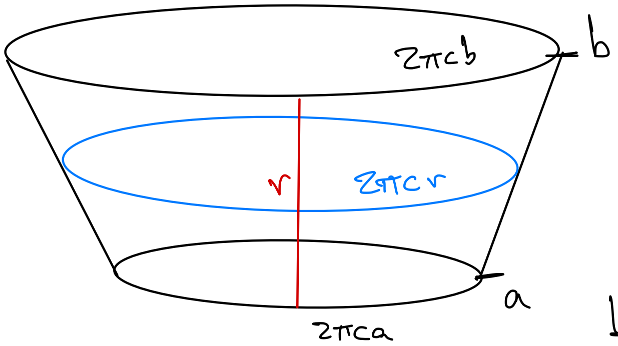
$$(g^f)_p(v,w) = (g_1)_{p_1}(v_1, w_1) + \boxed{f(p_1)}\,(g_2)_{p_2}(v_2, w_2), \quad \forall v = (v_1, v_2),\ w = (w_1, w_2) \in T_pM.$$

E.g., on $\underset{(r,\theta)}{\mathbb{R} \times S^1}$, we have the product (cylinder) metric $dr^2 + d\theta^2$,



or the warped product metric $dr^2 + f(r)^2 d\theta^2$, where the circle $S^1_{r_0} = \{(r_0, \theta) : \theta \in S^1\}$ has length $2\pi f(r_0)$.





Setting $f(r) = c\cdot r$ for some $c>0$ we have a cone metric

$$dr^2 + c^2 r^2 d\theta^2 \quad \text{on } [a,b]\times S^1$$

Letting $a \searrow 0$, and $b \nearrow +\infty$, we get

$$g^c = dr^2 + c^2 r^2 d\theta^2 \quad \text{on } (0,+\infty)\times S^1 \cong \mathbb{R}^2\setminus\{0\}$$

Q: For what value(s) of $c>0$ does g^c extend smoothly to $\mathbb{R}^2$?

A: In Euclidean coordinates (x,y) around $0\in\mathbb{R}^2$, we have the diffeo $\phi(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$ for $r>0$; and

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} dx = \dfrac{\partial x}{\partial r}dr + \dfrac{\partial x}{\partial \theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta \\[2mm] dy = \dfrac{\partial y}{\partial r}dr + \dfrac{\partial y}{\partial \theta}d\theta = \sin\theta\, dr + r\cos\theta\, d\theta \end{cases}$$

so
$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{\text{Rotation by } \theta} \begin{pmatrix} dr \\ r\,d\theta \end{pmatrix}, \quad \text{hence} \quad \begin{pmatrix} dr \\ r\,d\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{Rotation by } -\theta} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

invert the matrix

i.e.

$$\begin{cases} dr = \cos\theta\, dx + \sin\theta\, dy = \dfrac{x}{r}dx + \dfrac{y}{r}dy = \dfrac{x}{\sqrt{x^2+y^2}}dx + \dfrac{y}{\sqrt{x^2+y^2}}dy \\[2mm] d\theta = -\dfrac{\sin\theta}{r}dx + \dfrac{\cos\theta}{r}dy = -\dfrac{y}{r^2}dx + \dfrac{x}{r^2}dy = -\dfrac{y}{x^2+y^2}dx + \dfrac{x}{x^2+y^2}dy \end{cases}$$

hence

$$\begin{cases} dr^2 = \dfrac{x^2}{x^2+y^2}dx^2 + \dfrac{2xy}{x^2+y^2}dx\,dy + \dfrac{y^2}{x^2+y^2}dy^2 = \dfrac{x^2dx^2 + 2xy\,dx\,dy + y^2dy^2}{x^2+y^2} \\[2mm] d\theta^2 = \dfrac{y^2}{(x^2+y^2)^2}dx^2 - \dfrac{2xy}{(x^2+y^2)^2}dx\,dy + \dfrac{x^2}{(x^2+y^2)^2}dy^2 = \dfrac{y^2dx^2 - 2xy\,dx\,dy + x^2dy^2}{(x^2+y^2)^2} \end{cases}$$

Using the above, we compute:

$$(\phi^{-1})^* g^c = (\phi^{-1})^* dr^2 + c^2(x^2+y^2)(\phi^{-1})^* d\theta^2$$

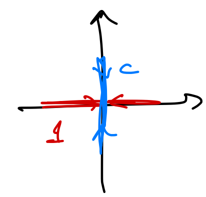
$$= \frac{x^2}{x^2+y^2}dx^2 + \frac{2xy}{x^2+y^2}dx\,dy + \frac{y^2}{x^2+y^2}dy^2$$

$$+ c^2\cancel{(x^2+y^2)}\left(\frac{y^2}{(x^2+y^2)^{\cancel{2}}}dx^2 - \frac{2xy}{(x^2+y^2)^{\cancel{2}}}dx\,dy + \frac{x^2}{(x^2+y^2)^{\cancel{2}}}dy^2\right)$$

$$= \frac{x^2+c^2y^2}{x^2+y^2}dx^2 + \frac{2xy(1-c^2)}{x^2+y^2}dx\,dy + \frac{y^2+c^2x^2}{x^2+y^2}dy^2$$

The above functions are smooth at $(x,y)=(0,0)$ if and only if $c=1$.

Indeed, e.g. $f(x,y)=\dfrac{x^2+c^2y^2}{x^2+y^2}$ has $\lim\limits_{x\to 0} f(x,0)=1$, $\lim\limits_{y\to 0} f(0,y)=c^2$.



Smoothness is invariant under diffeom., so

$$g^c = dr^2 + c^2r^2 d\theta^2, \quad \text{on} \quad (0,+\infty)\times S^1$$

extends smoothly to $r=0$, i.e., to $[0,+\infty)\times S^1$, if and only if $c=1$.

