Homework #1

Due: Feb 7, 2024

1. Prove that two Riemannian metrics g and h on the circle $\mathbb{S}^1$ are isometric if and only if $(\mathbb{S}^1, \mathrm{g})$ and $(\mathbb{S}^1, \mathrm{h})$ have the same length.

   Clearly, if $(\mathbb{S}^1, \mathrm{g})$ and $(\mathbb{S}^1, \mathrm{h})$ do not have the same length, then they are not isometric. For the converse, suppose $(\mathbb{S}^1, \mathrm{g})$ and $(\mathbb{S}^1, \mathrm{h})$ have the same length. Write $\mathrm{g} = f_1(\theta)^2\,\mathrm{d}\theta^2$ and $\mathrm{h} = f_2(\theta)^2\,\mathrm{d}\theta^2$, where $\theta\colon (0, 2\pi) \to \mathbb{S}^1$ is a coordinate chart for $\mathbb{S}^1 = [0, 2\pi]/\!\sim$ whose image is the complement of a point. By assumption, the lengths coincide, i.e.,

$$\int_0^{2\pi} f_1(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} f_2(\theta)\,\mathrm{d}\theta = 2\pi r, \text{ for some } r > 0.$$

   Let $\phi_i\colon [0, 2\pi] \to [0, 2\pi r]$ be the increasing smooth functions $\phi_i(\theta) = \int_0^\theta f_i(t)\,\mathrm{d}t$, which induce diffeomorphisms $\phi_i\colon \mathbb{S}^1 \to [0, 2\pi r]/\!\sim$, for $i = 1, 2$. Let $\mathrm{d}s^2$ be the metric on $[0, 2\pi r]/\!\sim$ induced by the Euclidean metric on $[0, 2\pi r]$. Then $\phi_1^*\mathrm{d}s^2 = \mathrm{g}$ and $\phi_2^*\mathrm{d}s^2 = \mathrm{h}$, so we have an isometry $(\phi_2^{-1} \circ \phi_1)^*\mathrm{h} = (\phi_1)^*((\phi_2^{-1})^*\mathrm{h}) = \mathrm{g}$.

2. Let $\mathrm{g}_{11}, \mathrm{g}_{12}, \mathrm{g}_{22}$ be real numbers such that $\mathrm{g}_{11} > 0$ and $\mathrm{g}_{11}\mathrm{g}_{22} - \mathrm{g}_{12}^2 > 0$. Prove that the "constant" Riemannian metric $\mathrm{g} = \mathrm{g}_{11}\,\mathrm{d}u^2 + 2\mathrm{g}_{12}\,\mathrm{d}u\mathrm{d}v + \mathrm{g}_{22}\,\mathrm{d}v^2$ on $\mathbb{R}^2$ is isometric to the "usual" Euclidean metric $\mathrm{g}_{\mathrm{Eucl}} = \mathrm{d}x^2 + \mathrm{d}y^2$ by finding an explicit linear diffeomorphism $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\phi^*\mathrm{g}_{\mathrm{Eucl}} = \mathrm{g}$.

   If $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear diffeomorphism given by

$$\phi(u, v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ b \end{pmatrix},$$

   then $\phi^*\mathrm{g}_{\mathrm{Eucl}} = (a^2 + c^2)\,\mathrm{d}u^2 + 2(ab + cd)\,\mathrm{d}u\mathrm{d}v + (b^2 + d^2)\,\mathrm{d}v^2$.

   Thus, solving $\phi^*\mathrm{g}_{\mathrm{Eucl}} = \mathrm{g}$ under the above assumptions, we find

$$\phi(u, v) = \frac{1}{\sqrt{\mathrm{g}_{11}}} \begin{pmatrix} \mathrm{g}_{11} & \mathrm{g}_{12} \\ 0 & \sqrt{\mathrm{g}_{11}\mathrm{g}_{22} - \mathrm{g}_{12}^2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

3. Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Find the coordinate expression (g_{ij}'s) of a Riemannian metric g such that the embedding $\phi\colon (U, \mathrm{g}) \to (\mathbb{R}^{n+1}, \mathrm{g}_{\mathrm{Eucl}})$ given by $\phi(x) = (x, f(x))$ is isometric. Show that the volume of (U, g) is

$$\int_U \sqrt{1 + \|\nabla f\|^2}\,\mathrm{d}x_1 \dots \mathrm{d}x_n,$$

   where $\|\nabla f\|^2 = \sum_i \left(\frac{\partial f}{\partial x_i}\right)^2$ is the square norm of the Euclidean gradient of f.

