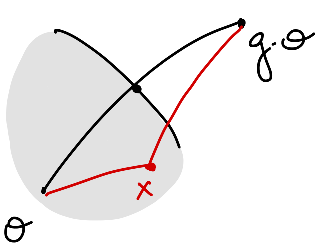
• Recall Toponogov triangle comparison (Triangle & Hinge) and comments on proof.

Preliminaries: $\Gamma = \pi_1 M$ acts freely on the universal covering $\tilde{M}$, and $\tilde{M}/\Gamma \underset{\text{diffeo}}{\cong} M$. Lifting a Riem. metric g from M, the projection map $p:(\tilde{M},\tilde{g}) \to (M,g)$ becomes a local isometry, and the action of Γ on $\tilde{M}$ is isometric, so $\tilde{M}/\Gamma \underset{\text{isom}}{\cong} M$.

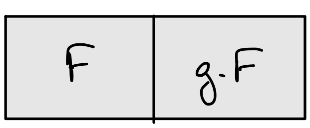
Def: A fundamental domain for the action $\Gamma \curvearrowright \tilde{M}$ is a subset $F \subset \tilde{M}$ s.t. $\Gamma \cdot F = \tilde{M}$ and $\operatorname{int}(g\cdot F) \cap \operatorname{int}(F) = \emptyset$, $\forall g \in \Gamma, g \neq e$.

Fix $F \subset \tilde{M}$ a fund. domain for the action $\Gamma \curvearrowright \tilde{M}$, e.g., fix $o \in \tilde{M}$ and take $F = \bigcap_{g \in \Gamma} \{x \in \tilde{M} : \operatorname{dist}(o,x) \leq \operatorname{dist}(o, g\cdot x)\}$ ← "Dirichlet Fundamental domain"



Exercise: Verify that F is a fund. domain.

Def: $g \in \Gamma$ is small if $g\cdot F \cap F \neq \emptyset$.



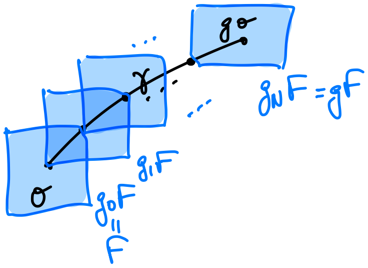
← $F \cap g.F \neq \emptyset \iff F$ and $g.F$ are adjacent (but their interiors are disjoint!)

Prop: If M is compact, then $\pi_1 M$ is finitely generated.

Pf. By the triangle inequality, $g\cdot F \subset B_{2\operatorname{diam}(F)}(o)$ for any small $g \in \Gamma$. Since $\{g\cdot(\operatorname{int} F)\}_{g\in\Gamma}$ are disjoint and have equal volume, only finitely many can fit inside $B_{2\operatorname{diam}(F)}(o)$, so there are only finitely many small elements $g \in \Gamma$.

Claim: Γ is generated by small elements.

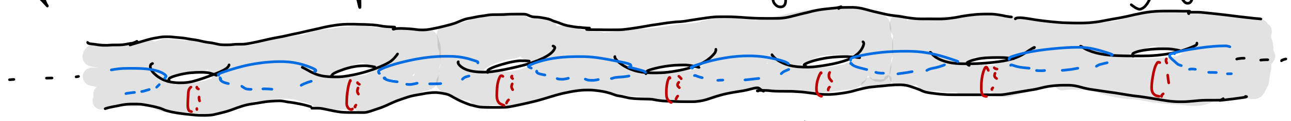
Indeed, given $g \in \Gamma$, choose a minimal geodesic γ from o to $g\cdot o$.



Then γ is covered by finitely many fundamental domains $g_0 F = F, g_1 F, g_2 F, \ldots, g_N F = gF$ s.t. $g_i F$ and $g_{i+1} F$ are adjacent, i.e., $g_i^{-1} g_{i+1} \in \Gamma$ is small. Thus, we have that

$$g = \underbrace{g_0^{-1} g_1}_{\text{small}} \underbrace{g_1^{-1} g_2}_{\text{small}} \underbrace{g_2^{-1} \cdots}_{} \cdots \underbrace{g_{N-1}}_{} \underbrace{g_{N-1}^{-1} g_N}_{\text{small}}$$ is a product of small elements. $\square$

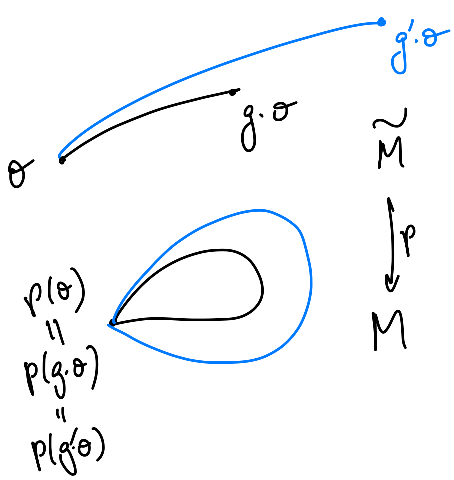
Rmk: If M is noncompact, then $\pi_1 M$ might not be finitely generated...



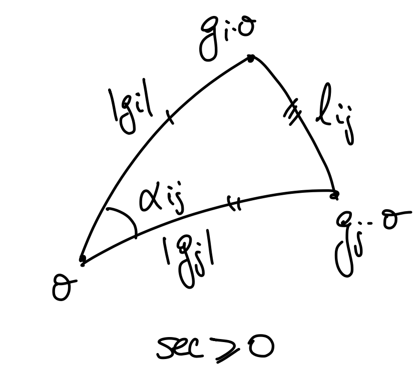
but these manifolds do not have metrics with $\sec \geq 0$:

Thm (Gromov 1978). If (M^n, g) has $\sec \geq 0$, then $\pi_1 M$ can be generated by $\leq \sqrt{2n\pi} \cdot 2^{n-2}$ elements. If (M^n, g) has $\sec \geq -K^2$ and $\operatorname{diam}(M) \leq D$, then $\pi_1 M$ can be generated by $\leq \frac{1}{2}\sqrt{2n\pi}\,\big(2 + 2\cosh(2KD)\big)^{\frac{n-1}{2}}$ $\leftarrow$ Note: If $K \to 0$, then this becomes $\sqrt{2n\pi}\cdot 2^{n-2}$.

Pf: (Case $K=0$). Fix $o \in \tilde M$ and consider the isometric action of $\Gamma = \pi_1 M$, by deck transformations (see preliminaries). Define displacement of $g \in \Gamma$: $|g| = \operatorname{dist}(o, g\cdot o)$. Clearly, a min. geod. from o to $g \cdot o$ in $\tilde M$ projects to geodesic loop based at $p(o) \in M$, which has minimal length in its homotopy class. For any given $R > 0$, there are only finitely many $g \in \Gamma$ with $|g| \leq R$, because otherwise an infinite seq. $g_i \in \Gamma$ with $|g_i| \leq R$ would produce an infinite seq. $g_i \cdot o$ of points in $B_R(o)$, which has a limit and contradicts the covering property.



Thus, we can define $g_1 \in \Gamma$ s.t. $|g_1| = \min_{g \in \Gamma} |g|$, and $g_2 \in \Gamma$ with $|g_2| = \min_{g \in \Gamma \setminus \langle g_1 \rangle} |g|$; inductively, define a sequence $g_1, g_2, \ldots \in \Gamma$ of generators ← "short basis" with $|g_1| \leq |g_2| \leq \ldots$ and $|g_{i+1}| = \min_{g \in \Gamma \setminus \langle g_1, \ldots, g_i \rangle} |g|$. (Keep adding elements g_i until a set of generators is achieved!)

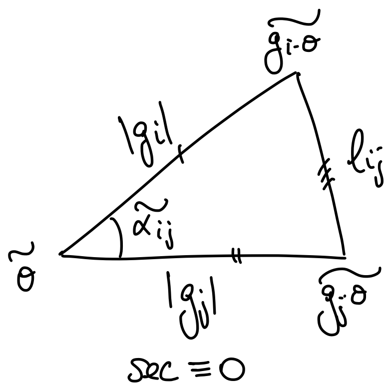


Set $l_{ij} = \mathrm{dist}(g_i\cdot o, g_j\cdot o)$ for all $i<j$. Then

$l_{ij} \geqslant |g_j|$, for otherwise $\bar{g} = g_i^{-1}\cdot g_j$ would have

$|\bar{g}| = l_{ij} < |g_j|$ and $\langle g_1, \dots g_i, \dots, g_j\rangle = \langle g_1,\dots,g_i,\dots,\bar{g}\rangle$

hence contradict the min. choice of g_j above.



Note that all sides of the triangles $o, g_i\cdot o, g_j\cdot o$ are min geodesics.

By Toponogov, applied to the hinge based at $g_i\cdot o$, we have that $\alpha_{ij} \geqslant \tilde{\alpha}_{ij}$.

Law of cosines in $\mathbb{R}^2$:

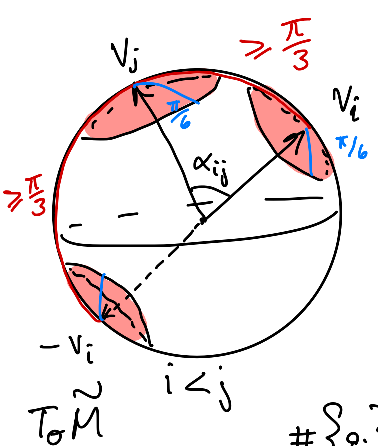
$$l_{ij}^2 = |g_i|^2 + |g_j|^2 - 2|g_i||g_j| \cos\tilde{\alpha}_{ij}$$

$|g_i| \leq |g_j| \leq l_{ij}$ if $i<j$

$$\Longrightarrow \quad \cos(\tilde{\alpha}_{ij}) = \frac{|g_i|^2 + |g_j|^2 - l_{ij}^2}{2|g_i|\cdot|g_j|} \leqslant \frac{|g_i|^2 + (|g_j|^2 - |g_j|^2)}{2\cdot |g_i|^2} = \frac{1}{2}$$

$$\Longrightarrow \quad \alpha_{ij} \geqslant \tilde{\alpha}_{ij} \geqslant \frac{\pi}{3}.$$

Let $v_i \in T_o\tilde{M}$ be the unit vector tangent to the min. geod. from o to $g_i\cdot o$. By the above, the distance (on the unit sphere in $T_o\tilde{M}$) between v_i and v_j is $\alpha_{ij} \geqslant \frac{\pi}{3}$, so the balls of radius $\frac{\pi}{6}$ centered at v_i and v_j must be disjoint. (This already proves there can be only finitely many v_i's, hence finitely many g_i's so $\Gamma = \pi_1 M$ is finitely generated.) Moreover, as $|g_i^{-1}| = |g_i|$, we must also have that distance from $-v_i$ to v_j is $\geqslant \frac{\pi}{3}$ if $i<j$, therefore the number of v_i's is:



$$\#\{g_i\} = \#\{v_i\} \leq \frac{\mathrm{Vol}(\mathbb{RP}^{n-1}(1))}{\mathrm{Vol}(B_{\pi/6}^{S^{n-1}}(v))}$$

Volume of the set of $\pm v \in S^{n-1} \subset T_o\tilde{M}$.

Volume of each disjoint ball around $\pm v_i \in S^{n-1}$.

Volume of spherical ball of radius r is $\geq$ volume of Euclidean ball of radius $\sin r$.

Standard computations give:

- $\mathrm{Vol}\left(B^{S^{n-1}}_{\pi/6}(v)\right) \geq \mathrm{Vol}\left(B^{\mathbb{R}^{n-1}}_{\underbrace{\sin \pi/6}_{=1/2}}(0)\right) = \dfrac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right) 2^{n-1}}$

(Γ = Gamma function)

log-concavity of Γ: $\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \leq \sqrt{\frac{n}{2}}$

- $\mathrm{Vol}\left(\mathbb{RP}^{n-1}(1)\right) = \frac{1}{2}\mathrm{Vol}\left(S^{n-1}(1)\right) = \dfrac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$

So
$$\#\{g_i\} = \#\{v_i\} \leq \frac{\pi^{n/2}\Gamma\left(\frac{n+1}{2}\right)2^{n-1}}{\Gamma\left(\frac{n}{2}\right)\cdot \pi^{\frac{n-1}{2}}} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\,2^{n-1} \leq \sqrt{2n\pi}\cdot 2^{n-2}.$$

For case $\sec \geq -\kappa^2$, adapt the argument above w/ Law of Cosines in the comparison space of constant curvature $-\kappa^2$:

$|g_i| \leq |g_j| \leq \ell_{ij}$ if $i<j$

$$\cos(\tilde{\alpha}_{ij}) = \frac{\cosh(\kappa|g_i|)\cosh(\kappa|g_j|) - \cosh(\kappa \ell_{ij})}{\sinh(\kappa|g_i|)\sinh(\kappa|g_j|)} \leq \frac{\cosh^2(\kappa|g_j|) - \cosh(\kappa|g_j|)}{\sinh^2(\kappa|g_j|)}$$

Need diameter bound here!

$$= \frac{\cosh(\kappa|g_j|)}{\cosh(\kappa|g_j|)+1} \leq \frac{\cosh(2\kappa D)}{\cosh(2\kappa D)+1}.$$

Thus, by Toponogov, $\alpha_{ij} \geq \tilde{\alpha}_{ij} \geq \arccos\left(\dfrac{\cosh(2\kappa D)}{\cosh(2\kappa D)+1}\right)$.

Estimate volume of spherical ball of the above radius (from below) by volume of Euclidean ball of radius $\sin\left[\frac{1}{2}\arccos\left(\frac{\cosh(2\kappa D)}{\cosh(2\kappa D)+1}\right)\right]$ to get estimate on $\#\{g_i\}$.

$\square$

<u>Rmk</u>: Bounds above are <u>never</u> sharp.

<u>Rmk</u>: Σ^2_g = orientable hyperbolic surface of genus g. Then $\sec \equiv -1$ and $\pi_1(\Sigma^2_g) = \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid [a_1,b_1]\cdots[a_g,b_g]=1\rangle$ has $2g$ generators. As $g \nearrow +\infty$, then keeping $\sec \equiv -1$ forces $\mathrm{diam} \nearrow +\infty$. (or if $\mathrm{diam} \equiv D$ then $\sec \equiv -\kappa^2 \searrow -\infty$)

Recent developments surrounding the above:

- If Γ is finitely generated, fix a finite generating set G, with $e \in G$ and $G^{-1} = G$. Then define <u>growth function</u> for Γ:

$$N_k^G = \#\{ g \in \Gamma : g = g_1 \cdots g_k ,\ \text{with } g_i \in G \}$$

# of group elements that can be written as product of k generators in the fixed generating set G.

- If G' is another choice of generating set for Γ as above, then $N_k^{G'} \geqslant N_{Ck}^{G}$ and $N_k^{G} \geqslant N_{Dk}^{G'}$ for some constants $C, D > 0$, so can ignore choice of gen. set G for questions below.

- <u>Q</u>: How does N_k grow with k? Polynomially? Exponentially?

> <u>Thm</u> (Milnor '68). If (M^n, g) is complete and has $Ric \geqslant 0$, then any finitely generated subgroup $\Gamma < \pi_1 M$ has $\underline{N_k \leq C \cdot k^n}$.

