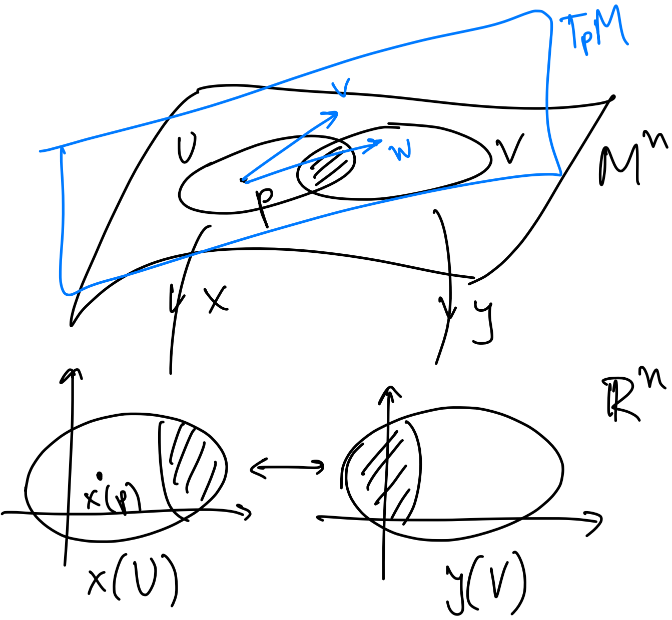
LECTURE NOTES FOR MATH 82000
COMPARISON GEOMETRY

Lecture 1 Review of Riemannian Geometry 1/26/2023



Def: A <u>Riemannian metric</u> g on a smooth manifold M^n is a (smoothly varying) inner product on the tangent spaces of M:

$$\forall p \in M, \quad g_p : T_pM \times T_pM \to \mathbb{R}$$

$$g_p(v,w) = g_p(w,v) \qquad \forall v, w \in T_pM$$

$$g_p(v,v) \geq 0 \qquad \forall v, w \in T_pM$$

$$g_p(v,v) = 0 \iff v = 0$$

and if X_p, Y_p are smooth vector fields, $p \mapsto g_p(X_p, Y_p)$ is smooth;

($X, Y \in \mathfrak{X}(M)$ sections of $TM \to M$.)

(symmetric bilinear forms)

i.e., g is a section of the bundle $\underline{Sym^2 TM} \to M$ whose image is contained in the open subset $Sym^2_{>0} TM \subset Sym^2 TM$ of positive-definite symmetric bilinear forms. We call (M^n, g) a <u>Riemannian manifold</u>.

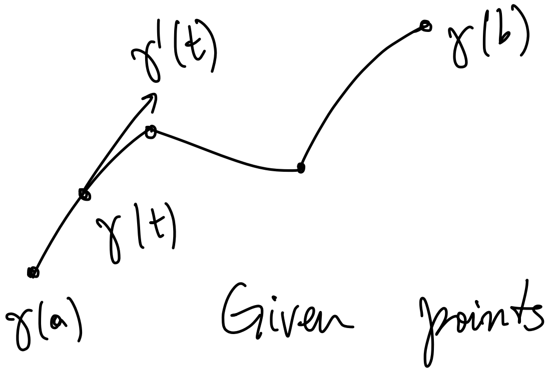
$$\boxed{\text{Prop: Every smooth manifold admits (many) Riemannian metrics.}}$$

Pf: Choose an atlas $\{x_\alpha : U_\alpha \to x_\alpha(U_\alpha)\}$ and a subordinate partition of unity $\rho_\alpha : U_\alpha \longrightarrow [0,1]$. On each $x_\alpha(U_\alpha)$, use either the Euclidean inner product $g^{(\alpha)}$, or, more generally, any inner product $g^{(\alpha)}(e_i, e_j) = \delta_{ij} + f_{ij}$, where f_{ij} are sufficiently small and $f_{ij} = f_{ji}$. Then set

$\left(f_{ij} : x_\alpha(U_\alpha) \to \mathbb{R},\ 1 \le i,j \le n \right)$

$$g(v,w) = \sum_\alpha \rho_\alpha\, g^{(\alpha)}\big(dx_\alpha(v), dx_\alpha(w)\big). \qquad \square$$

Def: Let $\gamma : [a,b] \to (M^n, g)$ be a piecewise smooth curve. The length of γ (w.r.t. g) is defined as



$$L_g(\gamma) = \int_a^b \underbrace{g_{\gamma(t)}\big(\gamma'(t), \gamma'(t)\big)^{1/2}}_{= \|\gamma'(t)\|}\, dt$$

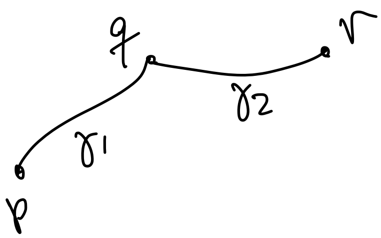
Given points $p, q \in M$, the <u>distance</u> (w.r.t. g) between p and q is defined as

$$\mathrm{dist}_g(p,q) = \inf \left\{ L_g(\gamma) : \begin{array}{l} \gamma : [a,b] \to M \text{ piecewise smooth} \\ \text{w/ } \gamma(a) = p,\ \gamma(b) = q \end{array} \right\}.$$

Prop. If (M^n, g) is a Riemannian manifold, then (M^n, dist_g) is a metric space, and the metric topology agrees with the manifold topology.

Pf: Clearly dist_g is nonnegative and symmetric.

For the triangle inequality, if γ_1 and γ_2 are curves with endpoints p, q and q, r, then define $\gamma_1 * \gamma_2$ by concatenating.



Clearly, $L_g(\gamma_1 * \gamma_2) = L_g(\gamma_1) + L_g(\gamma_2)$.

Suppose now γ_1 and γ_2 are such that

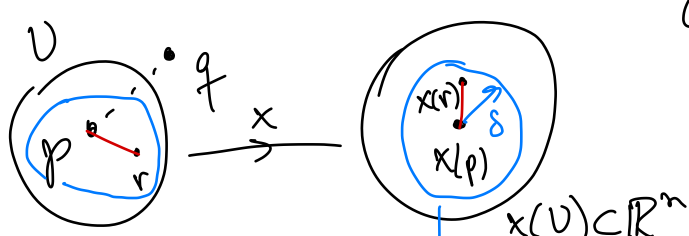
$$L_g(\gamma_1) < \mathrm{dist}_g(p,q) + \varepsilon$$
$$L_g(\gamma_2) < \mathrm{dist}_g(q,r) + \varepsilon$$

Then

$$\mathrm{dist}_g(p,r) \leq L_g(\gamma_1 * \gamma_2) = L_g(\gamma_1) + L_g(\gamma_2) < \mathrm{dist}_g(p,q) + \mathrm{dist}_g(q,r) + 2\varepsilon.$$

Letting $\varepsilon \to 0$ gives the triangle inequality.

Suppose $p, q \in M$ have $\mathrm{dist}_g(p,q) = 0$ but $p \neq q$. Then choose a chart $x : U \to x(U) \subset \mathbb{R}^n$ around $p \in M$. There exist $\delta > 0$ and $C > 0$ s.t.



$\mathrm{dist}_g(p,q) = 0$

Euclidean (open) ball of radius $\delta > 0$.

$$B_\delta(x(p)) \subset x(U) \quad \text{and}$$

$$g(v,v) \geq C^2 \| dx(v) \|^2,$$

for all $v \in T_r M$, $r \in x^{-1}(B_\delta(x(p)))$. Thus, for all such r, $\mathrm{dist}_g(p,r) \geq C \| x(p) - x(r) \|$.

So $q \notin x^{-1}(B_\delta(x(p)))$ and hence any curve from p to q must cross $x^{-1}(\partial B_\delta(x(p)))$ and thus have length $\geq C\cdot\delta$, contradicting $\mathrm{dist}_g(p,q)=0$.

Similarly, the topologies agree: because dist_g restricted to small charts is comparable to the Euclidean distance, open (metric) balls

$$B_\delta(p)=\{r\in M : \mathrm{dist}_g(p,r)<\delta\}$$

form a base for the (manifold) topology of M. $\square$

<u>Natural questions</u>: How does the Riemannian structure of (M^n,g) capture <u>completeness</u> of the metric space (M,dist_g)? When is the inf in $L_g(\gamma)$ attained by a curve?

<u>A</u>: Hopf–Rinow Theorem, coming soon.