$\boxed{\text{Prop: The metric } g^f = dr^2 + f(r)^2 d\theta^2 \text{ with } f(0)=0 \text{ extends smoothly to } r=0 \text{ iff } \dfrac{\left(\frac{f(r)}{r}\right)^2-1}{r^2} \text{ does, equivalently, iff } |f'(0)|=1 \text{ and } f^{(2k)}(0)=0,\ \forall k\geq 1}$

all even derivatives vanish.

Q: Do we "recognize" the metric $g^1 = dr^2 + r^2 d\theta^2$?

A: It is isometric to $g_{Eucl} = dx^2+dy^2$, since setting $c=1$ in the above we find $(\phi^{-1})^*(g^1) = g_{Eucl}$! Or, similar computations from ☆ previous page! yield:

$$\begin{cases} dx^2 = \cos^2\theta\, dr^2 - 2r\sin\theta\cos\theta\, dr\,d\theta + r^2\sin^2\theta\, d\theta^2 \\ dx\,dy = \cos\theta\sin\theta\, dr^2 + r\cos^2\theta\, dr\,d\theta - r\sin^2\theta\, dr\,d\theta - r^2\sin\theta\cos\theta\, d\theta^2 \\ dy^2 = \sin^2\theta\, dr^2 + 2r\sin\theta\cos\theta\, dr\,d\theta + r^2\cos^2\theta\, d\theta^2 \end{cases}$$

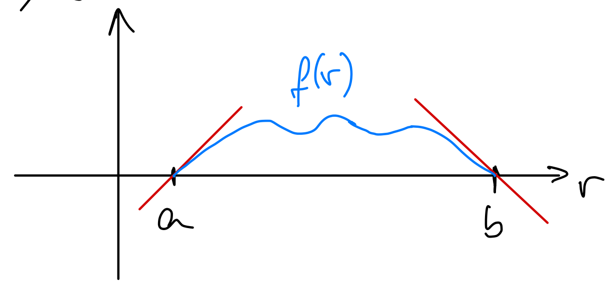
so $dx^2+dy^2 = dr^2+r^2d\theta^2$

these may be useful in your HW1...

Lecture 3 2/2/2024

The same considerations about extending $g^f = dr^2 + f(r)^2 d\theta^2$ smoothly to $r=a$ if $f(a)=0$ apply to extending it smoothly to $r=b$ if $f(b)=0$, namely, g^f extends smoothly to $[a,b]\times S^1/_\sim \cong S^2$ iff

$$f(a)=0 \qquad f(b)=0$$
$$f'(a)=1 \qquad f'(b)=-1$$
$$f^{(\text{even})}(a)=0 \qquad f^{(\text{even})}(b)=0$$

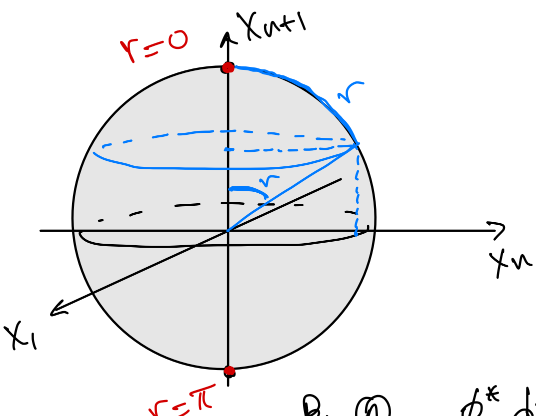


E.g., $f(r)=\sin r$ satisfies that on $[a,b]=[0,\pi]$, and the metric $g_{S^2}=dr^2+\sin^2 r\, d\theta^2$ is isometric to the round metric on S^2.

Def: $g_{S^n}=\phi^*(g_{\text{Eucl}})$ where $\phi: S^n \hookrightarrow \mathbb{R}^{n+1}$ is the usual inclusion of the unit sphere.

More generally, for $n\geq 2$, we can write the unit round metric g_{S^n} as a warped product on $(0,\pi)\times S^{n-1}\cong S^n\setminus\{\pm N\}$ by setting $g_{S^1}=d\theta^2$ and then $g_{S^n}=dr^2+\sin^2 r\, g_{S^{n-1}}$. Indeed, we have an isometric embedding $\phi_1: (S^1, g_{S^1}) \hookrightarrow (\mathbb{R}^2, g_{\text{Eucl}})$, $\phi_1(\theta)=(\sin\theta,\cos\theta)$, and we inductively set $\phi_n: (S^n, g_{S^n}) \hookrightarrow (\mathbb{R}^{n+1}, g_{\text{Eucl}})$ to be the extension to $r=0$ and $r=\pi$ of

$$\phi_n : (0,\pi)\times S^{n-1} \longrightarrow \mathbb{R}^{n+1}=\mathbb{R}^n\oplus\mathbb{R}$$
$$(r,p) \longmapsto \big((\sin r)\phi_{n-1}(p), \cos r\big)$$



Assume $\phi_{n-1}^*(dx_1^2+\dots+dx_n^2)=g_{S^{n-1}}$. Then

$$\begin{cases}(x_1,\dots,x_n)=(\sin r)\,\phi_{n-1}(p) & \text{Ⓐ}\\ x_{n+1}=\cos r & \text{Ⓑ}\end{cases}$$

By Ⓐ, $\phi_n^* dx_i = (\cos r)\underbrace{\phi_{n-1}(p)_i}_{p_i}\, dr + \sin r\, \phi_{n-1}^*(dx_i)$

$$\Rightarrow \phi_n^*(dx_i^2) = \cos^2 r\, p_i^2\, dr^2 + 2\cos r \sin r\,(p_i\,\phi_{n-1}^* dx_i)\,dr + \sin^2 r\,\phi_{n-1}^*(dx_i^2)$$