The pullback metric $\mathrm{g} = \phi^*(\mathrm{g}_{\mathrm{Eucl}})$ with respect to $\phi = (\phi_1, \dots, \phi_N) \colon M \to \mathbb{R}^N$ is

$$\mathrm{g}_{ij} = \sum_{a=1}^{N} \frac{\partial \phi_a}{\partial x_i} \frac{\partial \phi_a}{\partial x_j},$$

so, with $N = n+1$, we set $\phi_a(x) = x_a$ for $1 \leq a \leq n$ and $\phi_{n+1}(x) = f(x)$, and find that the pullback metric is

$$\mathrm{g}_{ij} = \delta_{ij} + \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}.$$

In other words, $\mathrm{g} = \mathrm{Id} + \nabla f \otimes \nabla f$ where, as a matrix, $\nabla f \otimes \nabla f = \nabla f \cdot (\nabla f)^T$ if ∇f is a column vector. From basic Linear Algebra,[1] $\det(\mathrm{Id} + vw^T) = 1 + \langle v, w \rangle$ for column vectors v, w, so

$$\det(\mathrm{g}) = \det\big(\mathrm{Id} + \nabla f \otimes \nabla f\big) = 1 + \|\nabla f\|^2,$$

hence the volume form of (U, g) is $\mathrm{vol}_{\mathrm{g}} = \sqrt{1 + \|\nabla f\|^2}\, \mathrm{d}x_1 \dots \mathrm{d}x_n$, so the formula for the volume follows.

4. A few different ways to see the unit round metric on the open hemisphere:

   (a) Use the previous exercise to find a coordinate expression for the metric $\mathrm{g}^{(\mathrm{a})}$ induced on the hemisphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z > 0\}$ and compute its volume.

   (b) Compute the volume of the unit ball in $\mathbb{R}^2$ with $\mathrm{g}^{(\mathrm{b})} = \frac{4}{(1+x^2+y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$.

   (c) Rewrite $\mathrm{g}^{(\mathrm{b})}$ in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ and reparametrize the radial direction by arclength to obtain an (isometric) metric $\mathrm{g}^{(\mathrm{c})} = \mathrm{d}\rho^2 + \sin^2\rho\, \mathrm{d}\theta^2$. Compute its volume once again, but now in the coordinates (ρ, θ).

   (a) Let $U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $f \colon U \to \mathbb{R}$ be $f(x, y) = \sqrt{1 - x^2 - y^2}$. Then, $\nabla f(x, y) = \left(\frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}\right)$, so by the previous exercise

$$\mathrm{g}^{(\mathrm{a})} = \left(1 + \frac{x^2}{1 - x^2 - y^2}\right) \mathrm{d}x^2 + \frac{2xy}{1 - x^2 - y^2}\, \mathrm{d}x\mathrm{d}y + \left(1 + \frac{y^2}{1 - x^2 - y^2}\right) \mathrm{d}y^2.$$

   Moreover, the volume form of $\mathrm{g}^{(\mathrm{a})}$ is

$$\mathrm{vol}_{\mathrm{g}^{(\mathrm{a})}} = \sqrt{1 + \frac{x^2 + y^2}{1 - x^2 - y^2}}\, \mathrm{d}x\mathrm{d}y = \sqrt{\frac{1}{1 - x^2 - y^2}}\, \mathrm{d}x\mathrm{d}y,$$

   from which we compute

$$\mathrm{Vol}(U, \mathrm{g}^{(\mathrm{a})}) = \iint_U \sqrt{\frac{1}{1 - x^2 - y^2}}\, \mathrm{d}x\mathrm{d}y = \int_0^{2\pi} \int_0^1 \sqrt{\frac{1}{1 - r^2}}\, r\mathrm{d}r\mathrm{d}\theta = 2\pi.$$

[1] See e.g., `https://en.wikipedia.org/wiki/Matrix_determinant_lemma`.