<u>Levi-Civita connection</u>

<u>Def</u>: A connection (or covariant derivative) on the tangent bundle TM of a smooth manifold M is a map

$$\nabla : \mathfrak{X}(M)\times\mathfrak{X}(M)\longrightarrow\mathfrak{X}(M) \quad \text{satisfying}$$

1) $\nabla_{\varphi X+\psi Y} Z = \varphi\nabla_X Z+\psi\nabla_Y Z$ (C^∞-bilinear in $\nabla_{(\cdot)}$)

2) $\nabla_X(\varphi Y+\psi Z)=\underset{\sim}{X(\varphi)}\,Y+\varphi\cdot\nabla_X Y+X(\psi)Z+\psi\cdot\nabla_X Z$

$X(\varphi)=d\varphi(X)$ ($\mathbb{R}$-bilinear in $\nabla(\cdot)$ & Leibniz rule)

Theorem (Levi–Civita). Given a Riemannian manifold (M^n, g), there exists a unique connection on TM such that

(3) $\nabla_X Y - \nabla_Y X = [X,Y]$ (torsion-free)

Lie bracket: $[X,Y]\varphi = X(Y(\varphi)) - Y(X(\varphi))$

(4) $X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ (compatible with g, or $\nabla g = 0$)

Proof: First, using partitions of unity, show that there exist connections on TM for any smooth manifold M.

Suppose ∇ is a connection on TM satisfying (3), (4).

Then

$$X\, g(Y,Z) = g(\nabla_X Y, Z) + \underline{g(Y, \nabla_X Z)}$$

$$Y\, g(Z,X) = \underline{g(\nabla_Y Z, X)} + g(Z, \nabla_Y X)$$

$$Z\, g(X,Y) = \underline{g(\nabla_Z X, Y)} + \underline{g(X, \nabla_Z Y)}$$

So

$$X\, g(Y,Z) + Y\, g(Z,X) - Z\, g(X,Y) = \underline{g([X,Z],Y)} + \underline{g([Y,Z],X)} + g([X,Y],Z) + 2g(\nabla_Y X, Z)$$

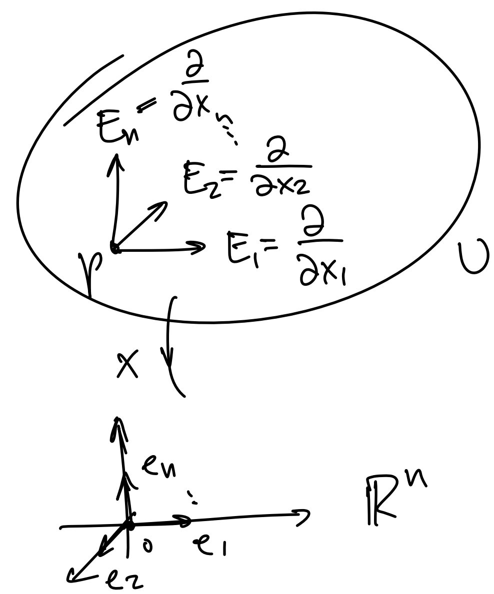
Thus

$$g(\nabla_Y X, Z) = \frac{1}{2}\Big(X\, g(Y,Z) + Y\, g(Z,X) - Z\, g(X,Y) - g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)\Big).$$

"Koszul Formula"

The above uniquely defines ∇. $\square$

<u>Christoffel Symbols</u> ← We'll try to avoid using them, but good to know what they are...



On a chart $x : U \to x(U)$, $U \subset M$, $x(U) \subset \mathbb{R}^n$, we have coordinate functions

$$x = (x_1, \dots, x_n) : U \to x(U) \subset \mathbb{R}^n$$

and coordinate vector fields $E_i = \frac{\partial}{\partial x_i}$

Recall:

$$T_pM = \text{span}\{E_i(p),\ i = 1, \dots, n\}.$$

$$\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k$$

$\Gamma_{ij}^k : U \to \mathbb{R}$ are called Christoffel symbols of ∇.

Setting $Y = E_i$ and $X = E_j$ in Koszul formula and solving for each $Z = E_k$, we find:

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km}$$

where $g_{ij} = g(E_i, E_j)$, and

(g^{km}) is the inverse matrix to (g_{ij})

<u>Note:</u> $[E_i, E_j] = 0$, $\forall i, j$ so last 3 terms vanish.

Clearly, ∇ determines Γ_{ij}^{k} and also vice-versa:

$X = \sum_i a_i E_i$, $Y = \sum_j b_j E_j$ on a chart $U \ni p$

$$\nabla_X Y = \sum_{i,j} a_i \nabla_{E_i}(b_j E_j) = \sum_{i,j} a_i E_i(b_j) E_j + a_i b_j \nabla_{E_i} E_j$$

$$= \sum_{i,j} a_i \frac{\partial}{\partial x_i} b_j \, E_j + a_i b_j \sum_k \Gamma_{ij}^{k} E_k$$

only need values at the point ("tensorial" in this entry)

need to know nearby values to compute derivative!

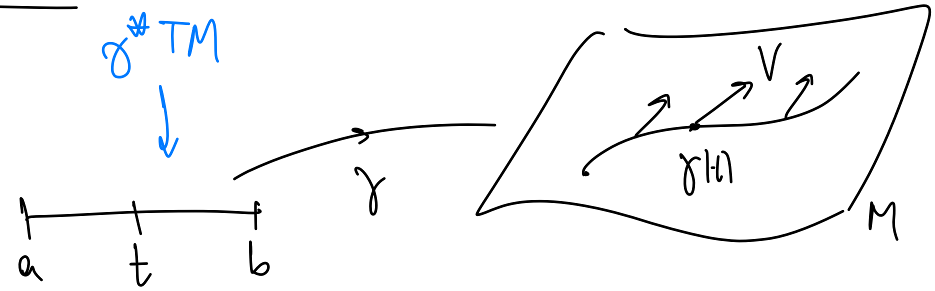
Note: The above depends on values of X only at p, but of Y on a neighborhood of p.

Vector fields along a curve

$\gamma : [a,b] \to M$

$V : [a,b] \to TM$

such that



$V_{\gamma(t)} \in T_{\gamma(t)} M$, $\forall t \in [a,b]$; i.e., V is a section of $\gamma^* TM$.

$V' := \nabla_{\gamma'} V$ is defined locally extending V.

No need to extend γ' b/c of note above!

often we write $\frac{DV}{dt}$

formally, ∇ induces a connection on $\gamma^* TM$.

e.g., in coordinates:

$$V = \sum_j V_j(t) E_j, \quad \dot\gamma(t) = \sum \dot\gamma_i(t) E_i$$

$$V' := \sum_j \frac{dV_j}{dt} E_j + \sum_{i,j,k} \dot\gamma_i(t) V_j(t) \Gamma_{ij}^{k}(\gamma(t)) E_k.$$

Def: The vector field V along $\gamma(t)$ is <u>parallel</u> if $V'(t) = 0$.

Def: A <u>geodesic</u> is a curve $\gamma(t)$ such that $\dot\gamma(t)$ is parallel;

equivalently, if $\dot\gamma = \sum_i \gamma_i(t) E_i$,

$$\frac{D\dot\gamma}{dt} = \sum_i \gamma_i''(t)\, E_i + \sum_{j,k} \gamma_i'(t)\, \gamma_j'(t)\, \Gamma_{ij}^k(\gamma(t))\, E_k = 0.$$

i.e., $\forall i$, $\gamma_i'' + \sum_{j,k} \gamma_j' \gamma_k' \Gamma_{jk}^i = 0$. "Geodesic ODE"

(System of n coupled 2^{nd} order nonlinear ODEs)

Immediate consequence of basic ODE theory:

> <u>Thm.</u> On a Riemannian manifold (M^n, g), given $p \in M$ and $v \in T_pM$, there exists a unique maximal geodesic $\gamma : (T_-, T_+) \to M$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Moreover, such γ depends smoothly on its initial conditions $(p, v) \in TM$.

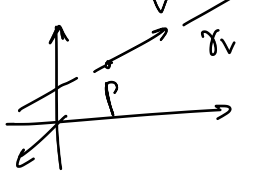
<u>Prop:</u> If $\gamma : I \to M$ is a geodesic, then $\|\dot\gamma\| = \text{const}$.