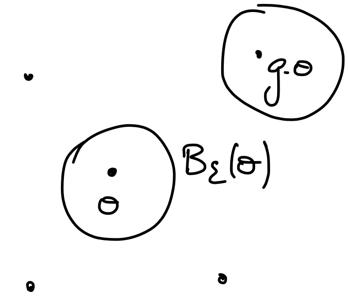
i.e., "polynomial growth"

<u>Pf</u>: Choose $o \in \widetilde{M}^n$, and let $V(r) = Vol(B_r(o))$. By Bishop Volume Comp.,

$$V(r) \leq Vol(B_r^{\mathbb{R}^n}(0)) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} r^n.$$ Let $G = \{g_1, \dots, g_p\}$ be the fixed generating set for $\Gamma < \pi_1 M$ and $\mu = \max_{1 \leq i \leq p} dist(o, g_i \cdot o)$.

Then $B_{\mu \cdot k}(o)$ has at least N_k^G distinct points of the form $g \cdot o$, with $g \in \Gamma$. Choose $\varepsilon > 0$ s.t. $g \cdot B_\varepsilon(o) \cap B_\varepsilon(o) = \phi$ if $g \neq e$. Then $B_{\mu \cdot k + \varepsilon}(o)$ has at least N_k^G disjoint subsets of the form $g \cdot B_\varepsilon(o)$, so

$$\mathrm{Vol}\Big(\coprod_{\substack{g=g_1\cdots g_k\\ g_i\in G}} g\cdot B_\varepsilon(o)\Big) = N_k^G\cdot V(\varepsilon) \leq V(\mu k+\varepsilon)$$



Thus $$N_k^G \leq \frac{V(\mu k+\varepsilon)}{V(\varepsilon)} \leq \frac{\tilde{C}\,(\mu k+\varepsilon)^n}{V(\varepsilon)} \leq C\cdot k^n. \qquad \square$$

Thm (Milnor '68). If (M,g) is a closed Riem. mfld with $\sec<0$, and $\pi_1 M=\langle G\rangle$, $|G|<\infty$, then $\underline{N_k^G \geq a^k}$ for some $a>1$.

← i.e., "exponential growth"

Ex: Fundamental group of hyperbolic manifold Σ^n has exponential growth; thus, cannot be π_1 of mfld w/ $\mathrm{Ric}\geq 0$.
So, cannot "improve" the above Thm to $\mathrm{scal}>0$, as $\Sigma^2\times S^{n-2}(\varepsilon)$ has $\mathrm{scal}>0$ for $n\geq 4$ and $\varepsilon>0$ suff. small, if Σ^2 is a hyperbolic surface.

- The following is currently still <u>open</u> for $n\geq 4$:

Conjecture (Milnor). If (M^n,g) is complete and has $\mathrm{Ric}\geq 0$, then $\pi_1 M$ is finitely generated.

- For $n=3$, it was proven by [Liu, 2013] and indep. [Pan, 2017].

↖ Inventiones paper, uses minimal surfaces

↖ Crelle paper, uses Cheeger–Colding theory and $\mathrm{Ricci}\geq 0$ limit spaces

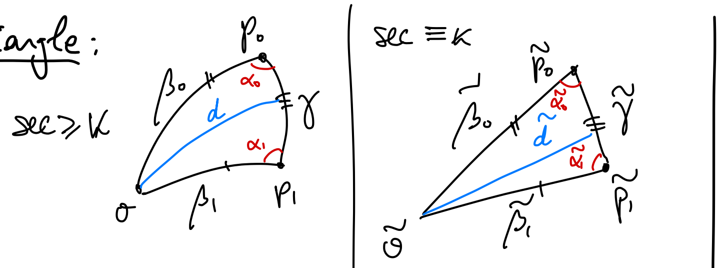
⇝ Geometric Group Theory

Thm (Gromov '81) If Γ is finitely generated and has polynomial growth, then Γ is virtually nilpotent: $\exists N \triangleleft \Gamma$ nilpotent with $[\Gamma : N]<\infty$.

Remark (Wilking 2000): If there is a counter-example M to Milnor's conj., then it has a covering space $\hat{M}\to M$ with $\pi_1\hat{M}$ abelian and not finitely generated.

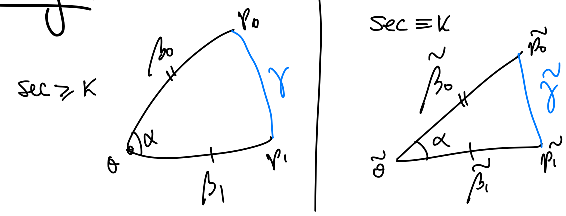
Recap: Toponogov triangle comparison for $\sec \geq K$.

Triangle:



$d \geq \tilde d$ $\quad \alpha_i \geq \tilde\alpha_i$

Hinge:



$\text{length}(\gamma) \leq \text{length}(\tilde\gamma)$ $\qquad$ rigidity...

Thm (Gromov '78). If (M^n, g) is a (possibly noncompact) complete manifold with $\sec \geq 0$, then $\pi_1 M$ can be generated by $\leq \sqrt{2n\pi} \cdot 2^{n-2}$ elements.
(Similar result if $\sec \geq -\kappa^2$, $\operatorname{diam} \leq D$.)

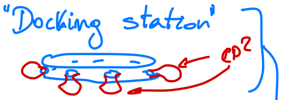
- Discuss Milnor's earlier contributions about $\pi_1 M$ if $\operatorname{Ric} \geq 0$ and growth function.

Using Bishop Volume Comparison, Toponogov Triangle Comparison, Critical point theory for distance functions ← more on this soon! and topological constructions, Gromov proved the following:

Thm (Gromov '1981).
i) If (M^n, g) is a complete mfld with $\sec \geq 0$, then $\sum_{k=0}^{n} b_k(M) \leq C(n)$.
ii) If (M^n, g) is a closed mfld with $\sec \geq -\kappa^2$ and $\operatorname{diam} \leq D$, then $\sum_{k=0}^{n} b_k(M) \leq C(n)^{1+\kappa D}$.

Cannot replace the hypothesis $\sec \geq 0$ to $\operatorname{Ric} > 0$ because:

"Docking station"



Thm (Sha-Yang '90s). $\forall \ell \in \mathbb{N}$, $\#^\ell S^2 \times S^2$ and $\#^k \mathbb{CP}^2 \#^\ell \overline{\mathbb{CP}^2}$ have $\operatorname{Ric} > 0$.

← also $\#^\ell S^n \times S^m$ for any $n, m \geq 2$, $\ell \geq 1$.

Thm. (Perelman '97). $\forall \ell \in \mathbb{N}$, $\#^\ell \mathbb{CP}^2$ has a metric with $\operatorname{Ric} > 0$, $\operatorname{diam} = 1$ and $\operatorname{Vol} \geq V > 0$.

Thus, since $b_2(\#^\ell S^2 \times S^2) = 2\ell$ and $b_2(\#^k \mathbb{CP}^2 \#^\ell \overline{\mathbb{CP}^2}) = k + \ell$, only finitely many of these manifolds can have $\sec \geq 0$. Currently,
- only S^4 and $\mathbb{CP}^2$ are known to have $\sec > 0$ and
- only $S^2 \times S^2$ and $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ are known to have $\sec \geq 0$.

Conjecturally, the above is the complete list of simply-connected 4-mflds with $\sec > 0$ and $\sec \geq 0$.

Note: $\operatorname{scal} > 0$ is preserved by $\#$; indeed by surgeries of codimension ≥ 3.

⚠ Related open question: is there a simply-connected closed mfld that admits $\operatorname{scal} > 0$ but does not admit $\operatorname{Ric} > 0$?

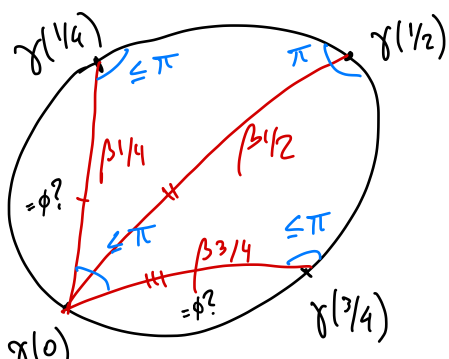
(double disk:)

Note: As $\ell \nearrow +\infty$, Perelman's $\#^\ell \mathbb{CP}^2$ converges to $B^4 \cup B^4$ flat

An application of Toponogov to closed geodesics in surfaces:

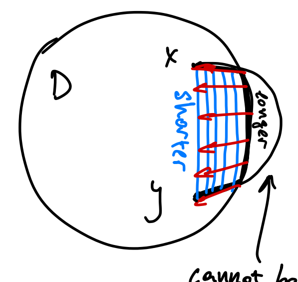
Thm (Toponogov). If (M^2, g) is a closed oriented surface with $\sec \geq K$ and γ is a simple closed geodesic, then $\text{length}(\gamma) \leq \frac{2\pi}{\sqrt{K}}$. Moreover, if $\text{length}(\gamma) = \frac{2\pi}{\sqrt{K}}$ then (M^2, g) is isometric to the round sphere $S^2(\frac{1}{\sqrt{K}})$.

Pf: Cut (M^2, g) along γ to obtain a disk D with geodesic boundary. (Note that γ bounds a disk because, by Gauss–Bonnet, $M^2 \underset{\text{homeo}}{\cong} S^2$.)
Assume $K = 1$, general case is obtained by rescaling.



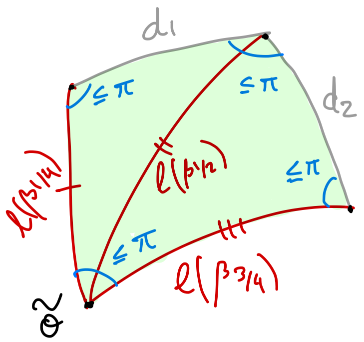
$\sec \geq 1$

Suppose $\gamma : [0,1] \to M$, so $\gamma(0) = \gamma(1)$ and $\dot\gamma(0) = \dot\gamma(1)$.
Since $\sec \geq 0$, the disk $D \subset M$ with $\partial D = \gamma$ is <u>convex</u>: min. geod. between $x, y \in D$ are contained in D. Indeed, if not, then can produce a variation of geodesics by application of Rauch II by "pushing inward" and these would have <u>shorter length</u>, contradicting minimality of the geodesic that departs from D.



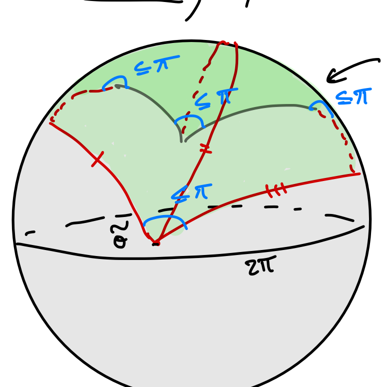
cannot be minimizing if $\sec \geq 0$

Let $\beta^{1/2}$ be min. geod. from $\gamma(0)$ to $\gamma(1/2)$, which is entirely contained in D. Since $\beta^{1/2}$ is minimizing, $\text{length}(\beta_{1/2}) \leq \pi$ by Myers. If $\text{length}(\beta_{1/2}) \geq \frac{1}{2}\text{length}(\gamma)$, then $\text{length}(\gamma) \leq 2\pi$ so we are done. If not, then let $\beta^{1/4}$ and $\beta^{3/4}$ be min. geod. from $\gamma(0)$ to $\gamma(1/4)$ and $\gamma(3/4)$, these are also entirely in D. By Toponogov (Hinge) applied to the hinges at $\gamma(0)$ with sides $\beta^{1/4}, \beta^{1/2}$ and $\beta^{1/2}, \beta^{3/4}$, we get a comparison quadrangle in $S^2(1)$ which has all internal angles $\leq \pi$ and is therefore <u>convex</u>, w/ side lengths:



$\sec \equiv 1$

- $\text{length}(\beta^{1/4})$,
- $d_1 \geq \text{length}(\gamma|_{[1/4, 1/2]})$
- $d_2 \geq \text{length}(\gamma|_{[1/2, 3/4]})$
- $\text{length}(\beta^{3/4})$



$\sec \equiv 1$

convex quadrangle must be contained in a hemisphere! Thus has perimeter $\leq 2\pi$.

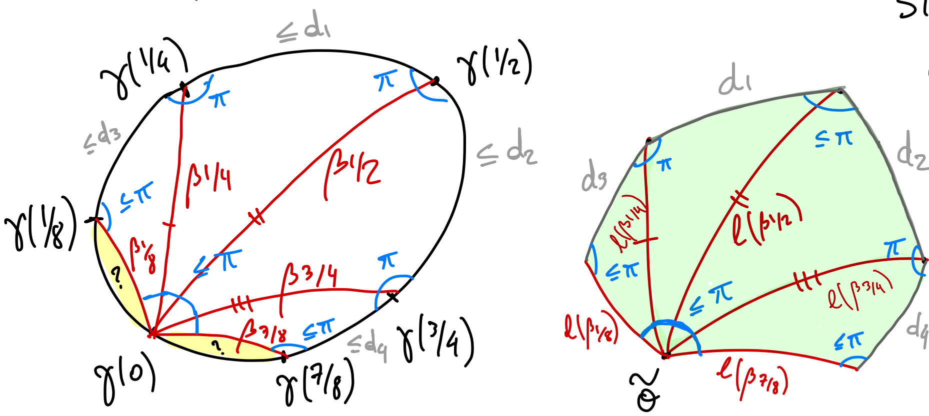
Since this quadrangle is convex, it must be contained in a hemisphere of $\mathbb{S}^2(1)$, thus its perimeter is $\overset{(*)}{\leq} 2\pi$; hence

$$\mathrm{length}(\beta_{1/4}) + \mathrm{length}(\gamma|_{[1/4,1/2]}) + \mathrm{length}(\gamma|_{[1/2,3/4]}) + \mathrm{length}(\beta_{3/4}) \leq$$

$$\text{Toponogov} \nearrow \leq \mathrm{length}(\beta_{1/4}) + d_1 + d_2 + \mathrm{length}(\beta_{3/4}) \overset{(*)}{\leq} 2\pi.$$

If $\beta_{1/4} = \gamma|_{[0,1/4]}$ and $\beta_{3/4} = \gamma|_{[3/4,1]}$, then the above proves $\mathrm{length}(\gamma) \leq 2\pi$. If, however, $\gamma|_{[0,1/4]}$ or $\gamma|_{[3/4,1]}$ are not min. then we further subdivide, see picture. Once again, we obtain a comparison polygon (hexagon) in $\mathbb{S}^2$ which is convex by Toponogov, since internal angles are $\leq \pi$ and has perimeter $\leq 2\pi$ b/c it is convex hence contained in a hemisphere of $\mathbb{S}^2(1)$. If $\beta_{1/8} = \gamma|_{[0,1/8]}$ and $\beta_{7/8} = \gamma|_{[7/8,1]}$, then we are done, since

$$\mathrm{length}(\gamma) \leq \begin{pmatrix}\text{perimeter of}\\ \text{comparison}\\ \text{hexagon}\end{pmatrix} \leq 2\pi$$



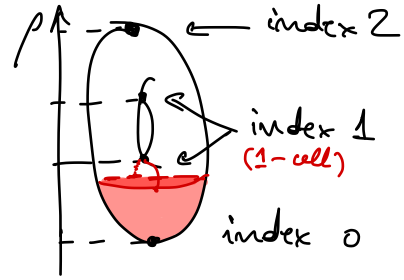
sec ≥ 1 $\qquad$ sec ≡ 1

If not, keep subdividing. Eventually, the min. geod. $\beta_{1/2^n}$ and $\beta_{1-1/2^n}$ will agree with $\gamma|_{[0,1/2^n]}$ and $\gamma|_{[1-1/2^n,1]}$ and then we
ie. region labelled "?" is empty
will have $\mathrm{length}(\gamma) \leq$ perimeter of comparison polygon $\leq 2\pi$ by Toponogov and convexity of the comparison polygon, which, itself, also follows from Toponogov (comparison angles are $\leq$ mfld angles $\leq \pi$).
The rigidity statement in the equality case follows from rigidity in Toponogov (we did not discuss this) applied to the disk $D \subset M$ and then to $M \setminus D$. $\square$

Rmk: This result has been reproven recently with PDE techniques, in a way that allows to show stability of the conclusion under Gromov-Hausdorff & Intrinsic Flat convergence (see paper of Hunter Stufflebeam)

Final application of Toponogov: critical point theory for distance functions

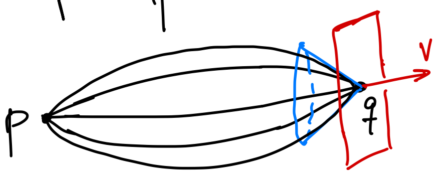
<u>Goal:</u> Apply Morse theory to $\rho(x) = \mathrm{dist}(x,o)$:



$$\begin{pmatrix}\text{critical points of}\\ \rho : M \to \mathbb{R}\end{pmatrix} \leftrightsquigarrow \big(\text{Topology of } M\big)$$

<u>Problem:</u> ρ is not smooth, $\nabla\rho$ does not have actual flow...