(b) The volume form of $\mathrm{g}^{(\mathrm{b})} = \frac{4}{(1+x^2+y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$ is

$$\mathrm{vol}_{\mathrm{g}^{(\mathrm{b})}} = \frac{4}{(1+x^2+y^2)^2}\,\mathrm{d}x\mathrm{d}y,$$

from which we compute

$$\mathrm{Vol}(U, \mathrm{g}^{(\mathrm{b})}) = \iint_U \frac{4}{(1+x^2+y^2)^2}\,\mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\int_0^1 \frac{4}{(1+r^2)^2}\,r\mathrm{d}r\mathrm{d}\theta = 2\pi.$$

(c) Using polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, we have

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and hence

$$\mathrm{d}x = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta$$
$$\mathrm{d}y = \sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta$$

and

$$\mathrm{d}x^2 = \cos^2\theta\,\mathrm{d}r^2 - 2r\sin\theta\cos\theta\,\mathrm{d}r\mathrm{d}\theta + r^2\sin^2\theta\,\mathrm{d}\theta^2$$
$$\mathrm{d}x\mathrm{d}y = \sin\theta\cos\theta\,\mathrm{d}r^2 + r(\cos^2\theta - \sin^2\theta)\,\mathrm{d}r\mathrm{d}\theta - r^2\sin\theta\cos\theta\,\mathrm{d}\theta^2$$
$$\mathrm{d}y^2 = \sin^2\theta\,\mathrm{d}r^2 + 2r\sin\theta\cos\theta\,\mathrm{d}r\mathrm{d}\theta + r^2\cos^2\theta\,\mathrm{d}\theta^2$$

Substituting the above into the expression for $\mathrm{g}^{(\mathrm{b})}$ we find

$$\frac{4(\mathrm{d}x^2 + \mathrm{d}y^2)}{(1+x^2+y^2)^2} = \frac{4}{(1+r^2)^2}(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2) = \left(\frac{2}{1+r^2}\right)^2\mathrm{d}r^2 + \left(\frac{2r}{1+r^2}\right)^2\mathrm{d}\theta^2.$$

To reparametrize the radial coordinate r by arclength, we introduce

$$\rho(r) = \int_0^r \frac{2}{1+t^2}\,\mathrm{d}t = 2\arctan r$$

so that $\mathrm{d}\rho = \frac{2}{1+r^2}\mathrm{d}r$ and hence $\mathrm{d}\rho^2 = \left(\frac{2}{1+r^2}\right)^2\mathrm{d}r^2$. Since $r = \tan\frac{\rho}{2}$, we find

$$\left(\frac{2}{1+r^2}\right)^2\mathrm{d}r^2 + \left(\frac{2r}{1+r^2}\right)^2\mathrm{d}\theta^2 = \mathrm{d}\rho^2 + \left(\frac{2\tan\frac{\rho}{2}}{1+\tan^2\frac{\rho}{2}}\right)^2\mathrm{d}\theta^2 = \mathrm{d}\rho^2 + \sin^2\rho\,\mathrm{d}\theta^2,$$

which is $\mathrm{g}^{(\mathrm{c})}$, as desired. Note that $0 < r < 1$ corresponds to $0 < \rho < \frac{\pi}{2}$. Finally, the volume form of the above metric is

$$\mathrm{vol}_{\mathrm{g}^{(\mathrm{c})}} = \sin\rho\,\mathrm{d}\rho\mathrm{d}\theta,$$

from which we compute

$$\mathrm{Vol}(U, \mathrm{g}^{(\mathrm{c})}) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\rho\,\mathrm{d}\rho\mathrm{d}\theta = 2\pi.$$

X. (Will not be graded) The metric tensors $\mathrm{g}^{(\mathrm{a})}$, $\mathrm{g}^{(\mathrm{b})}$, and $\mathrm{g}^{(\mathrm{c})}$ from the previous exercise are not *equal* to one another, but you have plenty of reason to suspect they are *isometric* to one another. In fact, $\mathrm{g}^{(\mathrm{b})}$ and $\mathrm{g}^{(\mathrm{c})}$ are isometric by construction, but it remains unclear (at this moment) why they are also isometric to $\mathrm{g}^{(\mathrm{a})}$. Try to find an explicit diffeomorphism ϕ of the unit ball in $\mathbb{R}^2$ such that $\phi^*(\mathrm{g}^{(\mathrm{a})})$ is equal to either $\mathrm{g}^{(\mathrm{b})}$ or $\mathrm{g}^{(\mathrm{c})}$.