<u>Pf:</u> $\frac{d}{dt}\|\dot\gamma(t)\|^2 = \frac{d}{dt}\langle \dot\gamma(t), \dot\gamma(t)\rangle = 2\langle \underbrace{\nabla_{\dot\gamma}\dot\gamma}_{=0}, \dot\gamma\rangle = 0.$ $\square$

<u>Examples of geodesics:</u>

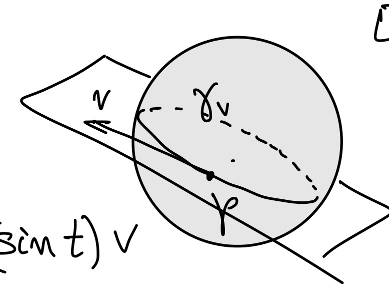
$\mathbb{R}^n$ straight lines

$\gamma_v(t) = p + tv$



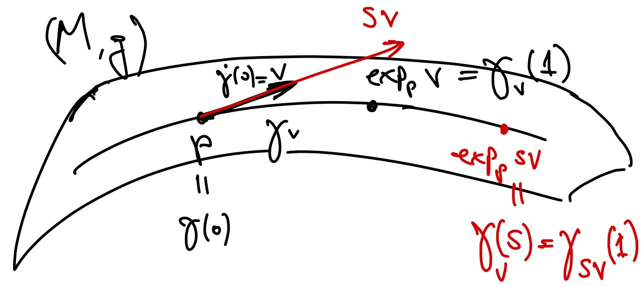
S^n great circles

$\gamma_v(t) = (\cos t)\, p + (\sin t)\, v$



Lecture 2   Exponential map, completeness, etc.   2/2/2023

Let (M,g) be a Riem. mfld; for every $v \in T_pM$, let $\gamma_v : (T_-, T_+) \longrightarrow M$ be the unique max. geodesic on M with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = v$.



Note: By uniqueness, for $|t|$, $|s|$ small,

$$\gamma_{sv}(t) = \gamma_v(st)$$

Def: The (Riem.) exponential map at $p \in M$ is

$$\exp_p : \mathcal{O}_p \subset T_pM \longrightarrow M$$
$$v \longmapsto \gamma_v(1)$$

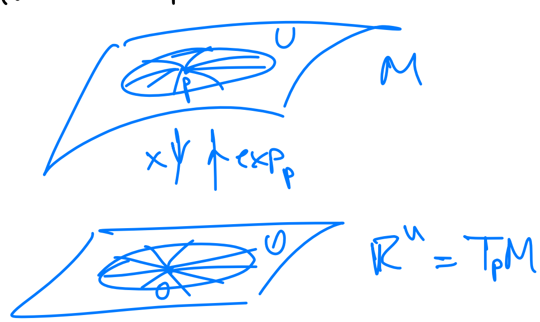
where $\mathcal{O}_p$ is the open neighborhood of $0 \in T_pM$ s.t. $\gamma_v(t)$ is defined up to $t = 1$ whenever $v \in \mathcal{O}_p$.

Prop: $d(\exp_p)_0 v = v$ for all $v \in T_pM = T_0(T_pM)$, or, in short, $d(\exp_p)_0 = \mathrm{id}$.

By the Inverse Function Theorem.

In particular, there are open subsets $\mathcal{O} \subset T_pM$ and $U \subset M$, with $0 \in \mathcal{O}$ and $p \in U$, s.t. $\exp_p|_{\mathcal{O}} : \mathcal{O} \to U$ is a diffeomorphism.

So $(\exp_p|_{\mathcal{O}})^{-1} : U \to \mathbb{R}^n$ defines a local chart, call these "geodesic normal coordinates"



Pf: $d(\exp_p)_0 v = \frac{d}{dt}(\exp_p)(tv)\Big|_{t=0} = \frac{d}{dt}\gamma_{tv}(1)\Big|_{t=0}$

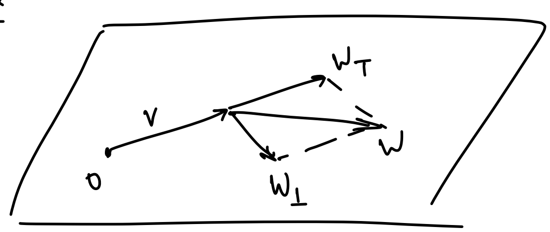
Chain rule; $df_p v = \frac{d}{dt} f(\gamma_v(t))\Big|_{t=0}$

$= \frac{d}{dt}\gamma_v(t)\Big|_{t=0} = \dot\gamma_v(0) = v.$ $\square$

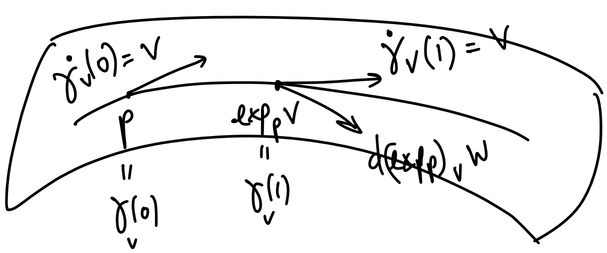
Gauss Lemma: $\exp_p$ is a radial isometry; more precisely

$$\langle d(\exp_p)_v v,\ d(\exp_p)_v w\rangle = \langle v, w\rangle, \quad \forall v, w \in T_pM = T_vT_pM$$

Pf:



$\exp_p \downarrow$ $T_pM = T_vT_pM$



Write $w = w_T + w_\perp$, where $\begin{cases} w_T = \alpha v. \\ \langle w_\perp, v\rangle = 0 \end{cases}$

Clearly,

$$d(\exp_p)_v v = \frac{d}{dt}(\exp_p)((t+1)v)\Big|_{t=0}$$

$$= \frac{d}{dt}(\exp_p)(tv)\Big|_{t=1}$$

$$= \frac{d}{dt}\gamma_v(t)\Big|_{t=1} = \dot\gamma_v(1) = \underbrace{P_p^{\gamma_v(1)}(v)}_{\text{parallel transport of } v\in T_pM \text{ along } \gamma_v \text{ to } \gamma_v(1).}.$$

$P_p^{\gamma_v(1)}: T_pM \to T_{\gamma_v(1)}M$

Thus

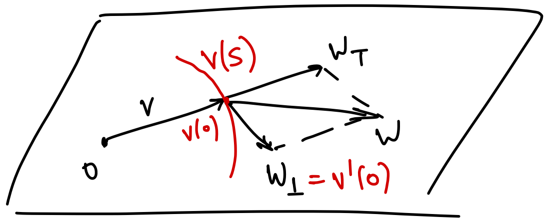
$$\langle d(\exp_p)_v v,\ d(\exp_p)_v w\rangle = \langle d(\exp_p)_v v,\ d(\exp_p)_v(\alpha v)\rangle + \langle d(\exp_p)_v v,\ d(\exp_p)_v w_\perp\rangle$$

$$= \alpha\langle P_p^{\gamma_v(1)} v,\ P_p^{\gamma_v(1)} v\rangle + \langle d(\exp_p)_v v,\ d(\exp_p)_v w_\perp\rangle$$

$$= \langle v, \underbrace{\alpha v}_{w_T}\rangle + \langle d(\exp_p)_v v,\ d(\exp_p)_v w_\perp\rangle$$

$$= \langle v, w\rangle + \langle d(\exp_p)_v v,\ d(\exp_p)_v w_\perp\rangle$$

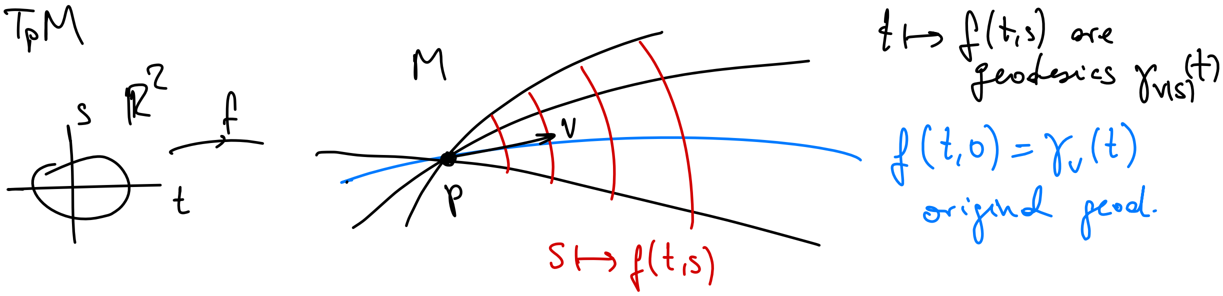
So we must show $\langle d(\exp_p)_v v, \ d(\exp_p)_v w_\perp \rangle = 0$.