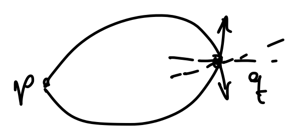
<u>Def:</u> A point $q \in M$ is <u>regular</u> for $\rho(x) = \mathrm{dist}(x,p)$ if $\exists\, v \in T_qM$ s.t. all min. geod. from p to q make an angle $> \frac{\pi}{2}$ with v.



v is called "gradient-like"

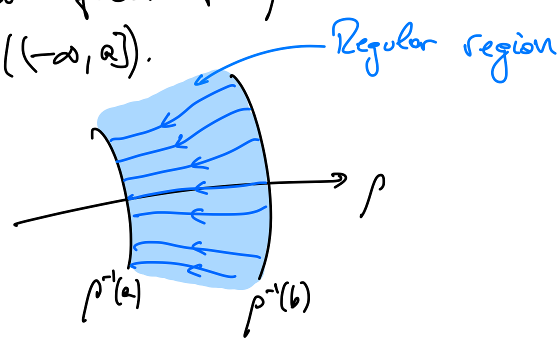
(moving in direction v takes q farther away from p!)

i.e. if $\gamma : [0,L] \to M$ is min. geod from q to p, then $\langle \dot\gamma(0), v\rangle < 0$. The point q is called <u>critical</u> if it is not regular; i.e., if there does not exist a direction v to move farther away from p.



cannot move farther away!

<u>Lemma.</u> Regular points for $\rho(x)$ form an open subset (cf. Sard's Theorem). Within any region between sublevelsets $\rho^{-1}([a,b])$ without critical points for ρ, can define a gradient-like vector field for ρ, which gives an isotopy from $\rho^{-1}((-\infty,b])$ to $\rho^{-1}((-\infty,a])$.



<u>Rmk:</u> No analogue to <u>Morse Lemma</u>, which says how to "build" $\rho^{-1}((-\infty,b])$ from $\rho^{-1}((-\infty,a])$ if there is a critical point of ρ in $\rho^{-1}((a,b))$; namely attach a cell of dimension given by the index of the critical point.

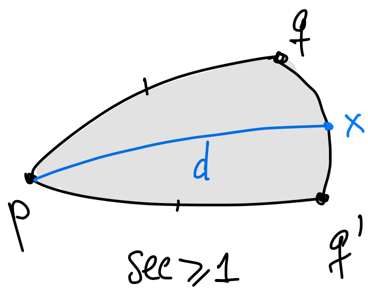
<u>Thm</u> (Grove–Shiohama '77). If (M^n,g) is a Riem. mfld. with $\sec \geq K > 0$ and $\mathrm{diam}(M,g) > \frac{1}{2}\mathrm{diam}\big(S^n(1/\sqrt{K})\big) = \dfrac{\pi}{2\sqrt{K}}$, then $M^n \underset{\text{homeo}}{\cong} S^n$.

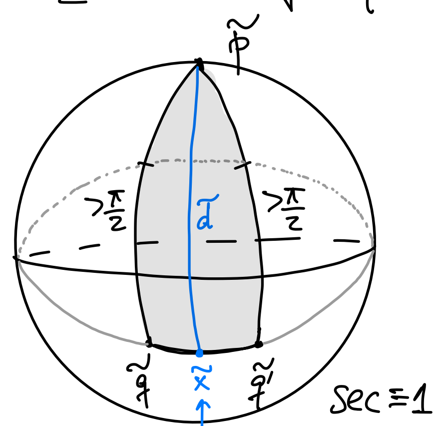
(Sharp b/c of $\mathbb{RP}^n(1/\sqrt{K})$.)

<u>Pf:</u> Up to rescaling, assume $K=1$, and let $p,q \in M$ be points s.t. $\mathrm{dist}(p,q) = \mathrm{diam}(M^n,g)$ realize the diameter of M.

Claim 1. If $q' \in M$ is s.t. $\mathrm{dist}(p,q') = \mathrm{diam}(M^n,g) > \frac{\pi}{2}$, then $q'=q$.

Pf: By Toponogov Triangle comp., if $q, q' \in M$ satisfy $\mathrm{dist}(p,q) = \mathrm{dist}(p,q') = \mathrm{diam}(M,g) > \frac{\pi}{2}$ then the distance from p to any x in the min. geod. joining q to q' would exceed the diameter of (M^n,g), thus $q = q'$.





$d \geq \tilde{d} > \mathrm{diam}(M,g)$

$\tilde{d} = \mathrm{dist}(\tilde{x},\tilde{p}) > \mathrm{dist}(\tilde{p},\tilde{q})$

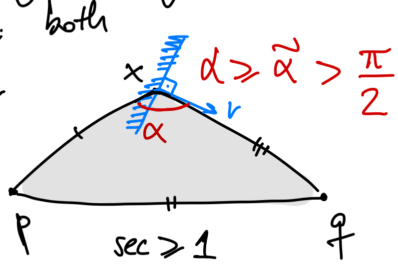
Claim 2. If $x \neq p$ and $x \neq q$, then x is regular for $\rho(x) = \mathrm{dist}(x,p)$.

Pf. If $x \neq p$ and $x \neq q$, then join x to p and to q by min. geod. of length ℓ_2 and ℓ; both are $\leq \ell_1 = \mathrm{diam}(M,g)$.

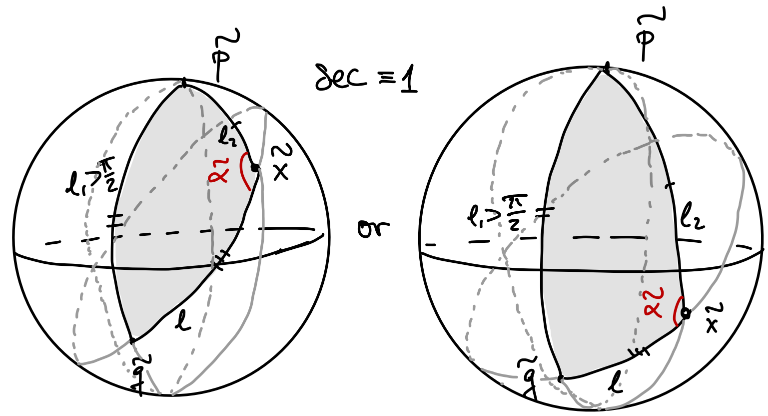
Since $\ell_1 > \frac{\pi}{2}$, the comparison angle $\tilde{\alpha} > \frac{\pi}{2}$ by spherical trigonometry, so $\alpha \geq \tilde{\alpha} > \frac{\pi}{2}$ by Toponogov.

Thus the velocity vector of the min. geod. from x to q proves x is regular.

what if multiple min. geod. from p to q?



$\alpha \geq \tilde{\alpha} > \frac{\pi}{2}$



$\tilde{\alpha} > \frac{\pi}{2}$ by law of cosines on S^2:

$$\cos\tilde{\alpha} = \frac{\cos \ell_1 - \cos \ell \cdot \cos \ell_2}{\sin \ell \cdot \sin \ell_2} < 0$$

By Lemma, can build a gradient-like vector field on $M \setminus B_\varepsilon(p) \cup B_\varepsilon(q)$ which is never zero, so $M \setminus (B_\varepsilon(p) \cup B_\varepsilon(q)) \underset{\text{homeo}}{\cong} [a,b] \times S^{n-1}$ and hence $M^n \underset{\text{homeo}}{\cong} S^n$. $\square$

Rmk: It is not known if such M is diffeomorphic to S^n, since above proof only shows it is a "twisted sphere". Currently, no exotic sphere is known to have sec>0. However, Gromoll-Meyer sphere Σ^7 has $\sec \geq 0$; actually $\sec > 0$ on open dense subset.

Cor. If (M^n,g) has $\sec \geq \kappa > 0$ and $\mathrm{Vol}(M,g) > \frac{1}{2}\mathrm{Vol}(S^n(1/\sqrt{\kappa}))$, then $M \underset{\text{homeo}}{\cong} S^n$.

Pf: Exercise (Bishop Volume Comparison!)

Thm (Perelman '94). $\forall n \geq 2$, $\exists \delta_n > 0$ s.t. if (M^n, g) has $\mathrm{Ric} \geq (n-1)g$ and $\mathrm{Vol}(M,g) \geq (1-\delta_n) \cdot \mathrm{Vol}(S^n(1))$, then $M^n \underset{\text{homeo}}{\cong} S^n$.

As mentioned before, the following is open:

Conjecture. If (M^n, g) has $\mathrm{Ric} \geq (n-1) \cdot g$ and $\mathrm{Vol}(M,g) > \frac{1}{2} \mathrm{Vol}(S^n(1))$, then (M^n, g) is homeomorphic? diffeomorphic? to S^n.

A trivial step towards it is to show that such (M^n, g) is simply-connected (Exercise we discussed earlier, using Bishop Volume Comparison).

Also, there can't be an "almost maximal" diameter theorem with $\mathrm{Ric} \geq n-1$:

Thm (Anderson). $\exists$ Riem. metrics on $\mathbb{CP}^n$ with $\mathrm{Ric} \geq (n-1)g$ and $\mathrm{diam} \geq \pi - \varepsilon$.

and many other manifolds which are not spheres...

Lecture 11 — Bochner technique — 4/20/2023

Def: Given an oriented Riem. manifold (M^n, g), define the orthogonal frame bundle $Fr(TM) \to M$ (sometimes written $Fr^{SO}(TM)$) where the fiber $Fr(TM)_p = \{(e_1, \dots, e_n) \in (T_pM)^n : g(e_i, e_j) = \delta_{ij}\}$ is the set of oriented orthonormal bases of T_pM.

Note this is a principal $SO(n)$-bundle, since $SO(n)$ acts freely and transitively on the set of oriented orthonormal bases of $\mathbb{R}^n$. As a bundle, $SO(n) \to Fr(TM) \to M$

(i.e. $G \to P \to B$ where $G \curvearrowright P$ is a free transitive right-action of a Lie gp, preserving the fibers and s.t. $\forall x \in B$, $y \in P_x$, $G \ni g \mapsto y \cdot g \in P_x \cong G$ is a diffeomorphism)

Example: $(M^n, g) = (S^n, g_{\text{round}})$ then $Fr(TM) \cong SO(n+1)$, since $SO(n) \to SO(n+1) \to S^n$ and

$$SO(n+1) \ni \underbrace{\begin{pmatrix} | & | & & | \\ p & e_1 & \cdots & e_n \\ | & | & & | \end{pmatrix}}_{\text{orthonormal columns}} \longmapsto (e_1, \dots, e_n) \text{ orthonormal frame of } T_pS^n = p^\perp.$$

Ex: $SO(2) \to Fr(TS^2) \to S^2$ is equivalent to the Hopf bundle $S^1 \to \mathbb{RP}^3 \to S^2$, obtained as $\underset{\overset{\shortparallel}{S^1}}{S^1/\mathbb{Z}_2} \to \underset{\overset{\shortparallel}{\mathbb{RP}^3}}{S^3/\mathbb{Z}_2} \to S^2$.

Note: M is parallelizable (i.e. $TM = M \times \mathbb{R}^n$ is trivial) if and only if $Fr(TM)$ has a global section.

Ex: S^n is parallelizable if and only if $n=0,1,3,7$ because $\mathbb{R}^{n+1}$ is a real division algebra iff $n=0,1,3,7$.

see [Bott-Milnor] 3-page paper "On the parallelizability of the spheres". 1958.

E.g., $n=3$ $\quad S^3 \subset \mathbb{H} \cong \mathbb{R}^4 = \operatorname{span}\{1,i,j,k\}$.

If $z \in S^3$, $\quad T_z S^3 \cong z^\perp = \operatorname{span}\{iz, jz, kz\}$

indeed, $z = a+bi+cj+dk$ then $iz = ai - b + ck - dj$ so $\langle z, iz\rangle = -ab+ab-cd+cd=0$ similarly $\langle z, jz\rangle = \langle z, kz\rangle = 0$, and $\{iz, jz, kz\}$ is o.n.b. of $T_z S^3 = z^\perp \subset \mathbb{H}$.

So we get a global section: $S^3 \ni z \mapsto (iz, jz, kz) \in Fr(TS^3)$, showing that $TS^3 = S^3 \times \mathbb{R}^3$.

Associated bundle construction

(Here we only discuss the particular case where the principal bundle $G \to P \to B$ is the frame bundle $SO(n) \to Fr(TM) \to M$)

Let E be a vector space and $\pi : SO(n) \to SO(E)$ be a representation of $SO(n)$, i.e., a linear action $SO(n) \curvearrowright E$. Then we can define the associated bundle

$$E \to E_\pi \to M, \quad \text{where} \quad E_\pi := Fr(TM) \times_\pi E = Fr(TM) \times E \big/ SO(n)$$

is the quotient space of the action $SO(n) \curvearrowright Fr(TM) \times E$

$$g \cdot (f, v) = (f \cdot g, \pi(g^{-1}) v) \quad f \in Fr(TM), v \in E.$$

Examples:

Vector space E	Representation $\pi : SO(n) \to SO(E)$	Associated bundle $E_\pi := Fr(TM) \times_\pi E$
$\mathbb{R}^n$	$id : SO(n) \to SO(n)$ defining representation $SO(n) \curvearrowright \mathbb{R}^n$	$\mathbb{R}^n \to TM \to M$ tangent bundle
$(\mathbb{R}^n)^*$	$SO(n) \curvearrowright (\mathbb{R}^n)^*$ $(A \cdot \phi)(v) = \phi(Av)$	$(\mathbb{R}^n)^* \to TM^* \to M$ cotangent bundle
$\Lambda^p \mathbb{R}^n$ $0 \leq p \leq n$	$\pi = \Lambda^p id \curvearrowright \Lambda^p \mathbb{R}^n$ $A \cdot (v_1 \wedge \dots \wedge v_p) = Av_1 \wedge \dots \wedge Av_p$	$\Lambda^p \mathbb{R}^n \to \Lambda^p TM \to M$ bundle of p-vectors

$\Lambda^p(\mathbb{R}^n)^*$ $0 \le p \le n$	$\pi = \Lambda^p \mathrm{id}^* \curvearrowright \Lambda^p(\mathbb{R}^n)^*$ $A\cdot(\phi_1 \wedge \dots \wedge \phi_p)(v_1 \wedge \dots \wedge v_p) = \phi_1(Av_1)\cdots\phi_p(Av_p)$	$\Lambda^p(\mathbb{R}^n)^* \to \Lambda^p TM^* \to M$ bundle of p-forms
$\mathrm{Sym}^p \mathbb{R}^n$	$\pi = \mathrm{Sym}^p \mathrm{id} \curvearrowright \mathrm{Sym}^p \mathbb{R}^n$ $A\cdot(v_1 \vee \dots \vee v_p) = Av_1 \vee \dots \vee Av_p$	$\mathrm{Sym}^p \mathbb{R}^n \to \mathrm{Sym}^p TM \to M$
$\mathrm{Sym}^p(\mathbb{R}^n)^*$	$\pi = \mathrm{Sym}^p \mathrm{id}^* \curvearrowright \mathrm{Sym}^p(\mathbb{R}^n)^*$ $A\cdot(\phi_1 \vee \dots \vee \phi_p)(v_1, \dots, v_p) = \phi_1(Av_1)\cdots\phi_p(Av_p)$	$\mathrm{Sym}^p(\mathbb{R}^n)^* \to \mathrm{Sym}^p TM^* \to M$

Similar for Sym^p_0 (traceless)

etc etc etc,

e.g., given two such bundles E_{π_1}, E_{π_2}, we can construct

$$E_{\pi_1} \oplus E_{\pi_2} = E_{\pi_1 \oplus \pi_2} \qquad \mathrm{Sym}^p E_\pi = E_{\mathrm{Sym}^p \pi}$$

$$E_{\pi_1} \otimes E_{\pi_2} = E_{\pi_1 \otimes \pi_2} \qquad \Lambda^p E_\pi = E_{\Lambda^p \pi}$$

and iterate these, e.g., $\Lambda^2 E_{\pi_1} \oplus \mathrm{Sym}^2\left(\mathrm{Sym}^4\left(\Lambda^2 E_{\pi_1} \otimes E_{\pi_2}^{\oplus 5}\right)\right) \dots$

<u>Note</u>: Since we have a Riem. metric, we often identify $TM^* \cong TM$ hence also $\Lambda^p TM \cong \Lambda^p TM^*$, $\mathrm{Sym}^p TM \cong \mathrm{Sym}^p TM^*$, etc.
(so $\Omega^p M = \Gamma(\Lambda^p TM)$.)