Note: $\|\phi_{n-1}(p)\|^2 = 1$ in $\mathbb{R}^n$, i.e., $p_1^2 + \cdots + p_n^2 = 1 \Rightarrow 2p_1\,\phi_{n-1}^*(dx_1) + \cdots + 2p_n\,\phi_{n-1}^*(dx_n) = 0$

So

$$\phi_n^*(dx_1^2 + \cdots + dx_n^2) = \cos^2 r \underbrace{\left(\sum_{i=1}^n p_i^2\right)}_{=1} dr^2 + 2\cos r \sin r \underbrace{\left(\sum_{i=1}^n p_i\,\phi_{n-1}^* dx_i\right)}_{=0} dr + \sin^2 r \underbrace{\phi_{n-1}^*(dx_1^2 + \cdots + dx_n^2)}_{g_{S^{n-1}}}$$

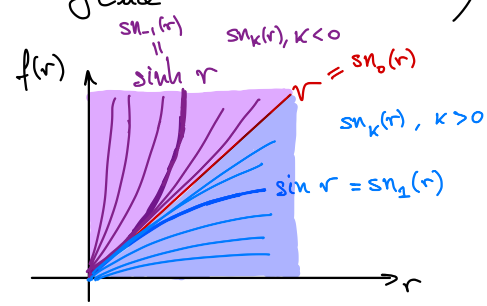
By (B), $\phi_n^*(dx_{n+1}) = -\sin r\, dr \Rightarrow \phi_n^*(dx_{n+1}^2) = \sin^2 r\, dr^2$

So $\phi_n^*(dx_1^2 + \cdots + dx_{n+1}^2) = \cos^2 r\, dr^2 + \sin^2 r\, g_{S^{n-1}} + \sin^2 r\, dr^2 = dr^2 + \sin^2 r\, g_{S^{n-1}}$. $\square$

Similarly, we can inductively endow $\mathbb{R}^n \setminus \{0\} \cong (0, +\infty) \times S^{n-1}$ with the metrics

- $dr^2 + r^2 \cdot g_{S^{n-1}}$ Euclidean metric (isometric to $g_{Euc} = dx_1^2 + \cdots + dx_n^2$)
- $dr^2 + \sinh^2 r\, g_{S^{n-1}}$ hyperbolic metric

both of which extend smoothly to $r = 0$.



The above warping functions $\begin{cases} \sin r & (S^n) \\ r & (\mathbb{R}^n) \\ \sinh r & (\mathbb{H}^n) \end{cases}$ fit in a 1-parameter family of warping functions $sn_K(r)$ s.t. $dr^2 + sn_K(r)^2 g_{S^{n-1}}$ extend smoothly to $r = 0$, namely the solutions to the 1-parameter family of ODEs

$$\begin{cases} sn_K''(r) + K\, sn_K(r) = 0, \\ sn_K(0) = 0, \quad sn_K'(0) = 1, \end{cases} \qquad K \in \mathbb{R}$$

Note this function vanishes at $r=0$ and $r = \pi/\sqrt{K}$, and extends smoothly to both points!

which can be computed explicitly to be $sn_K(r) = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}\, r), & K>0 \\ r, & K=0 \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}\cdot r), & K<0 \end{cases}$

$\boxed{\text{As we will see later, } dr^2 + sn_K(r)^2 g_{S^{n-1}} \text{ has constant sectional curvature } K.}$

(Leftovers from Lecture 2: Isometry group and induced metric on M/Γ.)

Lecture 4 2/7/2024

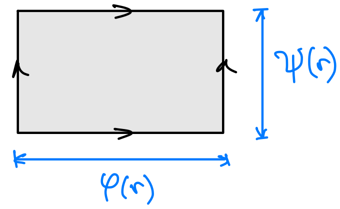
Doubly-warped products: $M = M_1 \times M_2 \times M_3$, $\varphi, \psi : M_1 \to \mathbb{R}$

$$g^{\varphi,\psi} = g_{M_1} \oplus \varphi^2 g_{M_2} \oplus \psi^2 g_{M_3}$$

E.g., on $M = (a,b) \times S^1 \times S^1$

consider $dr^2 + \varphi(r)^2 d\theta_1^2 + \psi(r)^2 d\theta_2^2$, i.e.

geometrically, this is a 1-par. family of flat tori



Q: When does the above metric extend smoothly to $r=a$ (or $r=b$)?

Note: At most 1 of φ, ψ can vanish at each endpoint, otherwise it would not be a manifold point (cone over a torus). ← cone over X is only a manifold if X is a sphere...

Similar considerations to (single) warped products yield:

A: If $\varphi(a)=0$, then $dr^2 + \varphi(r)^2 d\theta_1^2 + \psi(r)^2 d\theta_2^2$ extends smoothly to $r=a$ iff

$$\varphi(a) = 0 \qquad \psi(a) > 0$$
$$\varphi'(a) = 1 \qquad \psi^{(odd)}(a) = 0$$
$$\varphi^{(even)}(a) = 0$$

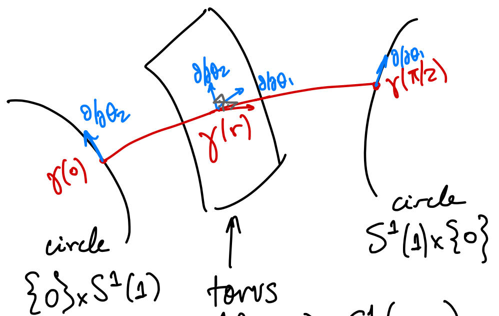
$\psi^{(odd)}(a) = 0$ ← i.e, has extension as even function

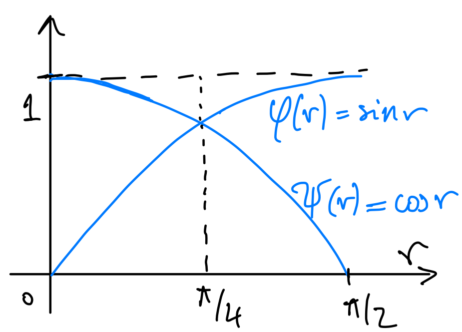
$\varphi^{(even)}(a) = 0$ ← i.e., has extension as odd function.