$T_pM = T_v T_p M$

Let $v(s) = (\cos s)\, v + (\sin s)\, w_\perp$ so $\begin{cases} v(0) = v \\ v'(0) = w_\perp \\ \|v(s)\| = \text{const.} \end{cases}$

and $f(t,s) = \exp_p(t\, v(s)) = \gamma_{v(s)}(t)$



$$\left.\begin{aligned} d(\exp_p)_v\, v &= \frac{\partial}{\partial t} \exp_p(t v(s)) \Big|_{\substack{t=1 \\ s=0}} = \frac{\partial f}{\partial t}(1,0) \\ d(\exp_p)_v\, w_\perp &= \frac{\partial}{\partial s} \exp_p(t v(s)) \Big|_{\substack{t=1 \\ s=0}} = \frac{\partial f}{\partial s}(1,0) \end{aligned}\right\} \Rightarrow \langle d(\exp_p)_v v, \ d(\exp_p)_v w_\perp \rangle = \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle (1,0).$$

Compute:

$$\frac{\partial}{\partial t} \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle = \langle \underbrace{\nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial t}}_{\substack{\| \\ 0}}, \frac{\partial f}{\partial s} \rangle + \left\langle \frac{\partial f}{\partial t}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial f}{\partial s} \right\rangle = \left\langle \frac{\partial f}{\partial t}, \nabla_{\partial/\partial s} \frac{\partial f}{\partial t} \right\rangle$$

metric compatibility of ∇

b/c $t \mapsto f(t,s) = \gamma_{v(s)}(t)$ are geodesics.

$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0$

and $\|\dot\gamma_{v(s)}(t)\| = \|\dot\gamma_{v(s)}(0)\| = \|v(s)\| = \text{const.}$

$$= \frac{1}{2} \frac{\partial}{\partial s} \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle = 0$$

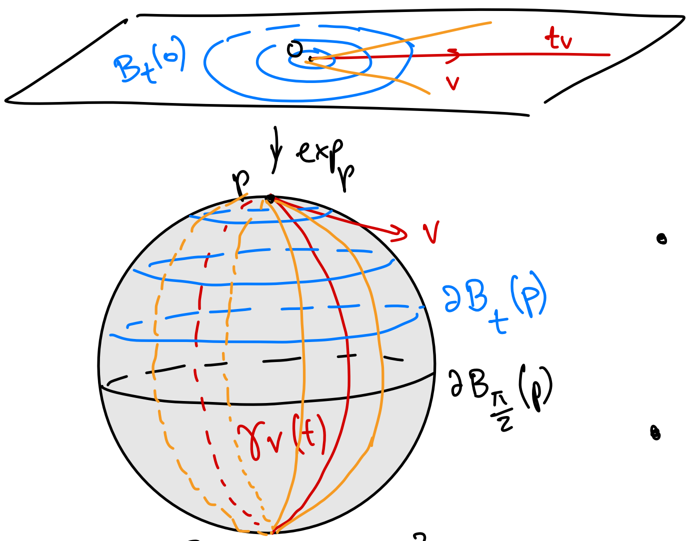
Therefore $t \mapsto \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle (t,0)$ is constant, and, computing at $t = 0$:

$$\frac{\partial f}{\partial s}(t,0) = \frac{\partial}{\partial s} (\exp_p)(t v(s)) \Big|_{s=0} = d(\exp_p)(t \underbrace{v(0)}_{v})(t \underbrace{v'(0)}_{w_\perp}) = d(\exp_p)_{tv}\, t w_\perp$$

$$\lim_{t \to 0} \frac{\partial f}{\partial s}(t,0) = \lim_{t \to 0} d(\exp_p)_{tv}\, t w_\perp = 0, \quad \text{so } \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle (1,0) = 0.$$

$\square$

Example: $S^n(1)$ unit round sphere, $\exp_p : T_pS^n \longrightarrow S^n$



$$\exp_p(B_r(0)) = B_r(p), \quad \forall r \in (0,\pi)$$

injectivity radius of $S^n(1)$ is π.

- $\exp_p\big|_{B_\pi(0)} : B_\pi(0) \longrightarrow S^n \setminus \{-p\}$ is a diffeomorphism
- Geodesic spheres / distance spheres have area distortion
  $\partial B_t(p)$ is a sphere of radius $\sin t$.
  $\partial B_t(0)$ is a sphere of radius t.

$$g = dt^2 + \sin^2 t \, d\theta^2$$

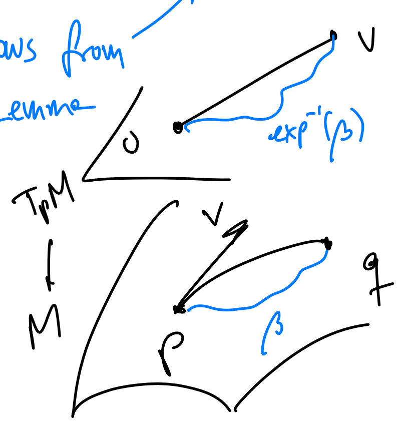
i.e. $\begin{pmatrix} 1 & 0 \\ 0 & \sin^2 t \end{pmatrix}$ on $\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta} \right\}$.

Rmk: In general, largest $r > 0$ s.t. $\exp_p\big|_{B_r(0)} : B_r(0) \to M$ is a diffeom. onto its image is called injectivity radius at p; denoted $\mathrm{inj}_p(M)$. Injectivity radius of (M,g) is $\mathrm{inj}(M) = \inf_p \mathrm{inj}_p(M)$.

If $r < \mathrm{inj}_p(M)$, then $B_r(p) = \exp_p(B_r(0))$; and for all $v \in B_r(0) \subset T_pM$, the geodesic $[0,1] \ni t \mapsto \exp_p(tv) \in B_r(p)$ is the unique minimizing geodesic from p to $q = \exp_p v$.

i.e. $L_g(\gamma) = \mathrm{dist}_g(p,q)$.

Pf. follows from Gauss Lemma



"cut locus"

More generally, $\mathrm{Cut}(p) \subset M$ is the image via $\exp_p$ of the set of $v \in T_pM$ s.t. $\exp_p(tv)$ is minimizing for $t \in [0,1]$ but not for $t = 1+\varepsilon$, for any $\varepsilon > 0$.

So $\mathrm{inj}_p(M) = \mathrm{dist}_g(p, \mathrm{Cut}(p))$. $\leftarrow$ $\mathrm{dist}(p,C) = \inf\{\mathrm{dist}(p,x) : x \in C\}$ as usual.

Also using Gauss Lemma, among other things, one proves:

published in Comm. Math. Helvetici!

Thm (Hopf-Rinow '1931). Let (M,g) be a connected Riem. mfld. TFAE:

(i) $\exists\, p_0 \in M$ s.t. $\exp_{p_0}$ is defined on all of $T_{p_0}M$

(ii) $\forall p \in M$, $\exp_p$ is defined on all of T_pM $\Big\{$ M is "geodesically complete", i.e. all geodesics can be extended to $(-\infty, +\infty)$

(iii) $K \subset M$ closed and bounded $\Rightarrow K$ compact ("Heine-Borel property")

(iv) (M, dist_g) is a complete metric space (i.e. Cauchy seq. converge.)

If any, hence all, of the above holds, then given any $p, q \in M$, there exists a minimizing geodesic γ from p to q, i.e., $L_g(\gamma) = \mathrm{dist}_g(p,q)$.

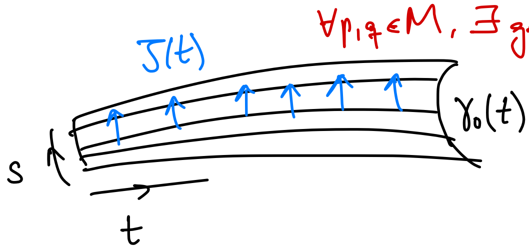
Q: Why isn't this also equivalent to (i)-(iv)?
A: E.g., $M = B_r(0) \subset \mathbb{R}^n$ is not complete, but $\forall p,q \in M$, $\exists$ geod. $\overline{pq}$.

Variations of geodesics & Jacobi fields

Consider a variation of geodesics

$$(-\varepsilon,\varepsilon)\times(T_-,T_+) \ni (s,t) \longmapsto \gamma(s,t) = \gamma_s(t) \in M$$

$t \longmapsto \gamma_s(t)$ is a geodesic, $\forall s \in (-\varepsilon,\varepsilon)$.