<u>Laplacians</u> $\Delta : \Gamma(E_\pi) \longrightarrow \Gamma(E_\pi)$

The above bundles often have a "natural" Laplace operator; e.g.,

- $\Omega^p M$: <u>Hodge Laplacian</u> $\Delta_H = d\delta + \delta d = (d+\delta)^2$
  $0 \le p \le n$
  where $d : \Omega^p M \to \Omega^{p+1} M$ exterior derivative
  $\delta : \Omega^p M \to \Omega^{p-1} M$ codifferential $\delta = (-1)^{n(p-1)+1} * d *$

- $\mathrm{Sym}^p TM$: <u>Lichnerowicz Laplacian</u>: $\Delta_L = \overline{\nabla}^* \overline{\nabla}$
  formal L^2-adjoint of D is D^*: $\int_M \langle D^*\phi, \psi\rangle = \int_M \langle \phi, D\psi\rangle$, $\forall \phi, \psi$
  where $\overline{\nabla} : \Gamma(\mathrm{Sym}^p TM) \to \Gamma(\mathrm{Sym}^{p+1} TM)$ is the (fully) symmetrized covariant derivative.

• $\mathrm{Sym}^2(\Lambda^2 TM)$: Lichnerowicz Laplacian. $\Delta_L = \overline{\nabla}^* \overline{\nabla}$

where $\overline{\nabla} : \Gamma(\mathrm{Sym}^2(\Lambda^2 TM)) \to \Gamma(TM^* \otimes \mathrm{Sym}^2(\Lambda^2 TM))$ is a symmetrized covariant derivative s.t. if $R \in \Gamma(\mathrm{Sym}^2(\Lambda^2 TM))$, then also $\Delta_L R \in \Gamma(\mathrm{Sym}^2(\Lambda^2 TM))$.

...

Q: Why care about these Laplacians?

A: Harmonic sections are geometrically/topologically relevant:

For example:

• Hodge theory: If (M^n, g) is a closed Riem. mfld, then

$$H^p_{dR}(M^n, \mathbb{R}) \cong \{\omega \in \Omega^p M : \Delta_H \omega = 0\}.$$

(de Rham cohomology) (Harmonic p-forms)

In particular, the p^{th} Betti number is $b_p(M) = \dim \operatorname{Ker}(\Delta_H|_{\Omega^p(M)})$

• Killing tensors: Let (M^n, g) be a Riem. mfld, and $\phi_t : M \to M$ a 1-parameter subgroup of diffeomorphisms, i.e., $\phi_0 = \mathrm{id}$, $\phi_{t+s} = \phi_t \circ \phi_s$. Then

$\phi_t : (M^n, g) \to (M^n, g)$ are isometries; i.e. $\phi_t^* g = g$ $\iff$ $X \in \mathfrak{X}(M)$, $X_p = \frac{d}{dt}\phi_t(p)\big|_{t=0}$ is a Killing field: $\mathcal{L}_X g = 0$, or, equivalently *, ∇X is skew symmetric, i.e.,

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$$

for all $Y, Z \in \mathfrak{X}(M)$; i.e., $\Delta_L X = 0$.

* indeed, letting $\theta(Y) = g(X, Y)$,

$$\underbrace{(\mathcal{L}_X g)(Y,Z)}_{\text{symmetric}} = 2g(\nabla_Y X, Z) - \underbrace{d\theta(Y,Z)}_{\text{skew symmetric}}$$

so $\mathcal{L}_X g \equiv 0$ iff ∇X is skew-symmetric

In particular, $\dim \mathrm{Iso}(M^n,g) = \dim \{X \in \Gamma(TM) : \Delta_L X = 0\}$.

<u>Rmk</u>: In fact, if (M^n,g) is complete, then $\mathrm{Iso}(M,g)$ is a Lie group and $\{X \in \mathfrak{X}(M) : \mathcal{L}_X g = 0\}$ is its Lie algebra. (Note $\mathcal{L}_{[X,Y]} g = [\mathcal{L}_X, \mathcal{L}_Y] g$.)

- <u>Harmonic curvature operators</u> (M^n,g) with $R : \Lambda^2 TM \to \Lambda^2 TM$ s.t. $\Delta R = 0$ are special cases of Yang-Mills fields.

<u>Bochner - Weitzenböck formulae</u>

Each of the above Laplacians on the associated bundle $E_\pi \to M$ satisfies

$$\Delta = \nabla^* \nabla + t\, K(R,\pi)$$

where $t \in \mathbb{R}$, $\nabla^*\nabla$ is the "connection Laplacian" induced by the connection in $E_\pi \to M$ determined by the Levi-Civita connection of $TM \to M$, and identifying $\Lambda^2 \mathbb{R}^n \cong \mathfrak{so}(n)$, letting $\{X_a\}$ be an orthonormal basis,

$$K(R,\pi) = -\sum_a d\pi(R \cdot X_a) \circ d\pi(X_a) = -\sum_{a,b} R_{ab}\, d\pi(X_a) \circ d\pi(X_b)$$

where $R = \sum_{a,b} R_{ab}\, X_a \otimes X_b \in \mathrm{Sym}^2(\Lambda^2 \mathbb{R}^n)$.

- In the above, $\pi : SO(n) \to SO(E)$, so $d\pi : \mathfrak{so}(n) \cong \Lambda^2 \mathbb{R}^n \to \mathfrak{so}(E)$

In particular, $d\pi(X) : E \to E$ is a skew-symmetric endomorphism for each $X \in \mathfrak{so}(n)$, hence $K(R,\pi) : E_\pi \to E_\pi$ is a <u>symmetric</u> endomorphism:

$$\langle K(R,\pi)\phi, \phi \rangle = -\sum_{a,b} R_{ab} \langle d\pi(X_a) \circ d\pi(X_b)\, \phi, \phi \rangle$$

$$= \sum_{a,b} R_{ab} \langle d\pi(X_a)\,\phi, d\pi(X_b)\,\phi \rangle$$

- Moreover, $\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \ni R \mapsto K(R,\pi) \in \mathrm{Sym}^2(E_\pi)$ is <u>linear</u> and <u>$SO(n)$-equivariant</u>, where $SO(n) \curvearrowright \mathrm{Sym}^2(\Lambda^2\mathbb{R}^n)$ via $A\cdot R = \sum_{a,b} R_{ab}\, \underbrace{Ad(A)X_a}_{} \otimes Ad(A)X_b$ and $SO(n) \overset{\pi}{\curvearrowright} \mathrm{Sym}^2(E_\pi)$ via $A\cdot T = d\pi(A)\circ T\circ d\pi(A^{-1})$.

ie. conjugation: $Ad(A)X = AXA^{-1}$

<u>Pf</u>: $$K(A\cdot R,\pi) = -\sum_{a,b} R_{ab}\, d\pi(Ad(A)X_a)\circ d\pi(Ad(A)X_b)$$
$$= -\sum_{a,b} R_{ab}\, d\pi(AX_aA^{-1})\circ d\pi(AX_bA^{-1})$$
$$= -\sum_{a,b} R_{ab}\, d\pi(A)\, d\pi(X_a)\, \cancel{d\pi(A^{-1})}\, \cancel{d\pi(A)}\, d\pi(X_b)\, d\pi(A^{-1})$$
$$= d\pi(A)\Big(-\sum_{a,b} R_{ab}\, d\pi(X_a)\circ d\pi(X_b)\Big) d\pi(A^{-1})$$
$$= A\cdot K(R,\pi).$$

- Clearly, $K(R,\pi_1\oplus\pi_2) = K(R,\pi_1)\oplus K(R,\pi_2)$ and $K(R,\pi^*) = K(R,\pi)^*$.

- Also from the above, <u>if $R\geq 0$, then $K(R,\pi)\geq 0$.</u>

<u>Pf</u>: Since $R:\Lambda^2 TM \to \Lambda^2 TM$ is symmetric, we can diagonalize it. Let $\{X_a\}$ be an o.n.b. of <u>eigenvectors</u>, ie. $RX_a = \nu_a X_a$. Since $R\geq 0$, we have $\nu_a \geq 0$, and:

Note: X_a need not be "decomposable" ie. $X_a = v\wedge w$ for some $v,w\in\mathbb{R}^n$. In general, $X_a = v_1\wedge w_1 + \dots + v_k\wedge w_k$.

$$\langle K(R,\pi)\phi,\phi\rangle = -\sum_a \langle d\pi(RX_a)\circ d\pi(X_a)\phi,\phi\rangle$$
$$= \sum_a \nu_a \cdot \|d\pi(X_a)\phi\|^2 \geq 0$$

<u>Note</u>: If π has no fixed vectors, ie. $\ker d\pi = \{0\}$, then $R>0$ implies $K(R,\pi)>0$. In general, $R>0$ only implies $K(R,\pi)\geq 0$.

Example: Defining representation $\pi = \mathrm{id} : SO(n) \to SO(n)$ is s.t.
$d\pi = \mathrm{id} : \mathfrak{so}(n) \to \mathfrak{so}(n)$

$$\boxed{K(R, \mathrm{id}) = \mathrm{Ric}_R}$$

Pf 1: Computation: $K(R, \mathrm{id}) = -\sum_a d\pi(R X_a)\, d\pi(X_a) = -\sum_a (R X_a) \circ X_a$

$$\langle K(R, \mathrm{id}) v, w\rangle = \sum_a \langle R X_a(v), X_a(w)\rangle = \cdots = \mathrm{Ric}_R(v, w)$$

Pf 2: $\underbrace{\mathrm{Sym}^2 \Lambda^2 \mathbb{R}^n}_{\substack{\cong \\ \mathbb{R} \oplus \mathrm{Sym}^2_0 \mathbb{R}^n \oplus W \oplus \Lambda^4}} \ni R \mapsto K(R, \mathrm{id}) \in \underbrace{\mathrm{Sym}^2 \mathbb{R}^n}_{\substack{\cong \\ \mathbb{R} \oplus \mathrm{Sym}^2_0 \mathbb{R}^n}}$ is $SO(n)$-equivariant.

traceless Ricci tensor $\mathring{\mathrm{Ric}} = \mathrm{Ric} - \frac{\mathrm{scal}}{n} \mathrm{Id}$.

so by Schur's Lemma, $K(R, \mathrm{id}) = a\, \mathrm{scal} \cdot \mathrm{Id} + b\, \mathring{\mathrm{Ric}}$ for some $a, b \in \mathbb{R}$.
Compute it at $R = \mathrm{Id}$ and another example with $\mathring{\mathrm{Ric}} \neq 0$ and $\mathrm{scal} = 0$, e.g.,
$R = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} \owedge \mathrm{Id}$ to find out $a = \frac{1}{n}$, $b = 1$, so that $K(R, \mathrm{id}) = \mathrm{Ric}_R$.

Kulkarni–Nomizu product

Bonus discussion of research related to algebraic nature of $K(R, \pi) \geq 0$:

Thm (Hitchin). $R > 0 \iff K(R, \pi) > 0$ for all nontrivial finite-dim irreducible $SO(n)$-representations $\pi : SO(n) \to SO(E)$.

Thm (B.–Mendes). $\sec_R > 0 \iff K(R, \mathrm{Sym}^p \mathbb{R}^n) \geq 0$, $\forall p \geq 2$.

Trivially, $\mathrm{Ric}_R > 0 \iff K(R, \mathrm{id}) \geq 0$

Actually, $K(R, \mathrm{id}) = \mathrm{Ric}_R$
$K(R, \pi_S) = \frac{\mathrm{scal}}{8} \mathrm{Id}$.

$\mathrm{scal}_R > 0 \iff K(R, \pi_S) \geq 0$, where $\pi_S : \mathrm{Spin}(n) \to S$ is the <u>spinor</u> <u>representation</u>.

Q: What other curvature conditions can be characterized in terms of $K(R, \pi) \geq 0$ for some family of representations π?

Relevance: • "Organize" algebraically/representation-theoretically curvature conditions.

• $\mathcal{S} = \{R \in \mathrm{Sym}^2 \Lambda^2 \mathbb{R}^n : K(R,\pi) \geq 0\}$ is a spectrahedron, so optimizing linear functions on $\mathcal{S}$ is "easy" with semidefinite programming.

Back to the Bochner technique: $\boxed{\Delta = \nabla^* \nabla + t\, K(R,\pi)}$

Suppose $t>0$ and $K(R,\pi)>0$, or $t<0$ and $K(R,\pi)<0$.

Then if $\phi \in \Gamma(E_\pi)$ is harmonic:

$$0 = \int_M \langle \Delta\phi, \phi\rangle = \int_M \langle \nabla^*\nabla\phi, \phi\rangle + t\langle K(R,\pi)\phi,\phi\rangle$$

$$= \int_M \underbrace{\|\nabla\phi\|^2}_{\geq 0} + \underbrace{t\langle K(R,\pi)\phi,\phi\rangle}_{\geq 0}$$

so $\phi \in \mathrm{Ker}\, K(R,\pi) = \{0\}$, i.e. $\phi = 0$. "All harmonic sections must vanish identically!"

Example: $E_\pi = TM$ $(\pi = \mathrm{id})$ $\quad t = 2$ and $K(R,\mathrm{id}) = \mathrm{Ric}$

$E_\pi = TM^*$ $(\pi = \mathrm{id}^*)$ $\quad t = -2$ and $K(R,\mathrm{id}^*) = \mathrm{Ric}^*$

Thus, the above implies the following:

Thm (Bochner 1946). If (M^n, g) is a closed manifold, then:

• if $\mathrm{Ric} > 0$, then all harmonic 1-forms on M vanish identically; in particular, $b_1(M) = 0$. (By Myers' Thm, $H^1_{dR}(M) = 0$ b/c $H^1(M) = (\pi_1 M)^{ab}$ is finite)

• if $\mathrm{Ric} < 0$, then all Killing vector fields vanish identically; in particular, $\mathrm{Iso}(M^n, g)$ is finite.