(similarly at $r=b$, or if ψ vanishes.)

Note: Same criteria hold replacing $(S^1, d\theta_i^2)$ with $(S^{n-1}, g_{S^{n-1}})$.

E.g., setting $\begin{cases} \varphi(r) = \sin r, \\ \psi(r) = \cos r, \end{cases}$ on $(a,b) = (0, \pi/2)$, we get a metric isometric to g_{S^3}.





$\gamma(r) = (\sin r, 0, \cos r, 0) \in \mathbb{R}^4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$
$((r_1,\theta_1), (r_2,\theta_2))$

cf. genus 1 Heegaard splitting of S^3.

More generally, for any $1 \leq k \leq n-2$, the metric $dr^2 + \sin^2 r\, g_{S^k} + \cos^2 r\, g_{S^{n-k-1}}$ on $(0,\pi/2) \times S^k \times S^{n-k-1} \subset S^n$ is isometric to g_{S^n}.

(i.e., restricted to $S^n \setminus ((S^k \times \{0\}) \cup (\{0\} \times S^{n-k-1})))$

In the above coordinates (r,θ_1,θ_2) on S^3, the Hopf fibration is

$$S^3 = [0,\pi/2]\times S^1\times S^1/\sim \ \ni (r,\theta_1,\theta_2) \overset{\pi}{\longmapsto} (r,\ \theta_1-\theta_2) \in [0,\pi/2]\times S^1/\sim \ \cong S^2(1/2)$$

$$dr^2+\sin^2 r\, d\theta_1^2+\cos^2 r\, d\theta_2^2 \qquad\qquad dr^2+\mathrm{sn}_4(r)^2 d\theta^2 = dr^2+\frac{\sin^2 2r}{4}d\theta^2$$

i.e., the Hopf circles are $\pi^{-1}(r,\theta)=\{(r,\ \theta+t,\ t) : t\in[0,2\pi]\}$ for each $(r,\theta)\in S^2(1/2)$

Let $p=(r,\theta_1,\theta_2)$ and recall that $\pi^{-1}(r,\theta)\subset S^3$ is a submanifold, with (b/c π is a submersion)

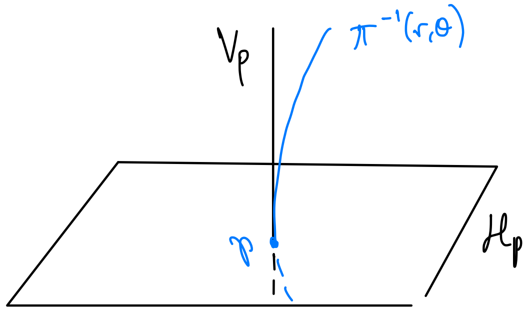
"vertical space" $V_p := T_p\left(\pi^{-1}(r,\theta)\right) = \operatorname{Ker} d\pi_p = \operatorname{span}\left\{\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2}\right\}$

$$d\pi_p\Big(\underbrace{a\frac{\partial}{\partial r}+b\frac{\partial}{\partial\theta_1}+c\frac{\partial}{\partial\theta_2}}_{\in T_pS^3}\Big) = a\frac{\partial}{\partial r}+(b-c)\frac{\partial}{\partial\theta}\in T_{\pi(p)}S^2(1/2)$$

"horizontal space" $H_p := T_p\left(\pi^{-1}(r,\theta)\right)^{\perp_g} = \operatorname{span}\left\{\frac{\partial}{\partial r},\ (\cot r)\frac{\partial}{\partial\theta_1}-(\tan r)\frac{\partial}{\partial\theta_2}\right\}$.

$$g\left(\frac{\partial}{\partial\theta_1}+\frac{\partial}{\partial\theta_2},\ \cot r\frac{\partial}{\partial\theta_1}-\tan r\frac{\partial}{\partial\theta_2}\right) = \cot r\left\|\frac{\partial}{\partial\theta_1}\right\|^2-\tan r\left\|\frac{\partial}{\partial\theta_2}\right\|^2 = \frac{\cos r}{\sin r}\cdot\sin^2 r-\frac{\sin r}{\cos r}\cos^2 r = 0;$$

others are clear. Note $\left\|\cot r\frac{\partial}{\partial\theta_1}-\tan r\frac{\partial}{\partial\theta_2}\right\|=1$, so this is a g-orthonormal basis.



$T_pS^3 = H_p\oplus V_p$ g-orthogonal direct sum.

and $d\pi_p|_{H_p} : H_p \longrightarrow T_{\pi(p)}S^2(1/2)$ is a (linear) isometry:

$$d\pi_p = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \text{ in the o.n.b. bases } \begin{matrix}\left\{\frac{\partial}{\partial r},\ (\cot r)\frac{\partial}{\partial\theta_1}-(\tan r)\frac{\partial}{\partial\theta_2}\right\} \text{ of } H_p\\[4pt] \left\{\frac{\partial}{\partial r},\ \frac{2}{\sin(2r)}\frac{\partial}{\partial\theta}\right\} \text{ of } T_{\pi(p)}S^2(1/2).\end{matrix}$$

b/c $d\pi_p\left(\cot r\frac{\partial}{\partial\theta_1}-\tan r\frac{\partial}{\partial\theta_2}\right) = (\cot r+\tan r)\frac{\partial}{\partial\theta} = \frac{\cos^2 r+\sin^2 r}{\sin r\cos r}\frac{\partial}{\partial\theta} = \frac{2}{\sin 2r}\frac{\partial}{\partial\theta}$;

and it is orthogonal to $\frac{\partial}{\partial r}\in T_{\pi(p)}S^2(1/2)$. Therefore, $\pi : S^3(1)\longrightarrow S^2(1/2)$ is a Riemannian submersion. In some sense, this is a dual notion to isometric immersion.