Def: The variational field $J(t) = \dfrac{d}{ds}\gamma_s(t)\Big|_{s=0}$ along $\gamma_0(t)$ is called a Jacobi field.

Prop: A vector field J along a geodesic γ is a Jacobi field if and only if it satisfies the Jacobi equation

2nd order linear ODE

$$J'' + R(J,\dot\gamma)\dot\gamma = 0,$$ where R is the curvature tensor: (more on this soon!)

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Pf: ($\Rightarrow$) If $J(t) = \frac{d}{ds}\gamma_s(t)\Big|_{s=0}$ where $\gamma_s(t)$ is a variation by geodesics, then

$$J''(t) = \frac{D^2 J}{dt^2} = \frac{D}{dt}\frac{D}{dt}\frac{d}{ds}\gamma_s(t) = \frac{D}{dt}\frac{D}{ds}\underbrace{\frac{d}{dt}\gamma_s(t)}_{\dot\gamma_s(t)}$$

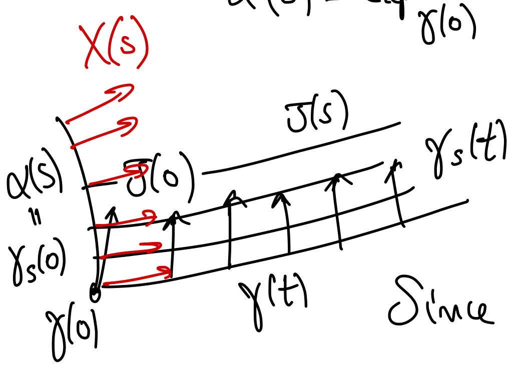
$$= \frac{D}{ds}\underbrace{\frac{D}{dt}\dot\gamma_s(t)}_{=0 \text{ b/c } \gamma_s(t) \text{ is geod.}} + R(\dot\gamma, J)\dot\gamma$$

(indeed: $R(\dot\gamma, J)\dot\gamma = \nabla_{\dot\gamma}\nabla_J\dot\gamma - \nabla_J\nabla_{\dot\gamma}\dot\gamma - \nabla_{\underbrace{[\dot\gamma, J]}_{}}\dot\gamma$

$$= \frac{D}{dt}\frac{D}{ds}\dot\gamma - \frac{D}{ds}\frac{D}{dt}\dot\gamma \qquad \left[\tfrac{\partial}{\partial s}, \tfrac{\partial}{\partial t}\right] = 0 \quad)$$

so $J'' + R(J, \dot\gamma)\dot\gamma = 0$ b/c $R(X,Y)Z = -R(Y,X)Z$.

($\Leftarrow$) If J satisfies $J'' + R(J, \dot\gamma)\dot\gamma = 0$, then let $\alpha(s) = \exp_{\gamma(0)} sJ(0)$ and let $X(s)$ be a vector field along $\alpha(s)$ with $X(0) = \dot\gamma(0)$, $X'(0) = J(0)$.



Set $\gamma_s(t) = \exp_{\alpha(s)} tX(s)$.

Since $t \mapsto \gamma_s(t)$ are geodesics, by ($\Rightarrow$), the vector field $\tilde J(t) = \frac{d}{ds}\gamma_s(t)\Big|_{s=0}$ satisfies $\tilde J'' + R(\tilde J, \dot\gamma)\dot\gamma = 0$.

Moreover, $\tilde J(0) = \frac{d}{ds}\gamma_s(0)\Big|_{s=0} = \alpha'(0) = J(0)$ and

$$\tilde{J}'(0) = \frac{D}{dt}\frac{d}{ds}\gamma_s(t)\Big|_{\substack{s=0\\t=0}} = \frac{D}{ds}\frac{d}{dt}\gamma_s(t)\Big|_{\substack{s=0\\t=0}} = \frac{D}{ds}X(s)\Big|_{s=0}$$

$$= X'(0) = J(0).$$

So $J(t) = \tilde{J}(t) = \frac{d}{ds}\gamma_s(t)\big|_{s=0}$ for all t by uniqueness of sol. to ODE w/ same initial conditions; hence J is the variational field of the family of geodesics $\gamma_s(t)$. $\square$

<u>Rmk:</u> The Jacobi field along $\gamma_v(t)$ with $J(0)=0$ and $J'(0)=w$ is given by $J(t) = d(\exp_p)_{tv}\, tw$, cf. end of Pf. of Gauss Lemma.

<u>Next class:</u> Comparison geometry via Jacobi fields (Rauch)

Before that, let's explore further the <u>curvature tensor</u>

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

No standardized sign convention ... We'll use this one, opposite of do Carmo, same as Eschenburg, Petersen ...

<u>Prop:</u> $R : \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ is a tensor, i.e., $(R(X,Y)Z)_p$ only depends on X_p, Y_p, Z_p; and we may thus consider R as a section of $TM^*\otimes TM^*\otimes TM^*\otimes TM$.

"(3,1)-tensor"

<u>Pf:</u> Follows from the claims:

$(X,Y) \longmapsto R(X,Y)Z$ is $C^\infty(M)$-bilinear (and skew-symmetric)

$Z \longmapsto R(X,Y)Z$ is $C^\infty(M)$-linear. $\square$

In (dreaded) coordinates

$$R\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \qquad [\partial_i,\partial_j]=0$$

$$= \nabla_{\partial_i}\left(\sum_\ell \Gamma_{jk}^{\ell}\partial_\ell\right) - \nabla_j\left(\sum_\ell \Gamma_{ik}^{\ell}\partial_\ell\right)$$

$$= \sum_\ell \frac{\partial \Gamma_{jk}^{\ell}}{\partial x_i}\partial_\ell + \sum_p \Gamma_{jk}^{\ell}\Gamma_{i\ell}^{p}\partial_p$$

$\ell \longleftrightarrow p$

$$- \sum_\ell \frac{\partial \Gamma_{ik}^{\ell}}{\partial x_j}\partial_\ell - \sum_p \Gamma_{ik}^{\ell}\Gamma_{j\ell}^{p}\partial_p$$

$$= \sum_\ell \underbrace{\left(\frac{\partial \Gamma_{jk}^{\ell}}{\partial x_i} - \frac{\partial \Gamma_{ik}^{\ell}}{\partial x_j} + \Gamma_{jk}^{p}\Gamma_{ip}^{\ell} - \Gamma_{ik}^{p}\Gamma_{jp}^{\ell}\right)}_{R_{ijk}{}^{\ell}}\partial_\ell$$

$\longleftarrow$ these are functions!

So that $R(X,Y)Z = \sum R_{ijk}{}^{\ell}\, a_i b_j c_k\, \partial_\ell$

if $X = \sum a_i \frac{\partial}{\partial x_i}$, $Y = \sum b_j \frac{\partial}{\partial x_j}$, $Z = \sum c_k \frac{\partial}{\partial x_k}$.

2/9/2023

Lecture 3.

Recall: Curvature tensor $R: \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$.

"Lowering indices", we get a $(4,0)$ - tensor

$$R: \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) \longrightarrow C^\infty(M)$$

$$R(X,Y,Z,W) = \langle R(X,Y)Z, W\rangle = \langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\rangle$$

i.e. $R\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_\ell}\right) = R_{ijk\ell} = \sum_p R_{ijk}{}^{p} g_{p\ell}$

which has the following symmetries:

$$\underbrace{R(\overbrace{X,Y}^{}, \overbrace{Z,W}^{})}_{}$$

symm. — skew — skew

$$R(X,Y,Z,W) = R(Z,W,X,Y)$$

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Y,X,W,Z)$$

1st Bianchi identity: $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$

Def (Curvature operator):
By the above, we can also consider R as a symmetric bilinear map

$$R: \Lambda^2 TM \longrightarrow \Lambda^2 TM:$$
$$(X \wedge Y) \longmapsto R(X \wedge Y)$$

Recall $\Lambda^2 V = V \otimes V / \sim$

$v \otimes w \sim -w \otimes v$

$v \wedge w := [v \otimes w]$

$$\langle R(\underbrace{X \wedge Y}_{\text{skew}}), \underbrace{Z \wedge W}_{\text{skew}} \rangle = -\langle R(X,Y)Z, W \rangle = \langle R(X,Y)W, Z \rangle$$

symm.

b/c of our sign conventions...