Lecture 12 Bochner technique II 4/27/2023

Recall basic elements of Bochner technique

(M^n, g) closed oriented Riem. mfld

or Spin(n) if M is spin, or G-bundle, $G \subset SO(n)$ if M^n has "special holonomy"!...

$Fr(TM)$ frame bundle ($SO(n)$ — principal bundle)

or $G \subset SO(n)$, or $Spin(n)$

$$\boxed{\begin{matrix}\pi : SO(n) \longrightarrow SO(E) \\ \text{unitary representation}\end{matrix}} \rightsquigarrow \boxed{\begin{matrix}E_\pi = Fr(TM) \times_\pi E \\ \text{associated bundle}\end{matrix}}$$

<u>Laplacian:</u> $\Delta = \underbrace{\nabla^* \nabla}_{\text{connection Laplacian}} + t \underbrace{K(R,\pi)}_{\text{curvature term}}$ acts on sections of E_π

both are determined by Levi-Civita connection, so ultimately, by metric g.

$$K(R,\pi) = -\sum_a d\pi(RX_a) \circ d\pi(X_a), \quad \{X_a\} \text{ o.n.b. of } \mathfrak{so}(n) \cong \Lambda^2 \mathbb{R}^n$$

$$d\pi : \mathfrak{so}(n) \to \mathfrak{so}(E)$$

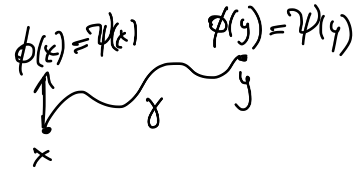
<u>Prop:</u> (i) If $t \cdot K(R,\pi) \geq 0$, then $\operatorname{Ker} \Delta = \{\phi \in \Gamma(E_\pi) : \nabla\phi \equiv 0\}$.

In particular, $\dim \operatorname{Ker} \Delta \leq \dim E$.

(ii) If $t \cdot K(R,\pi) > 0$, or, more generally, $t \cdot K(R,\pi) \geq 0$ and $\exists p \in M$ with $(t \cdot K(R,\pi))_p > 0$, then $\operatorname{Ker} \Delta = \{0\}$.

<u>Pf.</u> (i) If $\phi \in \operatorname{Ker} \Delta$, then $0 = \int_M \langle \Delta\phi, \phi \rangle = \int_M \underbrace{\|\nabla\phi\|^2 + \langle tK(R,\pi)\phi, \phi \rangle}_{\geq 0}$.

Thus $\nabla\phi \equiv 0$, and $\phi \in \operatorname{Ker} tK(R,\pi)$. Note $\phi \in \Gamma(E_\pi)$ is determined by its value at a point $x \in M$, $\phi(x) \in (E_\pi)_x$, since $\phi(y)$ is obtained by parallel transport along a path from x to y, so $\dim \operatorname{Ker} \Delta \leq \dim E$, namely, The linear map



$$ev_x : \operatorname{Ker}\Delta \longrightarrow (E_\pi)_x$$
$$\phi \longmapsto \phi(x)$$

is injective, since $\phi(x)=\psi(x)$ for $\phi,\psi\in\operatorname{Ker}\Delta$ implies $\phi=\psi$ everywhere on M, by parallel transport from x.
More precisely $\dim\operatorname{Ker}\Delta=\dim E'_\pi$, where $E'_\pi\subseteq E_\pi$ is the maximal parallel distribution in E_π.

(ii) If, furthermore, $\exists p\in M$ with $t\cdot K(R,\pi)>0$ on $p\in M$, hence on $B_\varepsilon(p)$ by continuity, then $\phi\in\operatorname{Ker}\Delta$ implies

$$0=\int_M\langle\Delta\phi,\phi\rangle=\int_M\|\nabla\phi\|^2+\langle tK(R,\pi)\phi,\phi\rangle\geqslant\int_{B_\varepsilon(p)}\|\nabla\phi\|^2+\underbrace{\langle tK(R,\pi)\phi,\phi\rangle}_{>0 \text{ unless } \phi=0}$$

thus $\phi\equiv 0$ on $B_\varepsilon(p)$, otherwise the above RHS would be >0.
Since $\nabla\phi=0$, in particular $\|\phi\|^2=$ const, it follows $\phi\equiv 0$ on M.

$\square$

<u>Applications to p-forms: $SO(n)\curvearrowright\Lambda^p\mathbb{R}^n$</u> Hodge Laplacian Δ has $t=2$

<u>Hodge theory:</u> $H^p_{dR}(M,\mathbb{R})\cong\operatorname{Ker}\Delta|_{\Omega^p(M)}$, so $b_p(M)=\dim\operatorname{Ker}\Delta|_{\Omega^p(M)}$

$p=1$: $\quad K(R,\Lambda^1\mathbb{R}^n)=\mathrm{Ric}_R$

<u>Bochner:</u> If (M^n,g) is a closed oriented Riem. mfld. then

(i) If $\mathrm{Ric}\geqslant 0$, then every harmonic 1-form is parallel.
In particular, $b_1(M)\leq n$ and $b_1(M)=n$ if and only if M^n is a flat torus.

(ii) If $\underbrace{\mathrm{Ric}\geqslant 0 \text{ and } \mathrm{Ric}_p>0}$, then every harmonic 1-form vanishes identically. In particular, $b_1(M)=0$. (cf. Myer's theorem)

<u>Rmk</u>: These manifolds also admit metrics w/ $\mathrm{Ric}>0$ <u>everywhere</u> [Ehrlich, 1976]

Pf: Only the last statement in (i) requires proof, the rest follows from the preceding discussion.

If $b_1(M)=n$, then there are n linearly independent parallel 1-forms on (M^n,g), hence n linearly independent parallel vector fields on (M^n,g). Thus (M^n,g) is flat. Pulling back these vector fields to $(\tilde{M},\tilde{g})=\mathbb{R}^n$, we have n constant vector fields that are invariant under the deck transformations action $\pi_1(M)\curvearrowright\mathbb{R}^n$. Therefore, $\pi_1(M)$ must consist entirely of translations, for any other isometry of $\mathbb{R}^n$ does not preserve n linearly independent constant vector fields. Thus $\pi_1 M$ is finitely generated, abelian, and torsion-free, so $\pi_1 M\cong\mathbb{Z}^k$ for some $k\leq n$. If $k<n$, then $\pi_1 M\curvearrowright\mathbb{R}^n$ would not be cocompact, so $k=n$ and $M^n=\mathbb{R}^n/\mathbb{Z}^n=T^n$. $\square$

Remark. There are many non-isometric flat tori in every dim., namely the moduli space of flat tori T^n is $O(n)\backslash GL(n)/GL(n,\mathbb{Z})$; which is an orbifold of dimension $n(n+1)/2$. Other closed flat manifolds are quotients of flat tori by a free action of a finite group, identified with the holonomy group (Bieberbach Thm).

Thm (Gromov 80, Gallot 81). If (M^n,g) is a closed oriented Riem. mfld, $\mathrm{Ric}\geq(n-1)\cdot K$ and $\mathrm{diam}(M)\leq D$, then $b_1(M)\leq C(n,K\cdot D^2)$, where $C(n,\varepsilon)$ is a function satisfying $\lim_{\varepsilon\to 0}C(n,\varepsilon)=n$. In particular, $\exists\,\varepsilon(n)>0$ s.t. $K\cdot D^2\geq-\varepsilon(n)$ implies $b_1(M)\leq n$.

i.e., can also handle some manifolds w/ slightly negative Ricci!

$p \geq 2$: $K(R, \Lambda^p \mathbb{R}^n)$ is more complicated to write down (but there are explicit formulas using Kulkarni–Nomizu product)

was a faculty member at CUNY GC!

Special cases: $R \underset{(\geq)}{>} 0 \Rightarrow K(R, \Lambda^p \mathbb{R}^n) \underset{(\geq)}{>} 0$ [Gallot, Meyer]

so if (M^n, g) is a closed oriented Riem. mfld with:

- $R > 0$, then $b_p(M) = 0$ for all $1 \leq p \leq n$; i.e., M^n is a rational homology sphere.
- $R \geq 0$, then $b_p(M) \leq \binom{n}{p}$ for all $1 \leq p \leq n$.

Note: There are examples of rational homology spheres w/ $sec > 0$ that are not spherical spaceforms S^n/Γ, e.g. Berger space $B^7 = SO(5)/SO(3)$ maximal.

These results have been improved substantially using Ricci flow:

$\boxed{\text{Thm (Böhm-Wilking 2006). If } R > 0, \text{ then } \widetilde{M} \underset{\text{diff}}{\cong} S^n.}$

(If $R \geq 0$, then $\widetilde{M}$ is isometric to a product of Euclidean space, sphere w/ $R \geq 0$, compact irreducible symmetric space, compact Kähler manifold biholomorphic to $\mathbb{C}P^n$ with $R \geq 0$ on real $(1,1)$-forms.)

A very recent refinement is the following:

Thm (Petersen–Wink, 2021). Given $1 \leq p \leq \lfloor \frac{n}{2} \rfloor$, suppose that R is $(n-p)$-positive, ie, the sum of any $n-p$ eigenvalues of R is positive. Then $b_p(M) = b_{n-p}(M) = 0$. In particular, if R is $\lceil \frac{n}{2} \rceil$-positive, then M^n is a rational homology sphere.

Of course, there are also versions for nonnegative curvature and $b_p(M) \leq \binom{n}{p}$.

Even more recently, the above was generalized to other representations:

Thm (B.–Goodman 2022). If $\pi: SO(n) \to SO(E)$ is irreducible, with highest weight λ, then $K(R, \pi) \geq \|\lambda\|^2 \cdot \left(\nu_1 + \cdots + \nu_{\lfloor r \rfloor} + (r - \lfloor r \rfloor) \nu_{\lfloor r \rfloor + 1} \right) \cdot \mathrm{Id}$, where $r = \frac{\langle \lambda, \lambda + 2\rho \rangle}{\|\lambda\|^2}$, ρ is the half-sum of positive roots, and $\nu_1 \leq \cdots \leq \nu_{\binom{n}{2}}$ the eigenvalues of R.

Similarly to the above improvements to slightly negative curvature:

Thm (Meyer–Gallot 1970s). If $R \geq \kappa \cdot \mathrm{Id}$ and $\mathrm{diam}(M) \leq D$, then

$$b_p(M) \leq \binom{n}{p} \cdot \exp\left(C(n, \kappa \cdot D^2) \cdot \sqrt{-\kappa D^2 p(n-p)} \right)$$

In particular, $\exists \varepsilon(n) > 0$ s.t. $\kappa \cdot D^2 \geq -\varepsilon(n)$ implies $b_p(M) \leq \binom{n}{p}$.

Petersen–Wink also improved the above, replacing $R \geq \kappa \cdot \mathrm{Id}$ by the weaker hypothesis $\nu_1 + \dots + \nu_{n-p} \geq (n-p) \cdot \kappa$, where $\nu_1 \leq \dots \leq \nu_{\binom{n}{2}}$ are eigenvalues of R.

Hopf Question: Does $S^2 \times S^2$ admit a metric w/ $\sec > 0$?

"Naive" Bochner technique approach would be to try to show that $\sec_R > 0$ implies $K(R, \Lambda^2 \mathbb{R}^4) > 0$ hence $b_2 M^4 = 0$. However, this is clearly false: $\mathbb{C}P^2$ has $\sec > 0$ and $b_2 = 1$.

Slightly more refined Bochner technique approach uses:

Finsler–Thorpe trick. $\boxed{R : \Lambda^2 \mathbb{R}^4 \to \Lambda^2 \mathbb{R}^4 \text{ has } \sec > 0} \iff \boxed{\exists \tau \in \mathbb{R} \text{ s.t. } R + \tau * > 0}$

and the computation that, splitting $\Lambda^2 \mathbb{R}^4 = \underbrace{\Lambda^2_+ \mathbb{R}^4}_{\text{"self-dual"}} \oplus \underbrace{\Lambda^2_- \mathbb{R}^4}_{\text{"anti-self-dual"}}$, then

$$* = \left(\begin{array}{c|c} \mathrm{Id} & 0 \\ \hline 0 & -\mathrm{Id} \end{array} \right)$$

e.g., $*(e_1 \wedge e_2 \pm e_3 \wedge e_4) = \pm (e_1 \wedge e_2 \pm e_3 \wedge e_4)$

$K(*, \Lambda^2 \mathbb{R}^4) = \pm 4 \, \mathrm{Id}$.

Thus, if $\sec > 0$ and $\tau > 0$, then, since $S \mapsto K(S, \pi)$ is linear and $S > 0 \Rightarrow K(S, \pi) > 0$, we get:

$$K(R, \Lambda^2_- \mathbb{R}^4) = K(R + \tau *, \Lambda^2_- \mathbb{R}^4) - \tau \cdot \underbrace{K(*, \Lambda^2_- \mathbb{R}^4)}_{-4\mathrm{Id}}$$

$$= K(\underbrace{R + \tau *}_{>0}, \Lambda^2_- \mathbb{R}^4) + 4 \tau \, \mathrm{Id} > 0$$

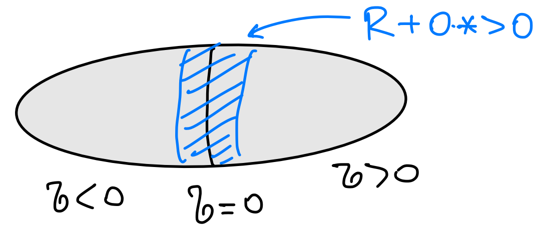
Such positivity implies vanishing of harmonic sections of $\Lambda^2_- TM$, called anti-self-dual 2-forms. Similarly, if instead $\tau < 0$, then use $\Lambda^2_+ TM$.

$$b_2^\pm(M) = \dim \operatorname{Ker} \Delta|_{\Omega^2_\pm(M)}, \qquad b_2(M) = b_2^+(M) + b_2^-(M).$$

As $S^2 \times S^2$ has $b_2^+ = b_2^- = 1$, it follows that: ← Same for any "indefinite" 4-mfld, e.g. $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

Thm (B.–Mendes, 2022). If $(S^2 \times S^2, g)$ has $\sec > 0$, then the subset $\{p \in S^2 \times S^2 : R_p \text{ is } \underline{\text{not}} \text{ positive-definite}\} \subset S^2 \times S^2$ has at least 2 connected components of non-empty interior.

Pf: Because $b_2^\pm > 0$, $\tau : M \to \mathbb{R}$ must change sign, so $\{\tau = 0\} \neq \emptyset$. On a neighborhood of $\{\tau = 0\}$, we have $R > 0$.



Other results towards answering the Hopf Question:

Thm (Hsiang–Kleiner, Grove–Wilking). If (M^4, g) is closed, simply-conn., $\sec > 0$ and $\underbrace{S^1 \curvearrowright M^4 \text{ isometrically}}_{\Leftrightarrow\ |\mathrm{Isom}(M)| = +\infty}$, then $M \underset{\text{eq. diff.}}{\cong} \underbrace{S^4, \text{ or } \mathbb{C}P^2}_{\text{have } \sec > 0 \text{ and } S^1 \curvearrowright M^4}$.

If (M^4, g) is closed, simply-conn., $\sec \geq 0$ and $S^1 \curvearrowright M^4$ isometrically, then $M^4 \underset{\text{eq. diff}}{\cong} S^4, \mathbb{C}P^2, \underbrace{S^2 \times S^2, \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \text{ or } \mathbb{C}P^2 \# \mathbb{C}P^2}$.