Def: A submersion $\pi : (M, g) \longrightarrow (N, h)$ is a Riemannian submersion if for all $p \in M$, the restriction of $d\pi_p : T_pM \to T_{\pi(p)}N$ to $\mathcal{H}_p = (\operatorname{Ker} d\pi_p)^{\perp_g}$ is a linear isometry.

Careful: The distribution of horizontal spaces in M need not be integrable!

Q: If $\pi : (M, g) \to N$ is a submersion, can we endow N with a Riem. metric so that π becomes a Riem. submersion?

A: In general, no. Need horizontal spaces along each $\pi^{-1}(x)$ to be isometric...

Write $g = g_{hor} \oplus g_{ver}$ w.r.t. $T_pM = \mathcal{H}_p \oplus \mathcal{V}_p$, where $\mathcal{V}_p = \operatorname{Ker} d\pi_p$, $\mathcal{H}_p = \mathcal{V}_p^{\perp_g}$.

Prop. Suppose $\pi : (M, g) \to N$ is a submersion with connected fibers $\pi^{-1}(x)$, $x \in N$. There exists a metric h on N s.t. π is a Riem. submersion iff $\mathcal{L}_V(g_{hor}) = 0$ $\forall V \in \mathcal{V}_p$, $\forall p \in M$.

Pf: See e.g. Gromoll–Walschap's book "Metric foliations and Curvature", Thm 1.2.1

Lie derivative of g_{hor} in the direction V:
$$(\mathcal{L}_V g_{hor})(X, Y) = V(g_{hor}(X, Y)) - g_{hor}([V, X], Y) - g_{hor}(X, [V, Y])$$

Cor. If $G \subset \operatorname{Isom}(M, g)$ acts on (M, g) in such way that M/G is a manifold (e.g., freely), then there exists a Riem. metric $\check{g}$ on M/G s.t. $\pi : (M, g) \to (M/G, \check{g})$ is a Riem. submersion.

Ex: $S^1 \curvearrowright S^{2n+1} \subset \mathbb{C}^{n+1}$ by unit complex multiplication, freely and isometrically, so there is a Riem. metric on $S^{2n+1}/S^1 = \mathbb{C}P^n$ s.t. $\pi : S^{2n+1}(1) \longrightarrow \mathbb{C}P^n$ is a Riem. submersion.

called "Fubini–Study metric" g_{FS}. Note: By the above, for $n=1$, $(\mathbb{C}P^1, g_{FS}) \underset{\text{isom}}{\cong} S^2(1/2)$.

Similarly for $S^3 \curvearrowright S^{4n+3} \subset \mathbb{H}^{n+1}$ by unit quaternion mult., and $S^{4n+3}/S^3 = \mathbb{H}P^n$.

Side Ex 1. Compute the length of the Hopf circle $\pi^{-1}(r, \theta) \subset S^3$.

Integrating the length of the tangent vector $\frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2}$, we find the length is:
$$\int_{\pi^{-1}(r,\theta)} \left\| \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} \right\| \mathrm{vol}_{\pi^{-1}(r,\theta)} = \int_0^{2\pi} (\sin^2 r + \cos^2 r)^{1/2} \, dt = 2\pi. \quad (\text{independent of } (r, \theta) \ldots)$$

Side Ex 2. Compute the volume of (S^3, g_{S^3}):
$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi/2} (\sin r \cos r) \, dr \, d\theta_1 \, d\theta_2 = 4\pi^2 \cdot \frac{\sin^2 r}{2} \Big|_0^{\pi/2} = 2\pi^2.$$

Connections:

Q: How do we differentiate vector fields with respect to each other in $\mathbb{R}^n$?

A: Vector fields $X, Y : \mathbb{R}^n \to \mathbb{R}^n$ are n-tuples of functions $a_i, b_j : \mathbb{R}^n \to \mathbb{R}$,

$$X = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_n \frac{\partial}{\partial x_n}$$

$$Y = b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2} + \cdots + b_n \frac{\partial}{\partial x_n}$$

(Note: Here we are identifying $T_p\mathbb{R}^n \cong \mathbb{R}^n$, which is only possible b/c $\mathbb{R}^n$ is a vector space...)

and each Y_j can be differentiated at $p \in \mathbb{R}^n$ in the direction $X(p)$:

$$\left(X(b_j)\right)_p = a_1(p)\frac{\partial b_j}{\partial x_1}(p) + \cdots + a_n(p)\frac{\partial b_j}{\partial x_n}(p) = \sum_{i=1}^n a_i(p)\frac{\partial b_j}{\partial x_i}(p)$$

so we write

$$\underbrace{D_X Y}_{\text{or } dY(X)\ldots} = X(Y) = \sum_{j=1}^n \underbrace{\left(\sum_{i=1}^n a_i \frac{\partial b_j}{\partial x_i}\right)}_{X(b_j)} \frac{\partial}{\partial x_j}$$

On a manifold M, vector fields are sections of TM, i.e. $X, Y : M \to TM$, so a "derivative" would be $dY(p) : T_pM \to T_{Y(p)}TM$ for each $p \in M$, i.e.

$dY : TM \to T(TM)$; and "$D_X Y$" $= dY(X)$ would land on $T(TM)$, not TM.

Note: $M \xrightarrow{X} TM \xrightarrow{dY} T(TM)$, and, unless $M = \mathbb{R}^n$, there's no canonical choice of "horizontal".