Recall: $\langle X \wedge Y, Z \wedge W \rangle = \langle X,Z \rangle \langle Y,W \rangle - \langle X,W \rangle \langle Y,Z \rangle$

$$= \det \begin{pmatrix} \langle X,Z \rangle & \langle Y,Z \rangle \\ \langle X,W \rangle & \langle Y,W \rangle \end{pmatrix}.$$

In particular, $\|X \wedge Y\|^2 = \|X\|^2 \|Y\|^2 - \langle X,Y \rangle^2$

Def (Sectional curvature):

$$\sec(X,Y) = \frac{\langle R(X,Y)Y, X \rangle}{\|X\|^2 \|Y\|^2 - \langle X,Y \rangle^2} = \frac{\langle R(X \wedge Y), X \wedge Y \rangle}{\|X \wedge Y\|^2} = \sec(\sigma)$$

Prop: $\sec(X,Y)$ only depends on $\sigma = \operatorname{span}\{X,Y\} \subset T_pM$.

Pf: Any other basis is obtained by performing finitely many of the following operations:

a) $\{X,Y\} \to \{Y,X\}$

b) $\{X,Y\} \to \{\lambda X, Y\}$ $\quad \lambda \in \mathbb{R}$

c) $\{X,Y\} \to \{X+\lambda Y, Y\}$ $\quad \lambda \in \mathbb{R}$.

All the above clearly preserve $\sec(X,Y)$; e.g., (c):

$$\langle R(X+\lambda Y, Y)Y, X+\lambda Y\rangle = \langle R(X,Y)Y,X\rangle \text{ b/c } R(Y,Y)=0$$
$$\langle R(\cdot,\cdot)Y,Y\rangle = 0.$$

$$\|X+\lambda Y\|^2\|Y\|^2 - \langle X+\lambda Y, Y\rangle^2 = \left(\|X\|^2 + 2\lambda\langle X,Y\rangle + \lambda^2\|Y\|^2\right)\|Y\|^2 - \left(\langle X,Y\rangle + \lambda\|Y\|^2\right)^2$$
$$= \|X\|^2\|Y\|^2 - \langle X,Y\rangle^2.$$

(or, more elegantly, note: $\|(X+\lambda Y)\wedge Y\|^2 = \|X\wedge Y + \lambda \underbrace{Y\wedge Y}_{=0}\|^2 = \|X\wedge Y\|^2$.) $\square$

Rmk: Given $\sigma \subset T_pM$, let $\Sigma = \exp_p(\sigma)$. Then $\sec(\sigma) = K_\Sigma$.

Gaussian curvature

$$\boxed{\text{Prop: } R_p \text{ is determined by } \sec : Gr_2 T_pM \to \mathbb{R}.}$$

Pf: "Polarization", using the symmetries of curvature tensor.

Suppose R' is s.t. $\dfrac{\langle R'(X,Y)Y,X\rangle}{\|X\wedge Y\|^2} = \dfrac{\langle R(X,Y)Y,X\rangle}{\|X\wedge Y\|^2} = \sec(X\wedge Y)$

for all X,Y; want to show $R'=R$.

By hypothesis, $\langle R'(\underline{X+Z},\underline{Y})\underline{Y},\underline{X+Z}\rangle = \langle R(\underline{X+Z},\underline{Y})\underline{Y},\underline{X+Z}\rangle$

So $$\langle R'(X,Y)Y,X\rangle + 2\langle R'(X,Y)Y,Z\rangle + \langle R'(Z,Y)Y,Z\rangle$$
$$= \langle R(X,Y)Y,X\rangle + 2\langle R(X,Y)Y,Z\rangle + \langle R(Z,Y)Y,Z\rangle$$

So $\langle R'(X,Y)Y,Z\rangle = \langle R(X,Y)Y,Z\rangle \quad \forall X,Y,Z$

Thus, $\langle R'(X,Y+W)(Y+W),Z\rangle = \langle R(X,Y+W)(Y+W),Z\rangle$

So $\langle R'(X,Y)Y,Z\rangle + \langle R'(X,Y)W,Z\rangle + \langle R'(X,W)Y,Z\rangle + \langle R'(X,W),W,Z\rangle =$
$$= \langle R(X,Y)Y,Z\rangle + \langle R(X,Y)W,Z\rangle + \langle R(X,W)Y,Z\rangle + \langle R(X,W)W,Z\rangle$$

So $\langle R'(X,Y)W,Z\rangle + \langle R'(X,W)Y,Z\rangle = \langle R(X,Y)W,Z\rangle + \langle R(X,W)Y,Z\rangle$

i.e. $$\langle R'(X,Y)W,Z\rangle - \langle R(X,Y)W,Z\rangle = \langle R(X,W)Y,Z\rangle - \langle R'(X,W)Y,Z\rangle$$
$$= \langle R'(W,X)Y,Z\rangle - \langle R(W,X)Y,Z\rangle$$
$\forall X,Y,Z,W$

Therefore $R'(X,Y)W - R(X,Y)W$ is invariant under cyclic perm. of (X,Y,W) and hence, by the 1st Bianchi identity,
$$3\left(R'(X,Y)W - R(X,Y)W\right) = 0, \quad \forall X,Y,W$$
so $R = R'$. $\square$

<u>Cor</u>: If $R: \Lambda^2 TM \to \Lambda^2 TM$ is s.t. $\sec(\sigma) = K$ for all σ, then $R = K \cdot \mathrm{Id}$, i.e.
$$\langle R(X,Y)Z,W\rangle = -\langle R(X\wedge Y), Z\wedge W\rangle = -K\langle X\wedge Y, Z\wedge W\rangle$$
$$= -K\left(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle\right)$$

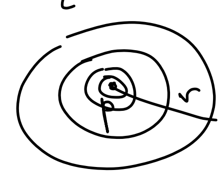
Pf: Check that RHS has $\sec \equiv K$, then use uniqueness. $\square$

Examples of (complete) Riem. mflds w/ $\sec \equiv K$:

	simply-connected	their quotients:
• $K>0$:	$S^n(1/\sqrt{K})$	$\mathbb{RP}^n$, Lens space...
• $K=0$:	$\mathbb{R}^n$	T^n, Klein bottle...
• $K<0$:	$\mathbb{H}^n(1/\sqrt{-K})$	Hyperbolic surfaces...

"Ball models"/"Warped product models":

Geometrically, in terms of distance spheres: $S(r) = \{x \in M : \mathrm{dist}(p,x) = r\}$



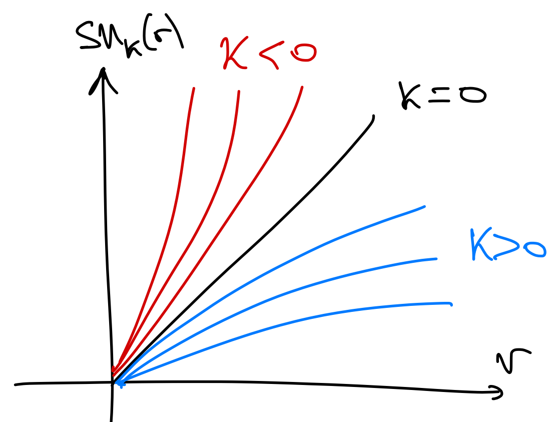
metric on n-dim mfld of constant curvature $\sec \equiv K$

$$g = dr^2 + \mathrm{sn}_K(r)^2 \, d\theta^2$$

metric of $S^{n-1}(1)$ unit round sphere

where $\mathrm{sn}_K(r) = \begin{cases} \sin(\sqrt{K}\, r) \\ r \\ \sinh(\sqrt{-K}\, r) \end{cases}$ is the solution to

the ODE $\begin{cases} \mathrm{sn}_K'' + K\, \mathrm{sn}_K = 0. \\ \mathrm{sn}_K(0) = 0 \\ \mathrm{sn}_K'(0) = 1 \end{cases}$



(see course webpage!)