Owing to spherical coordinates in $\mathbb{R}^3$ and some geometric intuition, namely the fact that ρ in $\mathrm{g}^{(\mathrm{c})}$ is the distance to the north pole, we are led to consider the diffeomorphism

$$\begin{gathered}\phi\colon \left(B^{(\mathrm{c})}, \mathrm{g}^{(\mathrm{c})}\right) \to \left(B^{(\mathrm{a})}, \mathrm{g}^{(\mathrm{a})}\right)\\ \phi(\rho,\theta) = (\cos\theta\sin\rho, \sin\theta\sin\rho)\end{gathered}$$

where, to be very precise, $B^{(\mathrm{a})} = \{(x,y)\in\mathbb{R}^2 : x^2+y^2<1\}\setminus\{(x,0): 0\le x<1\}$ and $B^{(\mathrm{c})} = \{(\rho,\theta): 0<\rho<\frac{\pi}{2}, 0<\theta<2\pi\}$. (Generally, one pretends ϕ is defined globally.)

Let us check that $\phi^*(\mathrm{g}^{(\mathrm{a})}) = \mathrm{g}^{(\mathrm{c})}$. Setting $(x,y)=\phi(\rho,\theta)$, that is,

$$\begin{aligned} x &= \cos\theta\sin\rho\\ y &= \sin\theta\sin\rho \end{aligned} \tag{1}$$

we have

$$\begin{aligned} \phi^*\mathrm{d}x &= \cos\theta\cos\rho\,\mathrm{d}\rho - \sin\theta\sin\rho\,\mathrm{d}\theta\\ \phi^*\mathrm{d}y &= \sin\theta\cos\rho\,\mathrm{d}\rho + \cos\theta\sin\rho\,\mathrm{d}\theta \end{aligned}$$

and hence

$$\begin{aligned} \phi^*\mathrm{d}x^2 &= \cos^2\theta\cos^2\rho\,\mathrm{d}\rho^2 - 2\cos\theta\cos\rho\sin\theta\sin\rho\,\mathrm{d}\rho\mathrm{d}\theta + \sin^2\theta\sin^2\rho\,\mathrm{d}\theta^2\\ \phi^*\mathrm{d}x\,\phi^*\mathrm{d}y &= \cos\theta\sin\theta\cos^2\rho\,\mathrm{d}\rho^2 + (\cos^2\theta-\sin^2\theta)\cos\rho\sin\rho\,\mathrm{d}\rho\mathrm{d}\theta\\ &\quad - \sin\theta\cos\theta\sin^2\rho\,\mathrm{d}\theta^2\\ \phi^*\mathrm{d}y^2 &= \sin^2\theta\cos^2\rho\,\mathrm{d}\rho^2 + +2\sin\theta\cos\rho\cos\theta\sin\rho\,\mathrm{d}\rho\mathrm{d}\theta + \cos^2\theta\sin^2\rho\,\mathrm{d}\theta^2. \end{aligned} \tag{2}$$

Replacing (1) in the first step below, and then (2) in the last step below (and patiently simplifying the result a lot),

$$\begin{aligned} \phi^*(\mathrm{g}^{(\mathrm{a})}) &= \phi^*\left(\left(1+\frac{x^2}{1-x^2-y^2}\right)\mathrm{d}x^2 + \frac{2xy}{1-x^2-y^2}\,\mathrm{d}x\mathrm{d}y + \left(1+\frac{y^2}{1-x^2-y^2}\right)\mathrm{d}y^2\right)\\ &= \left(1+\frac{\cos^2\theta\sin^2\rho}{\cos^2\rho}\right)\phi^*\mathrm{d}x^2 + \frac{2\cos\theta\sin\theta\sin^2\rho}{\cos^2\rho}\phi^*\mathrm{d}x\,\phi^*\mathrm{d}y\\ &\quad + \left(1+\frac{\sin^2\theta\sin^2\rho}{\cos^2\rho}\right)\phi^*\mathrm{d}y^2\\ &= \mathrm{d}\rho^2 + \sin^2\rho\,\mathrm{d}\theta^2, \end{aligned}$$

so we obtain the desired conclusion $\phi^*(\mathrm{g}^{(\mathrm{a})}) = \mathrm{g}^{(\mathrm{c})}$. (To make computations more concise, usually one omits the symbol "ϕ^*" in intermediate steps, e.g., in the left-hand side of (2), simply writing $\mathrm{d}x = \cos\theta\cos\rho\,\mathrm{d}\rho - \sin\theta\sin\rho\,\mathrm{d}\theta$ instead of $\phi^*\mathrm{d}x = \dots$.)