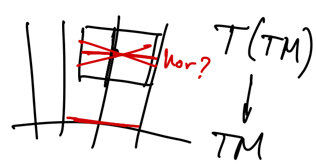
If $M = \mathbb{R}^n$, we identify $T_p\mathbb{R}^n \cong \mathbb{R}^n$ so $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ and $X : \mathbb{R}^n \to T\mathbb{R}^n$, $p \longmapsto (p, X(p))$. Then

$$dY : T\mathbb{R}^n \to T(T\mathbb{R}^n) \cong (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n)$$

$$(p, v) \longmapsto \left((p, Y(p)), \boxed{\left(Y(p), dY_p(v)\right)}\right)$$

Canonical "horizontal" part for bundle $T(TM) \to TM$, if $M = \mathbb{R}^n$!

$$\begin{tikzcd}
(p,v) \arrow[r, mapsto] & \left((p,Y(p)),(Y(p),dY_p v)\right) & \\
T\mathbb{R}^n \arrow[r, "dY"] \arrow[d] & T(T\mathbb{R}^n) \cong T\mathbb{R}^n \times T\mathbb{R}^n \arrow[d] & \cong \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \\
\mathbb{R}^n \arrow[r, "Y"] & T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n & \\
p \arrow[r, mapsto] & (p, Y(p)) &
\end{tikzcd}$$



so $dY(X) = dY\left((p, X(p))\right) = \left((p, Y(p)), (Y(p), \underline{dY_p(X(p))})\right) \simeq dY(X)_p = X(Y)_p$

On a manifold M, we could also choose coordinates near a point and compute derivatives of vector fields w.r.t. each other in $\mathbb{R}^n$, but this would depend on the choice of coordinates (charts). Issue: "difference quotient" is ill posed on M:

Can't define $dY(X)_p$ " $=$ " $\displaystyle\lim_{\substack{q \to p \\ \text{along } X}} \frac{Y(q) - Y(p)}{\operatorname{dist}(p,q)} \ldots$ do not belong to same vector space, and there's no canonical way to identify T_pM and T_qM, even for q near p.

Lecture 5 2/9/2024

Def: A connection ∇ on a vector bundle $E \to M$ is a map
$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \longrightarrow \Gamma(E)$$
$$(X, Y) \longmapsto \nabla_X Y \quad \text{s.t.}$$

(i) $X \longmapsto \nabla_X Y$ is $C^\infty(M)$-linear: $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y \quad \forall f_i \in C^\infty(M)$

(ii) $Y \longmapsto \nabla_X Y$ is $\mathbb{R}$-linear; $\nabla_X (c_1 Y_1 + c_2 Y_2) = c_1 \nabla_X Y_1 + c_2 \nabla_X Y_2. \ \forall c_i \in \mathbb{R}$

(iii) Leibniz rule $\nabla_X (f Y) = f \nabla_X Y + X(f)\, Y.$

Note: The above ∇ on $\mathbb{R}^n$ is a connection on $T\mathbb{R}^n \to \mathbb{R}^n$. Using partitions of unity, can easily show that every vector bundle $E \to M$ (e.g., $E = (TM^*)^{\otimes r} \otimes TM^{\otimes s}$) can be endowed with a connection (pull-back from $\mathbb{R}^n$ using bundle charts...). (will define this soon)

Prop: The value of $(\nabla_X Y)_p \in T_p M$ depends only on $X(p) \in T_p M$ and Y in a neighborhood of $p \in M$.

Pf: Use a bump function on neighborhood of $p \in M$ and (i)—(iii) to prove locality, then (i) to prove it only depends on X at p. (cf. Lee p. 89-91.)

Let us now specialize to $E = TM$, so $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$.

On a chart $x : U \subset M \to x(U) \subset \mathbb{R}^n$, using coordinate vectors $\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$

$$\boxed{\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial}{\partial x_k}}$$

Coefficients $\{\Gamma_{ij}^k\}$ of $\nabla_{\frac{\partial}{\partial x_i}} \partial/\partial x_j$ are called "Christoffel symbols" (of ∇).

Then, if $X = \sum a_i \frac{\partial}{\partial x_i}$, $Y = \sum b_j \frac{\partial}{\partial x_j}$ on U, we have

$$\nabla_X Y = \nabla_{\left(\sum_i a_i \frac{\partial}{\partial x_i}\right)} \left(\sum_j b_j \frac{\partial}{\partial x_j} \right) = \sum_i a_i \nabla_{\frac{\partial}{\partial x_i}} \left(\sum_j b_j \frac{\partial}{\partial x_j} \right)$$

$$= \sum_{i,j} a_i b_j \underbrace{\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}}_{\sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}} + \underbrace{\sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j}}_{X(b_j) \frac{\partial}{\partial x_j}}$$

as in $\mathbb{R}^n$... (replace index j with k)

$$= \sum_{i,k} \left[\underbrace{a_i \frac{\partial b_k}{\partial x_i}}_{\substack{\text{this is} \\ \text{"}X(Y)\text{"} \\ \text{in coordinates} \\ \text{cf. } \nabla \text{ in } \mathbb{R}^n}} + \underbrace{\sum_j a_i b_j \Gamma_{ij}^k}_{\substack{\text{this is the} \\ \text{"correction" to} \\ \text{make it coord.} \\ \text{invariant.}}} \right] \frac{\partial}{\partial x_k}$$

(In $\mathbb{R}^n$, the usual ∇ has $\Gamma_{ij}^k \equiv 0$, so that $\nabla_X Y = X(Y)$.)

Note: The above confirms that $(\nabla_X Y)_p$ only depends on $a_i(p)$ and b_j near p.

From ∇ we defined Γ_{ij}^k, and, conversely, given Γ_{ij}^k can define a unique connection ∇ with these prescribed Christoffel symbols (in charts, by formula above).

Prop: If ∇ and $\tilde{\nabla}$ are connections on TM, with Christoffel symbols Γ and $\tilde{\Gamma}$, then $A = \nabla - \tilde{\nabla}$ is a $(1,2)$-tensor $A_{ij}^k = \Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k$. The space of connections on TM is therefore an affine space $\{\underbrace{\nabla}_{\substack{\text{choose} \\ \text{a connection}}} + \underbrace{A}_{\substack{\text{difference to} \\ \text{any other connection.}}} : A \in \Gamma(TM^* \otimes TM^* \otimes TM)\}$

Pf: To check A is $C^\infty(M)$-linear in the remaining slot use Leibniz rule for ∇ and $\tilde{\nabla}$ and see that nonlinear terms cancel.