"Quadric models":

• $S^n(1/\sqrt{K}) = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = \frac{1}{K}\}$ $(K>0)$

w/ metric induced from Euclidean metric $dx_1^2 + \cdots + dx_{n+1}^2$

• $\mathbb{H}^n(1/\sqrt{-K}) = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = \frac{1}{K}\}$ $(K<0)$

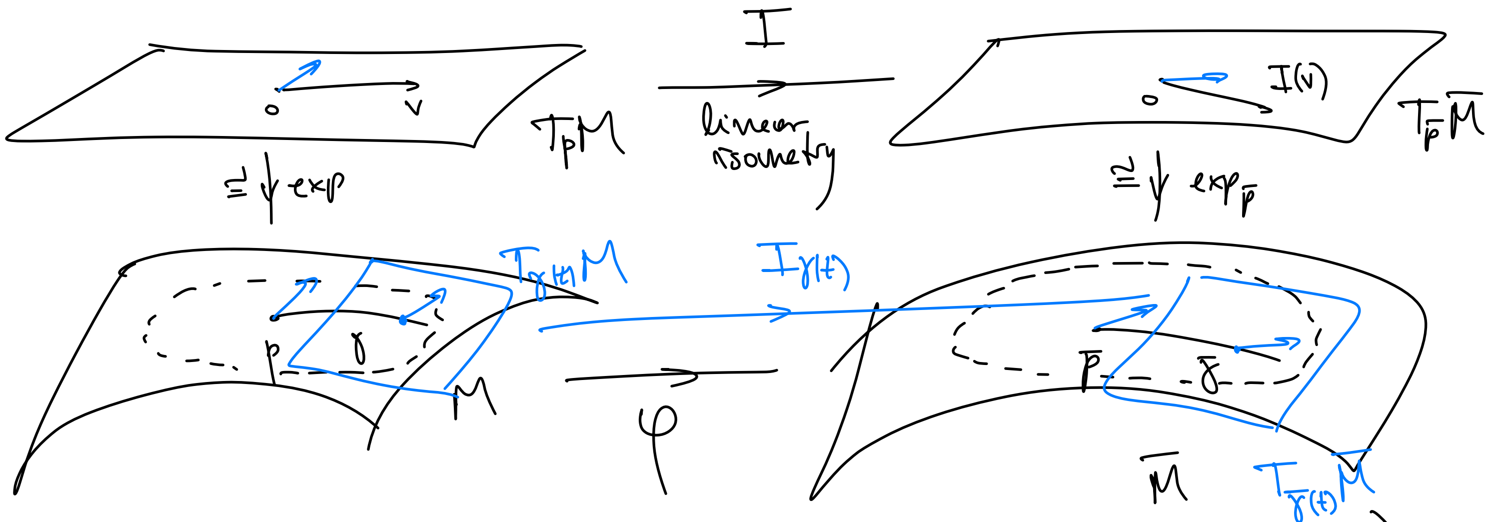
w/ metric induced from Lorenzian metric $dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2$

"hyperboloid model"

Upper half-space model: $(K \equiv -1)$

$\mathbb{H}^n(1) = \{(x,t) \in \mathbb{R}^n \times (0,+\infty)\}$ with metric induced from $\dfrac{dx_1^2 + \dots + dx_n^2 + dt^2}{t^2}$

$\boxed{\text{Cartan: Curvature is the only local invariant of a Riem mfld.}}$



$\varphi = \exp_{\bar p} \circ I \circ \exp_p^{-1}$ is a diffeom. (on geod. normal coord.)

Let $\bar\gamma = \varphi \circ \gamma$, $I_{\gamma(t)} : T_{\gamma(t)}M \to T_{\bar\gamma(t)}\bar M$ (Note: $I_{\gamma(t)}$ are linear isometries!)

$$I_{\gamma(t)} := P_{\bar p}^{\bar\gamma(t)} \circ I \circ P_{\gamma(t)}^{p}$$

parallel transport

Preserving curvature is the "Integrability condition" to become a local isometry:

$\boxed{\begin{array}{l}\underline{\text{Thm}} \text{ (Cartan). If for all geodesics } \gamma(t) \text{ starting at } p \in M, \\ \qquad I_{\gamma(t)}\big(R(X,Y)Z\big) = \bar R\big(I_{\gamma(t)}X, I_{\gamma(t)}Y\big) I_{\gamma(t)}Z \quad \forall |t| \text{ small} \\ \text{then } \varphi \text{ is a local isometry, and } d\varphi_{\gamma(t)} = I_{\gamma(t)}.\end{array}}$

Pf. Given q near p, and $X \in T_qM$, let $\gamma : [0,L] \to M$ be minimizing geodesic w/ $\gamma(0) = p$, $\gamma(L) = q$ and let $J : [0,L] \to TM$ be the Jacobi field along γ with $J(0) = 0$ and $J(L) = X$.

$\uparrow$ see Lemma 2 later

Let $\bar{J}(t) = I_{\gamma(t)}(J(t))$. By hypothesis, $\bar{J}(t)$ is a Jacobi field along $\bar{\gamma}$:

$\bar{J}'' = I(J'')$ b/c defined w/ parallel transport...

$$\bar{J}''(t) + \bar{R}(\bar{J}(t), \bar{\gamma}'(t))\bar{\gamma}'(t) = I_{\gamma(t)}\Big(J''(t) + R(J(t), \gamma'(t))\gamma'(t)\Big) = 0.$$

Clearly, $\|J(t)\| = \|\bar{J}'(t)\|$ b/c $I_{\gamma(t)}$ are linear isometries.

Moreover,
$$\begin{cases} J(t) = d(\exp_p)_{t\gamma'(0)}\, tJ'(0) \\ \bar{J}(t) = d(\exp_{\bar{p}})_{t\bar{\gamma}'(0)}\, t\bar{J}'(0) \end{cases}$$

See Lemma 1 later

$$tJ'(0) = d(\exp_p)^{-1}_{t\gamma'(0)} J(t)$$

Inverse Fct Thm

$$= d(\exp_p^{-1})_{\underbrace{\exp_p(t\gamma'(0))}_{\gamma(t)}} J(t)$$

so
$$\begin{aligned} \bar{J}(t) &= d(\exp_{\bar{p}})_{t\bar{\gamma}'(0)}\, t\bar{J}'(0) \\ &= d(\exp_{\bar{p}})_{t\bar{\gamma}'(0)}\, t\, I(J'(0)) \\ &= d(\exp_{\bar{p}})_{tI\gamma'(0)} \circ I \circ \underbrace{d(\exp_p^{-1})_{\gamma(t)} J(t)}_{} \\ &= d(\underbrace{\exp_{\bar{p}} \circ I \circ \exp_p^{-1}}_{\varphi})_{\gamma(t)} J(t) \\ &= d\varphi_{\gamma(t)} J(t) \end{aligned}$$

Computing at $t = L$, we have $\bar{J}(L) = d\varphi_{\gamma(L)} J(L) = d\varphi_q X$

and $\|d\varphi_q X\| = \|\bar{J}(L)\| = \|J(L)\| = \|X\|$ so $d\varphi_q$ is an isometry.

(I_q is linear isometry)

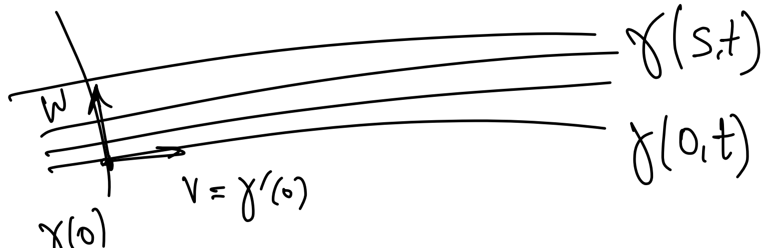
$\square$

<u>Lemma 1</u>. The Jacobi field along $\gamma(t)$ with $J(0)=0$ and $J'(0)=w$ is $J(t)=d(\exp_{\gamma(0)})_{t\gamma'(0)}\, tw$.

<u>Pf:</u> $J(t)=d(\exp_{\gamma(0)})_{t\gamma'(0)}\, tw$ is the variational field of a variation of γ by geodesics, so it is a Jacobi field.

Indeed: if $v=\gamma'(0)$, then

$$\gamma(s,t)=\exp_{\gamma(0)}\big(t(v+sw)\big)$$



$$\frac{\partial}{\partial s}\gamma(s,t)\Big|_{s=0}=d(\exp_{\gamma(0)})(tv)(tw)=d(\exp_{\gamma(0)})_{tv}\, tw=J(t)$$

Moreover, $J(0)=0$ and

$$J'(0)=\frac{D}{dt}\frac{\partial}{\partial s}\gamma(s,t)\Big|_{\substack{t=0\\ s=0}}=\frac{D}{ds}\frac{\partial}{\partial t}\gamma(s,t)\Big|_{\substack{t=0\\ s=0}}=\frac{D}{ds}(v+sw)\Big|_{s=0}=w$$

so by uniqueness of sol. to ODE with same initial condition, $J(t)$ is the claimed Jacobi field. $\square$

<u>Rmk:</u> There is a similar expression (using exp) for the unique Jacobi field along $\gamma(t)$ with arbitrary initial condition $J(0)$ and $J'(0)$, namely:

$$J(t)=\frac{\partial}{\partial s}\exp_{\alpha(s)} t\,w(s)\Big|_{s=0} \quad \text{where} \begin{cases} \alpha(s) \text{ is a curve s.t. } \alpha(0)=\gamma(0),\ \alpha'(0)=J(0) \\ w(s) \text{ is a vector field along } \alpha(s) \text{ with } w(0)=\gamma'(0) \text{ and } w'(0)=J'(0). \end{cases}$$

__Lemma 2.__ Let $\gamma : [0,L] \to M$ be a geodesic, $v \in T_{\gamma(0)}M$, $w \in T_{\gamma(L)}M$. If $L > 0$ is suff. small, there exists a unique Jacobi field J along γ with $J(0) = v$, $J(0) = w$.