← All are known to have $\sec \geq 0$ and $S^1 \curvearrowright M^4$

Cor: If $(S^2 \times S^2, g)$ has $\sec > 0$, then $|\mathrm{Isom}(S^2 \times S^2, g)| < \infty$.

Thm (Myers–Steenrod 1939, Palais 1957) If (M^n, g) is a complete Riem. mfld,

(i) $\phi: M \to M$ preserves distances, i.e. $\mathrm{dist}(\phi(x), \phi(y)) = \mathrm{dist}(x,y)$ $\forall x,y \in M$ $\iff$ ϕ is a (Riem.) isometry, i.e. ϕ is smooth and $\forall p \in M$, $v, w \in T_pM$, $g_{\phi(p)}(d\phi_p v, d\phi_p w) = g_p(v,w)$.

(ii) $\mathrm{Isom}(M,g) = \{\phi: M \to M \text{ isometry}\}$ is a Lie group, with Lie algebra $\mathfrak{isom}(M,g) = \{X \in \mathfrak{X}(M) : X \text{ is Killing, i.e., } g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0 \ \forall Y, Z\}$

we discussed these in previous lectures.

Ex:

- $\mathbb{R}^n$ has $\mathrm{Isom}(\mathbb{R}^n) = \underbrace{\mathbb{R}^n}_{\text{translations}} \rtimes \underbrace{O(n)}_{\text{rotations/reflections}}$. $F(x) = Ax + v$, $A \in O(n)$, $v \in \mathbb{R}^n$

- $S^n \subset \mathbb{R}^{n+1}$ has $\mathrm{Isom}(S^n) = O(n+1)$. $F(x) = Ax$, $x \in O(n+1)$

  Note: $O(n+1) = \{F: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} : \langle F(x), F(x)\rangle = \langle x, x\rangle\}$ Euclidean metric
  are the linear isometries of $\mathbb{R}^{n+1}$

- $H^n \subset \mathbb{R}^{n,1}$ has $\mathrm{Isom}(H^n) = O(n,1) = \{F: \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}, \langle F(x), F(x)\rangle = \langle x, x\rangle\}$
  where $\mathbb{R}^{n,1} = \{x = (x_1, \dots, x_n, x_{n+1})\}$ is the Minkowski space w/ the Lorentz metric $\langle \cdot, \cdot \rangle = dx_1^2 + \dots + dx_n^2 - dx_{n+1}^2$, $H^n = \{x \in \mathbb{R}^{n,1} : \langle x, x\rangle = -1\}$.
  (if $n=3$, these are often called "Lorentz transformations")

Prop: If (M^n, g) is connected and complete, then $\dim \mathrm{Isom}(M^n, g) \leq \frac{n(n+1)}{2}$.
Equality holds iff (M^n, g) has constant curvature (cf. above).

Rmk: The family $\{\mathrm{Isom}(\underbrace{M^n_\kappa}_{\text{model sp. w/ } \sec \equiv \kappa}), \kappa \in \mathbb{R}\}$ is actually a "bundle of Lie groups"! $\underset{\kappa<0}{O(n,1)} \longleftrightarrow \underset{\kappa=0}{\mathbb{R}^n \rtimes O(n)} \longleftrightarrow \underset{\kappa>0}{O(n+1)}$

Lemma (from Diff Geom Qual Course). If $\phi, \psi : (M,g) \to (M,g)$ are isometries and $\exists p \in M$ s.t. $\phi(p) = \psi(p)$ and $d\phi(p) = d\psi(p)$ (i.e. $d\phi_p(v) = d\psi_p(v)$, $\forall v \in T_pM$) then $\phi = \psi$. Equivalently, if X, Y are Killing fields on (M,g) and $\exists p \in M$ s.t. $X_p = Y_p$ and $(\nabla X)_p = (\nabla Y)_p$, then $X = Y$.

Pf: The set $\{x \in M : \phi(x) = \psi(x),\ d\phi(x) = d\psi(x)\}$ is clearly closed and nonempty. Moreover, as $\exp_x : T_xM \to M$ is a local diffeo and

$$\phi(\exp_x v) = \exp_{\phi(x)} d\phi(x) v = \exp_{\psi(x)} d\psi(x) v = \psi(\exp_x v)$$

it follows this set is also open, thus all of M by connectedness.

For Killing fields version, recall X is Killing $\iff \nabla X$ is skew $\iff \underbrace{\mathcal{L}_X g = 0}_{\text{linear in } X}$.

Since $\{X \in \mathfrak{X}(M) : \mathcal{L}_X g = 0\}$ is a vector space, it suffices to show $X_p = 0,\ (\nabla X)_p = 0 \Rightarrow X \equiv 0$. Again, by connectedness, can do this locally around p. Since $X_p = 0$, the flow ϕ_t^X fixes p; b/c $\frac{d}{dt} \phi_t^X(p) = X_p = 0$. Moreover, $d(\phi_t^X)_p = \mathrm{id}$, $\forall t \in \mathbb{R}$; indeed:

$$[X,Y]_p = (\nabla_X Y - \nabla_Y X)_p = 0 \text{ and so } 0 = (\mathcal{L}_X Y)_p = \lim_{t \to 0} \frac{d\phi_t^X v - v}{t}, \quad v = Y_p$$

Thus with $v = d\phi_{t_0}^X(Y)$,

$$0 = \mathcal{L}_X (d\phi_{t_0}^X Y)_p = \lim_{t \to 0} \frac{d\phi_t^X(p)(d\phi_{t_0}^X(p) Y) - d\phi_{t_0}^X(p) Y}{t} = \lim_{t \to 0} \frac{d\phi_{t+t_0}^X(p) Y - d\phi_{t_0}^X(p) Y}{t} = \lim_{t \to t_0} \frac{d\phi_t^X(p) Y - d\phi_{t_0}^X(p) Y}{t - t_0}$$

so $d\phi_t^X$ is constant in t, i.e. $d\phi_t^X(p) = d\phi_0^X(p) = \mathrm{id}$, as claimed.

By previous part, $\phi_t^X \equiv \mathrm{id}$, $\forall t$, since $\phi_t^X(p) = p$ and $d\phi_t^X(p) = \mathrm{id}$, hence $X \equiv 0$. $\square$

Pf of Prop. Let

$$F_p : \mathfrak{isom}(M^n, g) \longrightarrow T_pM \oplus \underbrace{\mathfrak{so}(T_pM)}_{=\{\text{skew linear maps on } T_pM\}}$$
$$X \longmapsto (X_p, (\nabla X)_p)$$

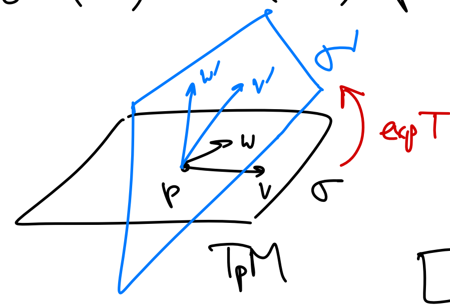
Note that F_p is a well-defined linear map (∇X is skew b/c X is Killing) and, by the Lemma, F_p is injective. Thus $\dim \mathfrak{isom}(M^n, g) \le n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

Moreover if $\dim \operatorname{isom}(M^n, g) = \frac{n(n+1)}{2}$, then F_p is also surjective $\forall p$, so $\forall\, T \in \mathfrak{so}(T_pM)$, $\exists X \in \mathfrak{X}(M)$ Killing field w/ $X_p = 0$, $(\nabla X)_p = T$.

The flow ϕ_t^X fixes p, and $d\phi_t^X(p) = \exp(tT) : T_pM \to T_pM$.

is a 1-param. subgroup of orthogonal transformations of T_pM, with arbitrary infinitesimal generator, so $\sec(\sigma) = \sec(\sigma')$ for all 2-planes $\sigma, \sigma' \subset T_pM$, i.e., $\sec \equiv k$.

(can find T s.t. $\exp(T)$ maps v, w to v', w', hence $\sigma = \operatorname{span}\{v,w\}$ to $\sigma' = \operatorname{span}\{v', w'\}$)



$\square$

Slowly tone down the symmetries:

e.g., $\boxed{\dim \operatorname{Isom}(M^n, g)}$ "degree of symmetry", $\boxed{\operatorname{rank} \operatorname{Isom}(M^n, g)}$ "symmetry rank", $\boxed{\dim M/G}$ "cohomogeneity", ...

(M^n, g) is a homogeneous space iff $G \curvearrowright M$ transitively, $G \subset \operatorname{Isom}(M^n, g)$.

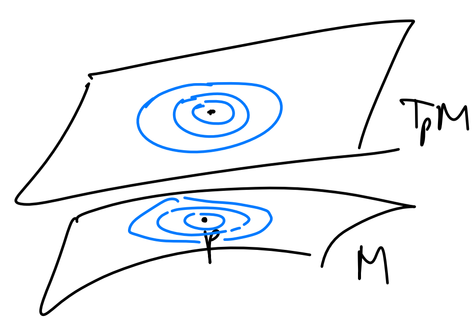
i.e. $M/G = \{pt\}$, so $\dim M/G = 0$ "cohomogeneity zero"

Isotropy at $p \in M$ is $G_p = H \subset G$, and $M = G(p) \cong G/H$

$G_p = \{g \in G : g \cdot p = p\}$

Isotropy action/representation: $H \curvearrowright T_pM$

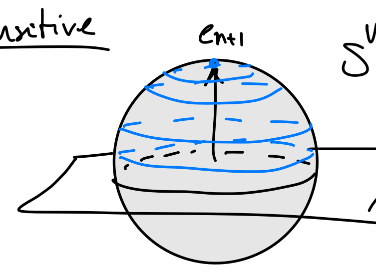
H-orbits



Example: $G = O(n+1) \curvearrowright S^n$, $A \cdot v = Av$ is transitive

could also use $SO(n+1)$ ...

$$H = G_{e_{n+1}} = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \in O(n+1) \right\} \cong O(n)$$



$S^n = O(n+1)/O(n)$

At other points, isotropy is conjugate to H.

• $U(n+1) \curvearrowright \mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ and hence $U(n+1) \curvearrowright S^{2n+1}$ (unit sphere)

This action is also transitive, and has isotropy (at $e_{n+1} \in \mathbb{C}^{n+1}$) $\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \in U(n+1) \right\} \cong U(n)$

Thus, $S^{2n+1} = U(n+1)/U(n)$. Also, could do the same with $SU(n+1)$ ..

• $Sp(n+1) \curvearrowright \mathbb{H}^{n+1} \cong \mathbb{R}^{4n+4}$ and hence $Sp(n+1) \curvearrowright S^{4n+3}$ (unit sphere)

This action is also transitive, and has isotropy (at $e_{n+1} \in \mathbb{H}^{n+1}$):

$$\left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \in Sp(n+1) \right\} \cong Sp(n)$$

Thus, $S^{4n+3} \cong Sp(n+1)/Sp(n)$.

<u>Theorem</u> (Montgomery–Samelson–Borel, 50s). The only connected groups acting transitively on a sphere S^d are given in the table below:

Group	Isotropy	sphere	Isotropy repr. decomposition
$SO(n+1)$	$SO(n)$	S^n	$\mathbb{R}^n$ (irred.) def. rep.
$SU(n+1)$	$SU(n)$	S^{2n+1}	$\mathbb{C}^n \oplus \mathbb{R}$
$U(n+1)$	$U(n)$	S^{2n+1}	$\mathbb{C}^n \oplus \mathbb{R}$
$Sp(n+1)$	$Sp(n)$	S^{4n+3}	$\mathbb{H}^n \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$
$Sp(n+1)Sp(1)$	$Sp(n)Sp(1)$	S^{4n+3}	$\mathbb{H}^n \oplus \mathrm{Im}\,\mathbb{H}$
$Sp(n+1)U(1)$	$Sp(n)U(1)$	S^{4n+3}	$\mathbb{H}^n \oplus \mathbb{C} \oplus \mathbb{R}$
$Spin(9)$	$Spin(7)$	S^{15}	$\mathbb{O} \oplus \mathrm{Im}\,\mathbb{O}$
$Spin(7)$	G_2	S^7	$\mathbb{R}^7$ (irred.)
G_2	$SU(3)$	S^6	$\mathbb{R}^6$ (irred.)

fixed (trivial)

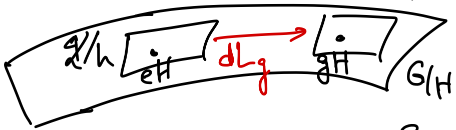
irred.

The above leads to the classification of homogeneous metrics on spheres, recalling that there is a natural correspondence:

$$\left\{ \begin{array}{c} G\text{-invariant homog.} \\ \text{metrics on } G/H \end{array} \right\} \longleftrightarrow \left\{ \mathrm{Ad}(H)\text{-inv. inner products on } \mathfrak{g}/\mathfrak{h} \right\}$$

$\mathrm{Ad}(H)$ is precisely the isotropy representation at $eH \in G/H$.

Indeed, $Ad(H)$-invariance is the requirement to coherently define a tensor on G/H by using <u>left-translations</u> from $T_{eH}\, G/H \cong \mathfrak{g}/\mathfrak{h}$.



[see e.g. Cheeger-Ebin, or Alexandrino-Bettiol]

Thus, e.g. on S^{2n+1}, there is a 2-parameter family of metrics invariant under the transitive $U(n+1)$-action:

<u>Def.</u> ("Berger metric"). Let g be the (unit) round metric on S^{2n+1}, and write $g = g|_{hor} + g|_{ver}$ according to horizontal/vertical spaces for the Hopf fibration $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. Then

$$g_{s,t} = s^2 g|_{hor} + t^2 g|_{ver}, \quad s,t>0,$$ is $U(n+1)$-invariant.

Up to global rescaling (homothety), consider $g(t) = g_{1,t}$. Geometrically,

$$t S^1 \longrightarrow (S^{2n+1}, g(t)) \longrightarrow \mathbb{C}P^n$$

it is obtained "shrinking" the fibers of the Hopf bundle, i.e., rescaling by t the vertical directions.

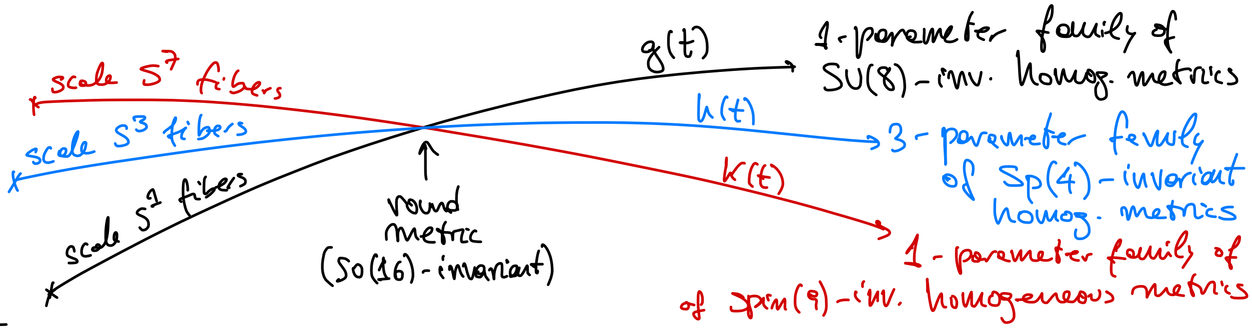
Similarly for $\underline{S^3} \to (S^{4n+3}, h(t)) \to \mathbb{H}P^n$ and $S^7 \to (S^{15}, k(t)) \to S^8(\tfrac{1}{2})$.

can use any left-inv. metric on $S^3 \cong SU(2)$

<u>Cor.</u> Up to homotheties, homog. metrics on spheres are the above: 1-param. family on S^{2n+1}, 3-param. family of S^{4n+3}, 1-par. family on S^{15}.