Note: From a connection ∇ on TM, can define induced connections ∇ (same symbol...) on TM^* by setting for a 1-form $\omega \in \Gamma(TM^*) \cong \Omega^1(M)$,

$$(\nabla_X \omega)(Y) = X(\omega(Y)) - \omega(\underbrace{\nabla_X Y}_{\text{connection on } TM}) \qquad \forall X, Y \in \mathfrak{X}(M)$$

and, moreover, on all tensor bundles over M, i.e., on $E = (TM^*)^{\otimes r} \otimes TM^{\otimes s}$, via

$$\begin{aligned}(\nabla_X T)(Y_1, \dots, Y_r, \omega_1, \dots, \omega_s) &= X(T(Y_1, \dots, Y_r, \omega_1, \dots, \omega_s)) \\ &\quad - \sum_{k=1}^r T(Y_1, \dots, \underbrace{\nabla_X Y_k}_{\text{the original connection } \nabla \text{ on } TM}, \dots, Y_r, \omega_1, \dots, \omega_s) \\ &\quad - \sum_{k=1}^s T(Y_1, \dots, Y_r, \omega_1, \dots, \underbrace{\nabla_X \omega_k}_{\text{defined above}}, \dots, \omega_s).\end{aligned}$$

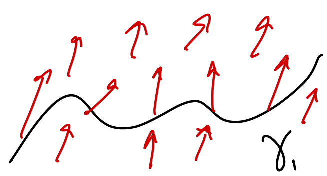
Note: Can think of $\nabla T \in \Gamma((TM^*)^{\otimes r+1} \otimes TM^{\otimes s})$, setting $(\nabla T)(X, Y_1, \dots Y_r, \omega_1, \dots, \omega_s) = (\nabla_X T)(Y_1, \dots, Y_r, \omega_1, \dots, \omega_s)$. So $\nabla : \Gamma(E) \to \Gamma(TM^* \otimes E)$.

Def: Given a connection ∇ on $E \to N$ and a smooth map $f: M \to N$, the pullback connection $f^*\nabla$ on $f^*E \to M$ is determined by

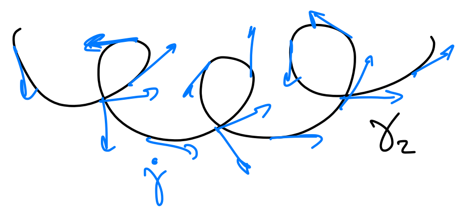
$$(f^*\nabla)_X (f^*s) = f^*\left(\nabla_{df(X)} s\right)$$

<u>Important example:</u> Let $\gamma: (a,b) \to M$ be a smooth curve in M, and γ^*TM be the pullback bundle of TM via γ, i.e., the vector bundle over (a,b) whose sections $V \in \Gamma(\gamma^*TM)$ are vector fields along γ:

$$\begin{tikzcd} & TM \arrow[d] \\ (a,b) \arrow[ur, "V"] \arrow[r, "\gamma"] & M \end{tikzcd} \qquad V: (a,b) \to TM \quad \text{s.t. } V(t) \in T_{\gamma(t)}M \quad \forall t \in (a,b).$$



e.g., restricting an ambient vector field X to γ, we obtain a vector field $X(\gamma(t))$ along γ.



e.g. $\dot\gamma \in \Gamma(\gamma^*TM)$ is a vector field along γ, which may not be the restriction of an ambient vector field.

Given a connection ∇ on TM, use the pullback connection $\gamma^*\nabla$ to define the <u>covariant derivative</u> of vector fields along γ:

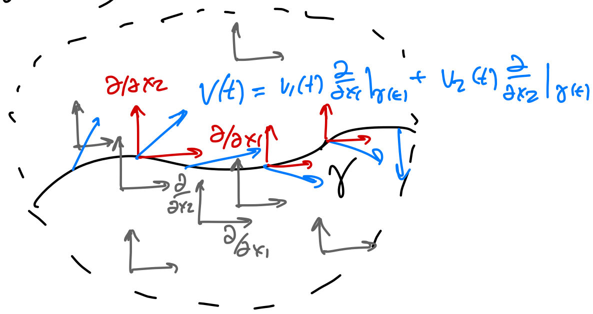
$$V \in \Gamma(\gamma^*TM) \rightsquigarrow \frac{D}{dt} V := (\gamma^*\nabla)_{\partial/\partial t} V \in \Gamma(\gamma^*TM)$$

satisfying the following properties:

- $\dfrac{D}{dt}(c_1V_1 + c_2V_2) = c_1 \dfrac{D}{dt} V_1 + c_2 \dfrac{D}{dt} V_2, \quad \forall c_1, c_2 \in \mathbb{R}$
- $\dfrac{D}{dt}(fV) = f' \cdot V + f \cdot \dfrac{D}{dt} V, \qquad \forall f \in C^\infty((a,b), \mathbb{R})$
- If V is the restriction of an ambient vector field $\tilde V$ on M, then $\dfrac{D}{dt} V = \nabla_{\dot\gamma(t)} \tilde V$.