E.g., on S^{15}:

$$S^{15} = \frac{SO(16)}{SO(15)} = \frac{SU(8)}{SU(7)} = \frac{Sp(4)}{Sp(3)} = \frac{Spin(9)}{Spin(7)}$$

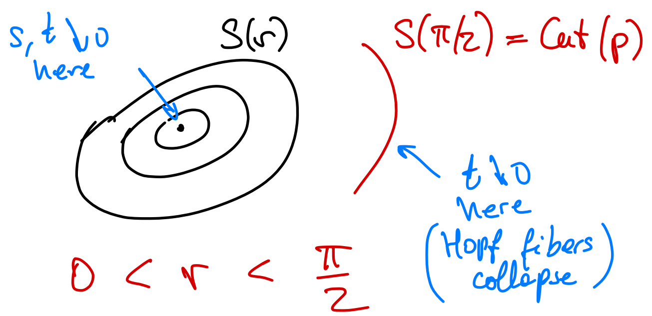


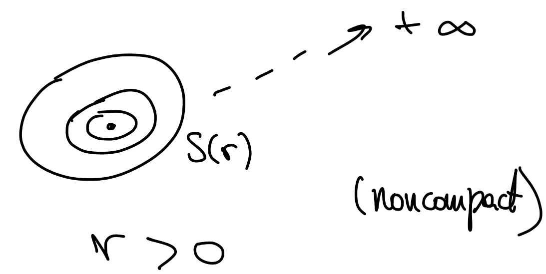
Geometric realization of Berger metrics (Bourguignon–Karcher)

Consider distance spheres $S_p(r) = \{x \in M : \operatorname{dist}(x,p) = r\}$ on

$$\underbrace{\mathbb{C}P^n,\ \mathbb{H}P^n \text{ and } \mathbb{C}aP^2}_{\text{compact rank one symmetric spaces}}; \text{ and on } \underbrace{\mathbb{C}H^n,\ \mathbb{H}H^n \text{ and } \mathbb{C}aH^2}_{\text{non compact rank one symmetric spaces}}$$

compact rank one symmetric spaces $\longleftrightarrow$ duals $\longleftrightarrow$ non compact rank one symmetric spaces



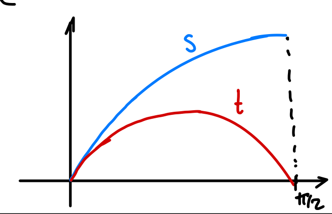


Berger metrics $g_{s,t}, h_{s,t}, k_{s,t}$, given by $s^2 g|_{hor} + t^2 g|_{ver}$, where $g|_{hor} + g|_{ver}$ is the unit round metric, are realized for all $s,t>0$ as distance spheres $S(r)$:

$S(r) \subset \mathbb{C}P^{n+1}$ is isometric to $(S^{2n+1}, g_{s,t})$ $\quad \begin{cases} s = \sin r \\ t = \sin r \cos r \end{cases}$

$S(r) \subset \mathbb{H}P^{n+1}$ is isometric to $(S^{4n+3}, h_{s,t})$

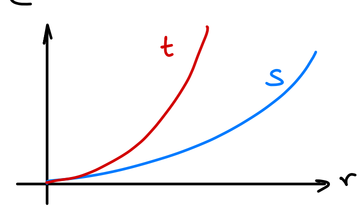
$S(r) \subset \mathbb{C}aP^2$ is isometric to $(S^{15}, k_{s,t})$



$S(r) \subset \mathbb{C}H^{n+1}$ is isometric to $(S^{2n+1}, g_{s,t})$ $\quad \begin{cases} s = \sinh r \\ t = \sinh r \cdot \cosh r \end{cases}$

$S(r) \subset \mathbb{H}H^{n+1}$ is isometric to $(S^{4n+3}, h_{s,t})$

$S(r) \subset \mathbb{C}aH^2$ is isometric to $(S^{15}, k_{s,t})$



Recall $\mathbb{C}P^n = S^{2n+1}/S^1$ is the orbit space of $S^1 \curvearrowright \mathbb{C}^{n+1}$ given by

$$e^{i\theta} \cdot (z_0, \dots, z_n) = (e^{i\theta} z_0, \dots, e^{i\theta} z_n)$$

$U(n+1) \curvearrowright S^{2n+1}$ commutes with $S^1 \curvearrowright S^{2n+1}$ so descends to $U(n+1) \curvearrowright \mathbb{C}P^n$

Similarly, $Sp(n+1) \curvearrowright S^{4n+3}$ commutes w/ $S^3 \curvearrowright S^{4n+3}$ so descends to $Sp(n+1) \curvearrowright \mathbb{H}P^n$
$$\mathbb{H}P^n = S^{4n+3}/S^3$$

Thm (Oniscik '60s). The only groups (connected) acting transitively on a projective space $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}aP^2$ are given in the table below:

Group	Isotropy	Proj. space	Isotropy repr.
$SU(n+1)$	$S(U(n)U(1))$	$\mathbb{C}P^n$	$\mathbb{C}^n$ (irred.) def. rep.
$Sp(n+1)$	$Sp(n)Sp(1)$	$\mathbb{H}P^n$	$\mathbb{H}^n$ (irred.) def. rep.
$Sp(n+1)$	$Sp(n)U(1)$	$\mathbb{C}P^{2n+1}$	$\mathbb{H}^n \oplus \mathbb{C}$ (irred.)
F_4	$Spin(9)$	$\mathbb{C}aP^2$	$\mathbb{C}a^2$ (irred.)

Note: If $H \subset K \subset G$, then there is a natural fibration
$$K/H \longrightarrow G/H \longrightarrow G/K$$
$$gH \longmapsto gK$$

The above give the additional Hopf-like bundles:

$$S^1 \to S^{2n+1} \longrightarrow \mathbb{C}P^n \leftrightsquigarrow U(1) \to SU(n+1)/SU(n) \to SU(n+1)/S(U(n)U(1))$$

$$\begin{tikzcd}
S^3 \arrow[r] \arrow[d, "/S^1"'] & S^{4n+3} \arrow[r] \arrow[d, "/S^1"'] & \mathbb{H}P^n \arrow[r, leftrightsquigarrow] \arrow[d, equal] & Sp(1) \arrow[r] \arrow[d, "/U(1)"'] & Sp(n+1)/Sp(n) \arrow[r] \arrow[d, "/U(1)"'] & Sp(n+1)/Sp(n)Sp(1) \arrow[d, equal] \\
\mathbb{C}P^1 \arrow[r] & \mathbb{C}P^{2n+1} \arrow[r] & \mathbb{H}P^n \arrow[r, leftrightsquigarrow] & Sp(1)/U(1) \arrow[r] & Sp(n+1)/Sp(n)U(1) \arrow[r] & Sp(n+1)/Sp(n)Sp(1)
\end{tikzcd}$$

$$S^7 \to S^{15} \longrightarrow S^8(1/2) \leftrightsquigarrow Spin(8)/Spin(7) \to Spin(9)/Spin(7) \to Spin(9)/Spin(8)$$

Note: $\mathbb{C}aP^2$ is <u>not</u> the base of a sphere bundle $S^d \to \mathbb{C}aP^2$!

Classification

very closely related to Cartan's classification of simple Lie Groups $\underbrace{A_n, B_n, C_n, D_n}_{\text{classical}}$, $\underbrace{E_6, E_7, E_8, F_4, G_2}_{\text{exceptional}}$

What does it mean to classify homogeneous spaces?

- Given M, classify all $G \curvearrowright M$ transitive actions (may e.g. assume G connected).
  up to diffeo, homeo...?
  done above for spheres and projective spaces, up to diffeo. [cf. Straume for homeo.]
- Given n, classify all n-dim. homog. spaces.
  (done for $n \leq 6$) E.g., for $n=2$, only have: S^2, $\mathbb{RP}^2$, T^2, K^2
  Klein bottle

Why homogeneous?

- Good test ground, all PDEs/ODEs become algebraic,
  can compute cohomology ring systematically (Borel).

Note: (M, g) homog., $f: M \to \mathbb{R}$ invariant under isometries $\Rightarrow f \equiv$ const.
For instance, on a homog. space, the following scalar-valued geometric quantities are trivially constant: scal, $|\mathrm{Ric}|^2$, $|R|^2$, $|\nabla R|^2$, ...

All nonscalar geometric quantities, e.g., sec, Ric, ... are algebraic.

Ex: Suppose we want to find Einstein metrics i.e. $\mathrm{Ric} = \lambda g$.
On a homog. space, this becomes a matrix equation on $T_{eH} G/H$.
In fact, if the isotropy repr. is irreducible, then every homog. metric is Einstein:

$H \curvearrowright T_{eH} G/H$ irred. $\overset{\text{Schur's Lemma}}{\rightsquigarrow}$ $T: T_{eH} G/H \longrightarrow T_{eH} G/H$
H-equivariant linear map
then $T = \lambda \cdot \mathrm{Id}$. $(\lambda \in \mathbb{R})$

Technically, this works for complex representations...
$H \curvearrowright V$ irred., V real vector sp. $\rightsquigarrow H \curvearrowright V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$
and $H \curvearrowright V_{\mathbb{C}}$ is irreducible if $V_{\mathbb{C}}$ is of "real type"
Then, by Schur's Lemma, $\mathrm{Hom}^H(V, V) = \mathbb{R} \cdot \mathrm{id}$, as desired
Note: Adjoint repr. of compact semisimple Lie gps are always as above!
(else, $H \curvearrowright V_{\mathbb{C}}$ splits as $W \oplus W^*$, $W^* \cong W$ "quaternionic type"
$W^* \not\cong W$ "complex type")

As both g and Ric define H-equivariant linear maps, they must be multiples.

As a consequence, "the" homog. metrics on S^n (even), $\mathbb{C}P^n$ (even), $\mathbb{H}P^n$, $\mathbb{C}aP^2$ are <u>unique</u> up to homotheties, and <u>Einstein</u>

- Round metric:

S^n : $Ric = (n-1)g$ $\qquad$ $sec \equiv 1$

- Fubini–Study metric:

$\mathbb{C}P^n$; $Ric = 2(n+1)g$ $\qquad$ $1 \leq sec \leq 4$ $\qquad$ $\mathbb{C}P^1 \underset{isom}{\cong} S^2(1/2)$.

$\mathbb{H}P^n$: $Ric = 4(n+2)g$ $\qquad$ $1 \leq sec \leq 4$ $\qquad$ $\mathbb{H}P^1 \underset{isom.}{\cong} S^4(1/2)$

$\mathbb{C}aP^2$: $Ric = 36g$ $\qquad$ $1 \leq sec \leq 4$ $\qquad$ $\mathbb{C}aP^1 \underset{isom}{\cong} S^8(1/2)$.

Among the remaining homog. metrics on compact rank one symmetric spaces (S^n, ~~$\mathbb{R}P^n$~~, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}aP^2$), we have:

Jensen metric $g_J = g|_{hor} + \frac{1}{2n+3} g|_{ver}$ on S^{4n+3} is Einstein
($Sp(n+1)$ — invariant)

Bourguignon – Karcher metric $g_{BK} = g|_{hor} + \frac{3}{11} g|_{ver}$ on S^{15} is Einstein
($Spin(9)$ – invariant)

Ziller metric $g_Z = g_{FS}|_{hor} + \frac{1}{n+1} g_{FS}|_{ver}$ on $\mathbb{C}P^{2n+1}$ is Einstein
($Sp(n+1)$ – invariant)

Ziller showed these are all possibilities. (Math. Ann. 1982)

<u>Next step</u> down the symmetry ladder, as measured by cohomogeneity:

<u>Cohomogeneity one manifolds</u> are those with $G \curvearrowright M$, $G \subset Isom(M,g)$

$$\dim M/G = 1. \quad \left(\Longrightarrow M/G \underset{isom.}{\cong} \mathbb{R}, [0,+\infty), S^1, \text{ or } [0,L].\right)$$

more about this next time...

$G \curvearrowright M$ isom. action, cohomogeneity is $\dim M/G$.

"isotropy" or "stabilizer" at $p \in M$.

- cohomogeneity 0: $M/G = \{pt\}$ i.e. $M = G(p) \cong G/H$ where $H = G_p < G$

Adjoint action $G \curvearrowright \mathfrak{g}$, $Ad_g(x) = dL_g \circ dR_{g^{-1}}(x) = \frac{d}{dt} g \underbrace{\exp(tx)}_{\text{Lie exponential}} g^{-1}\big|_{t=0}$

$ad_x(y) = d(Ad_e)_x(y) = [x,y]$ ($L_g(h) = gh$, $R_g(h) = hg$ left/right translations)

$\pi : G \longrightarrow G/H$ is G-equivariant: $g_1(gH) = (g_1 g)H$.
$g \longmapsto gH$

$\pi: G \to G/H$
$d\pi: \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$

if $h \in H$, then $hgH = hgh^{-1}H$, so $Ad_h : \mathfrak{g} \longrightarrow \mathfrak{g}$ leaves $\mathfrak{h}$ invariant, and, differentiating $h\exp(tx)H = h\exp(tx)h^{-1}H$ in $t=0$, we get $dL_h(x) = d\pi[Ad_h(x)]$

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ Ad-invariant complement, $\mathfrak{m} \cong T_{eH}\, G/H$. $= [Ad_h(x)]_{\mathfrak{m}}$ ⊛

$Ad_H \circlearrowright \mathfrak{m}$, $H \circlearrowright T_{eH}\,G/H$ isotropy repr.

$$\left\{\begin{matrix} G\text{-inv. metrics} \\ \text{on } G/H \end{matrix}\right\} \xleftrightarrow{\cong} \left\{\begin{matrix} Ad_H\text{-inv. inner} \\ \text{products on } \mathfrak{m} \end{matrix}\right\}$$

$$g \longmapsto \langle\cdot,\cdot\rangle = g|_{T_{eH}G/H \times T_{eH}G/H} \text{ is } Ad_H\text{-inv. by ⊛}$$

Conversely, if $\langle\cdot,\cdot\rangle$ is Ad_H-inv., let $g_{gH}(x,y) = \langle dL_{g^{-1}}x, dL_{g^{-1}}y\rangle$, $x,y \in T_{gH}G/H$.
then it is well-defined (indep. of representative g in coset gH) and left-inv., so G-homogeneous. $\square$

Prop: $T(G/H) = G \times_H \mathfrak{m}$ is the associated bundle to H-princ. bdl $H \to G \to G/H$, using $Ad_H \curvearrowright \mathfrak{m}$.

PDEs on M that are G-invariant become algebraic; e.g.,

Ricci flow $\frac{\partial g_t}{\partial t} = -2\,Ric(g_t)$ $\longleftrightarrow$ Evolution equation (ODE)

Fix $\langle\cdot,\cdot\rangle = g_0|_{T_{eH}G/H \times T_{eH}G/H}$
and let $P_t : \mathfrak{m} \to \mathfrak{m}$ be symm. automorphism s.t. $g_t(x,y) = g_0(P_t x, y)$

RF: $\frac{d}{dt} P_t = \dots$ ODE in P_t

If M is compact, then $G \subset \mathrm{Isom}(M,g)$ is compact, so it admits a bi-invariant metric $Q: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ s.t. L_g and R_g are isometries.

Can take $\mathfrak{m} = \mathfrak{h}^\perp$ w.r.t. Q; then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is Ad_H-inv.