Similarly for any other tensors along γ, i.e., sections of $\gamma^*\left((TM^*)^{\otimes r} \otimes TM^{\otimes s}\right)$

In coordinates, with $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$, we have that a vector field along γ, say $V \in \Gamma(\gamma^* TM)$, can be written as



$$V(t) = \sum_{j=1}^n v_j(t) \frac{\partial}{\partial x_j}\Big|_{\gamma(t)}$$

for some functions $v_j : (a,b) \to \mathbb{R}$, so

$$\frac{D}{dt} V(t) = \sum_j v_j'(t) \frac{\partial}{\partial x_j}\Big|_{\gamma(t)} + v_j(t) \underbrace{\frac{D}{dt}\left(\frac{\partial}{\partial x_j}\Big|_{\gamma(t)}\right)}$$

$= \nabla_{\dot\gamma(t)} \frac{\partial}{\partial x_j}$ b/c $\frac{\partial}{\partial x_j}\Big|_{\gamma(t)}$ is the restriction of $\frac{\partial}{\partial x_j}$...

$$= \sum_j v_j'(t) \frac{\partial}{\partial x_j}\Big|_{\gamma(t)} + \sum_{i,j,k} a_i'(t) \cdot v_j(t) \cdot \Gamma_{ij}^k(\gamma(t)) \frac{\partial}{\partial x_k}\Big|_{\gamma(t)}$$

$$= \sum_k \underbrace{\left(v_k' + \sum_{i,j} a_i' v_j \Gamma_{ij}^k \right)} \frac{\partial}{\partial x_k}\Big|_\gamma$$

So these are the coefficients of $\frac{D}{dt}V$ as a vector field along γ.

Write $\gamma(t)$ in coordinates;

$\gamma(t) = (a_1(t), \ldots, a_n(t))$ so

$$\dot\gamma(t) = \sum_{i=1}^n a_i'(t) \frac{\partial}{\partial x_i}\Big|_{\gamma(t)}$$

and hence:

$$\nabla_{\dot\gamma(t)} \frac{\partial}{\partial x_j} = \sum_{i=1}^n a_i'(t) \nabla_{\partial/\partial x_i} \frac{\partial}{\partial x_j} = \sum_{i=1}^n a_i'(t) \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

<u>Def:</u> A vector field V along γ is <u>parallel</u> if $\frac{D}{dt} V \equiv 0$. (Similarly for any tensor along γ.)

<u>Prop:</u> Given a (piecewise) smooth curve $\gamma : (a,b) \to M$, a connection on TM, and a vector $v \in T_{\gamma(t_0)} M$, for some $t_0 \in (a,b)$, there exists a unique parallel vector field $V : (a,b) \to TM$ along γ s.t. $V(t_0) = v$. Moreover, $V(t)$ depends smoothly on the initial data v.

Pf: Existence, uniqueness and smooth dependence on initial data for the first-order linear ODE system

$$\begin{cases} v_k'(t) + \sum_{i,j} \underline{a_i'(t)}\, v_j(t)\, \Gamma_{ij}^k(\gamma(t)) = 0 . & k = 1, \dots, n \\ \qquad\qquad\quad \text{coeff. of } \dot\gamma & \\ v_k(t_0) = v_k \leftarrow \text{coeff. of initial data} & \end{cases}$$

$\square$

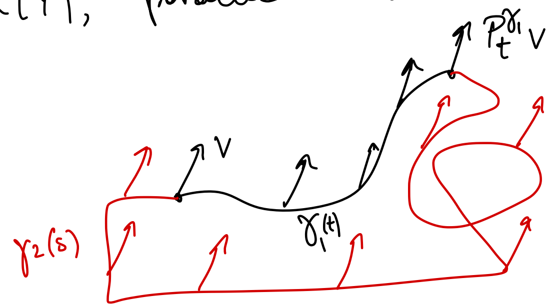
Def: The above $V(t)$ is called the parallel transport (or translation) w.r.t. ∇ of $v \in T_{\gamma(t_0)}M$ along γ. Sometimes write $P_t^\gamma : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$, $P_t^\gamma v := V(t)$.

Dependence on ∇ is not indicated in this notation...

Ex: In $\mathbb{R}^n$, w.r.t. $\nabla_X Y = X(Y)$, parallel translation is "constant", and independent of the path:

$$P_t^\gamma v = v, \quad \forall v \in \mathbb{R}^n, \ \forall \gamma.$$

$$\underset{T_{\gamma(t)}\mathbb{R}^n \cong \mathbb{R}^n}{\cap}$$



None of these features hold in general... (see e.g., HW 2.)

Def: Let M be a smooth manifold and ∇ be a connection on TM. A curve $\gamma : (a,b) \to M$ is a $\underline{\nabla\text{-geodesic}}$ in M if $\nabla_{\dot\gamma} \dot\gamma = 0$.

In coordinates, if $V = \dot\gamma(t) = \sum_i \underset{V_i(t)}{\underset{\|}{a_i'(t)}} \frac{\partial}{\partial x_i}\Big|_{\gamma(t)}$, then by above computation:

$$\nabla_{\dot\gamma} \dot\gamma = \sum_k \left(a_k''(t) + \sum_{i,j} a_i'(t) a_j'(t)\, \Gamma_{ij}^k(\gamma(t)) \right) \frac{\partial}{\partial x_k}\Big|_{\gamma(t)}$$

so $\gamma(t) = (a_1(t), \dots, a_n(t))$ is a ∇-geodesic iff

$$\boxed{a_k'' + \sum_{i,j} a_i' a_j' \Gamma_{ij}^k = 0. \qquad k = 1, \dots, n}$$

"Geodesic equation"
System of second order quasilinear ODEs

(Intuition: $\nabla_{\dot\gamma}\dot\gamma$ is the acceleration of γ; so $\nabla_{\dot\gamma}\dot\gamma = 0 \iff \gamma$ free fall)

Prop: Given $p \in M$ and $v \in T_pM$, and a connection ∇ on TM, there exists a unique maximal ∇-geodesic $\gamma : I \to M$, with $t_0 \in I$ and $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = v$; which depends smoothly on $(p,v) \in TM$.

Pf: Again, existence, uniqueness, and smooth dependence for second-order ODEs. $\square$

E.g., in $\mathbb{R}^n$ with $D_X Y = X(Y)$, we know that $\Gamma_{ij}^k \equiv 0$, so geodesics are straight lines.