Fact 1: (G,Q) has $\sec \geq 0$

Indeed, $\nabla_X Y = \frac{1}{2}[X,Y]$ $\forall X,Y \in \mathfrak{g}$, so $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$

$$\sec(X \wedge Y) = \frac{1}{4} \frac{\|[X,Y]\|^2}{\|X \wedge Y\|^2} \geq 0.$$

Fact 2: If $\pi: (M,g) \to (N,\check{g})$ is a Riem. submersion, then

$$\sec_N(X \wedge Y) \geq \sec_M(\overline{X} \wedge \overline{Y})$$

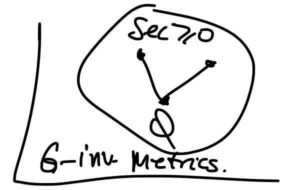
So: every compact homogeneous space G/H has $\sec \geq 0$.

Thm. The moduli space of G-inv. metrics on G/H with $\sec \geq 0$ is path-connected.

normal homog. metric.

Pf. In some sense, it is "starshaped" around $Q|_{\mathfrak{m} \times \mathfrak{m}}$.

(inverse linear path)



compact

Thm. G/H admits a G-inv. metric w/ $\mathrm{Ric} > 0 \iff |\pi_1(G/H)| < \infty$

Pf.

$(\Rightarrow)$ Bonnet-Myers

$(\Leftarrow)$ $\mathrm{Ric}(X,X) \geq 0$ $\forall X \in \mathfrak{m}$ w.r.t normal homog. metric $Q|_{\mathfrak{m} \times \mathfrak{m}}$.

$$= 0 \iff X \in \mathfrak{m} \cap \underbrace{Z(\mathfrak{g})}_{\text{center of Lie algebra.}} = \{0\}.$$

if $\pi_1(G/H)$ is finite.

$\square$

Sometimes can deform $\sec \geq 0$ to $\sec > 0$ among homogeneous metrics

rarely... "Cheeger deformation"

Thm (Berger, Wallach, Aloff-Wallach, Bérard-Bergery, Wilking-Ziller).
If $M^n = G/H$ is a compact homog. space with $\sec > 0$, and $\pi_1 M = \{1\}$, then it is isometric to:

1. CROSS: S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}aP^2$ (homog. metrics discussed last time!)
2. Wallach flag mfld: $W^6 = SU(3)/T^2$, $W^{12} = Sp(3)/Sp(1)Sp(1)Sp(1)$, $W^{24} = F_4/Spin(8)$
3. Aloff-Wallach space: $W^7_{k\ell} = SU(3)/S^1_{k,\ell}$, $W^7_{1,1} = SU(3)\,SO(3)/U(2)$
4. Berger space: $B^7 = SO(5)/SO(3)$, $B^{13} = SU(5)/S^1 \cdot Sp(2)$.

Note: If $\dim M \geq 25$, then M is a CROSS!

Note: Some (but not all!) homog. metrics on the above have $\sec > 0$, and finding out exactly which (modulo space) is not always easy; but has been done in most (all?) cases.

Thm (Hsiang-Hsiang '69). If $M^n = G/H$ compact homog. sp. is homeomorphic to S^n, then $M^n \underset{\text{diffeo}}{\cong} S^n$.

(i.e., no exotic spheres can be homogeneous!) but they can be "biquotients" or cohomogeneity one!

Def: $\operatorname{deg\,symm}(M) = \max\{\dim G : G \subset \operatorname{Diff}(M) \text{ compact subgroup}\}$.

e.g. $\operatorname{deg\,symm}(S^n) = \dfrac{n(n+1)}{2} = \dim O(n+1)$.

Namely, [HH'69] show that if Σ^n, $n \geq 40$, is an exotic sphere, then

$$\operatorname{deg\,symm}(\Sigma^n) \leq \frac{1}{8}(n^2+7)$$

smaller than dim of any gp acting transitively on n-dim sphere (see table from last lecture).

bound a parallelizable $(n+1)$-mfld.

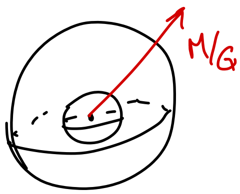
$= \iff \Sigma^n \in bP_{n+1}$ is a Kervaire sphere
$(n \equiv 1 \bmod 4)$

Improvements by [Straume '94], e.g., $\operatorname{degsymm}(\Sigma^n) = \frac{1}{8}(n^2-4n+11)$ if $n \equiv 3 \bmod 4$, and $\Sigma^n \in bP_{n+1}$
and $\operatorname{degsymm}(\Sigma^n) < \frac{3}{2}(n+1)$ if $\Sigma^n \notin bP_{n+1}$.

• <u>cohomogeneity 1</u>: $\dim M/G = 1 \implies M \underset{\text{isom}}{\cong} \underbrace{S^1, [0,L]}_{M \text{ compact}}, \underbrace{\mathbb{R}, [0,+\infty)}_{M \text{ noncompact}}$

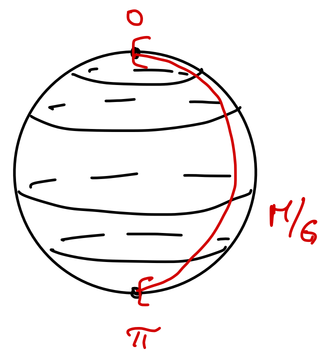
assume G is compact throughout!

e.g.: $M = \mathbb{R}^n$, $G = O(n)$, $M/G = [0,+\infty)$ "radial"



G-orbits are $G(p) = \{x \in \mathbb{R}^n : \|x\| = \|p\|\}$ round spheres (principal orbits)

$G(0) = \{0\}$ (singular orbit)



$M = S^n$, $G = O(n)$, $M/G = [0,\pi]$

G-orbits: parallels (principal orbit)

$\{\pm N\}$ (singular orbits)

In general, $\pi : M \to M/G$

principal orbits $\longleftrightarrow$ interior points

non-principal orbits $\longleftrightarrow$ boundary points

(singular or exceptional)

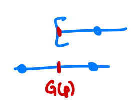
b/c of the "Slice Theorem"

$\mathrm{Tub}(G(p)) \cong G \times_{G_p} D_p$

slice

$\Sigma_{G(p)} = \partial D_p / G_p$

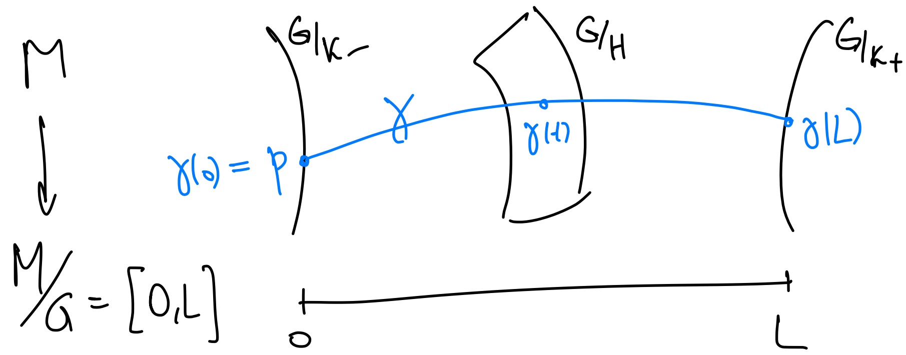
space of directions in M/G $\leftarrow$ in cohom 1 this is either 1 or 2 pts, according to being boundary or interior point in M/G



So if $M/G = S^1$ or $\mathbb{R}$, then <u>all orbits</u> are principal, i.e., all isotropies are conjugate, say to $H = G_p < G$, and hence M is the total space of a bundle

$$G/H \to M \to S^1 \quad \text{or} \quad G/H \to M \to \mathbb{R}$$

More interesting cases are $M/G = [0,L]$ or $[0,+\infty)$, where not all orbits are principal. We focus on the first case; the second is quite similar if one imagines $L \to +\infty$, so the orbit $\pi^{-1}(L)$ "disappears" at ∞.



Let $p \in \pi^{-1}(0)$, $K_- = G_p$, and $\gamma : [0,L] \to M$ be a horizontal geodesic w/ $\gamma(0) = p$. Then
$G_{\gamma(t)} = H$ if $t \in (0,L)$
$G_{\gamma(L)} = K_+$ if $t = L$.

<u>Riem. metric:</u> $g = dt^2 + g_t$, where $(g_t)_{t \in (0,L)}$ is a 1-param. family of G-inv. metrics on G/H.

<u>Prop:</u> The groups $H < K_\pm < G$ are such that $K_\pm / H = S^{d_\pm}$, and the "group diagram"

$$\begin{tikzcd} & G & \\ K_- \arrow[ur, dash] & & K_+ \arrow[ul, dash] \\ & H \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

<u>determines</u> M up to G-equiv. diffeom. (up to $a \in N(H)_0$, $b \in G$, and replacing $K_\pm, H$ with bK_-b^{-1}, bHb^{-1}, $abK_+b^{-1}a^{-1}$). Conversely, given $H < K_\pm < G$ w/ $\underbrace{K_\pm/H = S^{d_\pm}}$, there exists a cohom 1 mfld given by

we know the possible $H, K_\pm$ (if <u>connected</u>)

$$M = \left(G \times_{K_-} D^{d_- + 1}\right) \cup_{G/H} \left(G \times_{K_+} D^{d_+ + 1}\right)$$

whose gp. diagram is as above.

Just like for homog. sp. G/H, the gp. diagram $H < K_\pm < G$ can be used to compute the topology of M; e.g., $H^*(M, \mathbb{F})$ etc.

e.g., $\chi(M) = \chi(G/K_-) + \chi(G/K_+) - \chi(G/H)$.

<u>Exercise</u> (Hopf–Samelson Thm). $\chi(G/H) \geq 0$ for any compact homog. sp. and $> 0 \iff \operatorname{rk} H = \operatorname{rk} G$

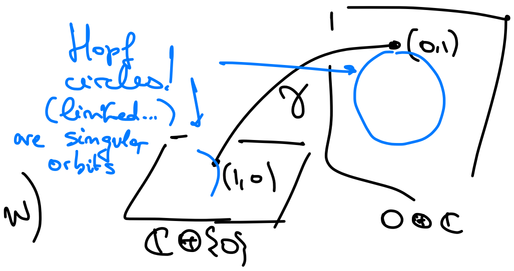
<u>Ex:</u> $SO(n) \curvearrowright S^n$

$$\begin{tikzcd} & SO(n) & \\ SO(n) \arrow[ur, dash] & & SO(n) \arrow[ul, dash] \\ & SO(n-1) \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

$K_\pm = G$ (sing. orbits are <u>fixed</u> pts!)

$g_{S^n} = dt^2 + \underbrace{\sin^2 t \cdot g_{S^{n-1}}}_{g_t =}$ unit round metric

$\underline{\text{Ex:}}$ $T^2 \curvearrowright S^3 \subset \mathbb{C}^2$



$$\begin{tikzcd} & T^2 \arrow[dl, dash] \arrow[dr, dash] & \\ \{1\}\times S^1 \arrow[dr, dash] & & S^1\times\{1\} \arrow[dl, dash] \\ & \{1\} & \end{tikzcd}$$

$$(e^{i\theta}, e^{i\varphi})\cdot(z,w) = (e^{i\theta}\cdot z, e^{i\varphi} w)$$

$$\gamma(t) = (\cos t, \sin t) \qquad K_- = G(\gamma(0)) = \{1\}\times S^1$$
$$K_+ = G(\gamma(\pi/2)) = S^1\times\{1\}$$

unit round metric:
$$g = dt^2 + \underbrace{\sin^2 t\, dx_1^2 + \cos^2 t\, dx_2^2}_{g_t}$$

$\underline{\text{Ex:}}$ Brieskhorn variety $M_d^{2n-1} \subset \mathbb{C}^{n+1}$ defined by

$$\begin{cases} z_0^d + z_1^2 + \dots + z_n^2 = 0 \\ |z_0|^2 + |z_1|^2 + \dots + |z_n|^2 = 1 \end{cases}$$

$$n, d \text{ odd} \implies M_d^{2n-1} \underset{\text{homeo}}{\cong} S^{2n-1}$$

$$2n-1 \equiv 1 \bmod 8 \implies M_d^{2n-1} \underset{\text{diff}}{\not\cong} S^{2n-1}$$ is an <u>exotic</u> sphere!
(Kervaire sphere)

<u>Calabi, Hsiang–Hsiang</u>: $G = SO(2)\cdot SO(n) \curvearrowright M_d^{2n-1}$ cohom 1 action:

$$(e^{i\theta}, A)\cdot(z_0, z_1, \dots, z_n) = \left(e^{2i\theta} z_0, e^{id\theta} A(z_1, \dots, z_n)^t\right)$$

principal isotropy: $H = \mathbb{Z}_2 \times SO(n-2) = \begin{cases} (\varepsilon, \mathrm{diag}(\varepsilon, \varepsilon, A)), & d \text{ odd} \\ (\varepsilon, \mathrm{diag}(1,1,A)), & d \text{ even} \end{cases}$

where $\varepsilon = \pm 1$, $A \in SO(n-2)$

thus $G/H \underset{\text{diff}}{\cong} S^1 \times T_1 S^{n-1}$, where $T_1 S^{n-1} = \dfrac{SO(n)}{SO(n-2)}$ is the unit tangent bundle of S^{n-1}.

singular orbits are: $K_- = SO(2)\, SO(n-2) = \left(e^{-i\theta}, \mathrm{diag}(R(d\theta), A)\right)$

$$K_+ = \begin{cases} O(n-1) = (\det B, \mathrm{diag}(\det B, B)) & d \text{ odd} \\ \mathbb{Z}_2 \times SO(n-1) = (\varepsilon, \mathrm{diag}(1, B')) & d \text{ even} \end{cases}$$
$\varepsilon = \pm 1$, $B \in O(n-1)$, $B' \in SO(n-1)$.

Thm (Grove–Ziller, 2002). A compact cohom. 1 mfld (M,g) has an invariant metric with $Ric>0$ iff $\pi_1(M)$ is finite.

Thm (Verdiani, Grove–Wilking–Ziller, Verdiani–Ziller). Apart from CROSSes, compact simply-connected cohom 1 mflds with $sec>0$ are equiv. diffeom. to:

1. Berger space $B^7 = SO(5)/SO(3)$
2. Eschenburg spaces $E^7_p = SU(3)//S^1_p$
3. Bazaikin spaces $B^{13}_p = SU(5)//S^1_p \cdot Sp(2)$

these are defined as biquotients, but also admit $sec>0$ invariant under cohom 1 action

4. Candidates P^7_k, Q^7_k

aside from P_1, P_2, Q_1, these are infinite families currently not known to have $sec>0$ invariant under cohom 1 action. Also not known if some Q_k's are diffeom. to Eschenburg spaces (?)

Note: $P_2 \underset{\text{homeo}}{\cong} T_1 S^4$ but not diffeomorphic to it: "exotic $T_1 S^4$"!

What about cohom >1 for $sec>0$?

Thm. (Wilking, 2006). If (M^n,g) has $sec>0$ and cohom $=k$, with $n>18(k+1)^2$, then M is homotopy equivalent to a CROSS.

i.e. new examples w/ large dim. can only occur w/ large cohom:
if not h.e. to CROSS, then $n \leq 18(k+1)^2$.

PDEs in Cohom 1: PDE in 1 "space" variable

very useful framework to generalize results about "radial" solutions

e.g., Ricci flow becomes PDE in $\{t, r\}$.

t: time of flow
r: cohom 1 variable

$\frac{\partial}{\partial t} g_t = -2Ric_{g_t} \rightsquigarrow$ PDE on components of cohom 1 metric...

need to argue that Ansatz is indeed preserved; e.g., γ remains horizontal geod. etc.

cf. [B.–Krishnan] in 4-dim. case.