__Pf:__ Let $\mathcal{J} = \{ J \text{ is a Jacobi field along } \gamma,\ J(0) = 0 \}$;

$$\overset{\text{Lemma 1}}{=} \left\{ J(t) = d(\exp_{\gamma(0)})_{t\gamma'(0)}\, t J'(0) \right\}$$

this is a vector space

$\dim \mathcal{J} = \dim T_pM$

and $ev_t : \mathcal{J} \longrightarrow T_{\gamma(t)}M$

$$J \longmapsto J(t)$$

If $t > 0$ is small, then ev_t is injective: otherwise $J_1, J_2 \in \mathcal{J}$, $J_1(t) = J_2(t)$ but $J_1 \neq J_2$. Then $J_1 - J_2 \in \mathcal{J}$ satisfies $0 = (J_1 - J_2)(t) = d(\exp_{\gamma(0)})_{t\gamma'(0)}\, t (J_1 - J_2)'(0)$ and for t small $d(\exp)_{t\gamma'(0)}$ is invertible, so $(J_1 - J_2)'(0) = 0$, hence $(J_1 - J_2)(0) = 0$ and $(J_1 - J_2)'(0) = 0$ so $J_1 \equiv J_2$. $\rightarrow\leftarrow$

Since $ev_t : \mathcal{J} \to T_{\gamma(t)}M$ is linear and $\dim \mathcal{J} = \dim T_{\gamma(t)}M$, ev_t is bijective. So $\exists\, J_1 \in \mathcal{J}$ with $J_1(t) = w$.

By the same argument starting from $\gamma(t)$, $\exists\, J_2$ a Jacobi field along γ with $J_2(0) = v$ and $J_2(t) = 0$.

Thus $J = J_1 + J_2$ satisfies $J(0) = v$ and $J(t) = w$. $\square$

Rmk: In Lemma 2, can replace "$L>0$ suff. small" with "there is no $0<t_*\leq L$ s.t. $\gamma(t_*)$ is conjugate to $\gamma(0)$ along $\gamma(t)$".

Def: A point $q=\gamma(t_*)$ is conjugate to $p=\gamma(0)$ along $\gamma(t)$ if there exists a Jacobi field $J(t)$ along $\gamma(t)$ s.t. $J(0)=0$ and $J(t_*)=0$.   Ex: Antipodal points on S^n.

Prop: $q=\exp_p(t_*v)$ is conjugate to p along $\gamma(t)=\exp_p tv$ iff t_*v is a critical point of $\exp_p: T_pM\to M$.

Global version of Cartan's Thm:

> Thm (Cartan-Ambrose-Hicks). If M and $\overline{M}$ are complete and M is simply-connected, can extend the above argument along piecewise geodesics to get $\varphi: M\to\overline{M}$ which is a local isometry hence a covering map.

Pf. Essentially the same, using uniform continuity of homotopy between two paths (piecewise geodesics) in M with same endpoints.

> Thm (Killing-Hopf). If M^n is simply-connected and has $\sec\equiv k$, then
>
> $$M^n \overset{\text{isom}}{\cong} \begin{cases} S^n(1/\sqrt{k}) & \text{if } k>0 \\ \mathbb{R}^n & \text{if } k=0 \\ H^n(1/\sqrt{-k}) & \text{if } k<0 \end{cases}$$
>
> "constant curvature model spaces"

Pf: Combine the above with simple topological arguments.

$\rightsquigarrow$ Space form problem: Which π act freely and isometrically on the above, so that M/π is a smooth manifold with $\sec\equiv k$ and $\pi_1\cong\pi$?

Lecture 4 Comparison results for Jacobi fields 2/16/2023

Prop: If $\gamma : [0,L] \to M$ is a geodesic with $\gamma(0)=p$, $\dot\gamma(0)=v$, $w \in T_vT_pM$ has $\|w\|=1$ and $J(t)$ is the Jacobi field along $\gamma(t)$ with $J(0)=0$ and $J'(0)=w$, $\|w\|=1$, (i.e., $J(t)=d(\exp_p)_{tv}\, tw$), then

$$\|J(t)\|^2 = t^2 - \frac{1}{3}\langle R(v,w)w,v\rangle t^4 + O(t^5)$$

Pf:

$$\langle J, \underset{0}{J}\rangle(0) = 0$$

$$\langle J,J\rangle'(0) = 2\langle \underset{0}{J}, J'\rangle(0) = 0$$

$$\langle J,J\rangle''(0) = 2\underbrace{\langle J',J'\rangle(0)}_{\|w\|^2=1} + 2\langle J'', \underset{0}{J}\rangle(0) = 2$$

Also, $J''(0) = -R(\underset{0}{J},\dot\gamma)\dot\gamma\,(0) = 0$ so

$$\langle J,J\rangle'''(0) = 6\langle J', \underset{0}{J''}\rangle(0) + 2\langle J''', \underset{0}{J}\rangle(0) = 0$$

Moreover, for any vector field W along γ,

$$\left\langle \frac{D}{dt} R(J(t),\dot\gamma(t))\dot\gamma(t), W\right\rangle = \frac{d}{dt}\underbrace{\langle R(J,\dot\gamma)\dot\gamma, W\rangle}_{=\langle R(W,\dot\gamma)\dot\gamma,J\rangle} - \langle R(J,\dot\gamma)\dot\gamma, W'\rangle$$

$$= \left\langle \frac{D}{dt} R(W,\dot\gamma)\dot\gamma, J\right\rangle + \underbrace{\langle R(W,\dot\gamma)\dot\gamma, J'\rangle}_{=\langle R(J',\dot\gamma)\dot\gamma, W\rangle}$$

So at $t=0$:

$$- \langle R(J,\dot\gamma)\dot\gamma, W'\rangle$$

$$\frac{D}{dt} R(J,\dot\gamma)\dot\gamma = R(J',\dot\gamma)\dot\gamma \quad \text{(all other terms are zero at } t=0\text{.)}$$

Thus:

$$\langle J,J\rangle''''(0) = 8\langle J', J'''\rangle(0) + 6\langle \underset{0}{J''}, \underset{0}{J''}\rangle(0) + 2\langle J'''', \underset{0}{J}\rangle(0)$$

$$= -8\langle J', R(J',\dot\gamma)\dot\gamma\rangle(0) = -8\langle R(w,v)v,w\rangle$$

$$= -8\langle R(v,w)w,v\rangle.$$

$J'' = -R(J,\dot\gamma)\dot\gamma$

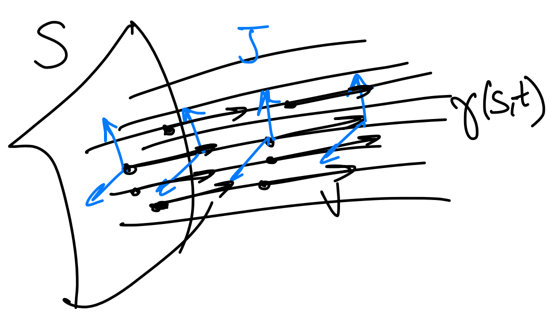
so $J''' = -R(J',\dot\gamma)\dot\gamma$

$\square$

<u>Goal</u> for our first comparison results (<u>Rauch</u>) is to promote the geometric information

"curvature controls length of Jacobi fields"

from the above "infinitesimal at $t=0$" version to the more global version "until the first conjugate point."



Let $S\subset M$ be a submanifold, and

$\gamma : S\times(-\varepsilon,\varepsilon)\longrightarrow M$

a family of geodesics:

$\forall s\in S,\quad t\longmapsto \gamma(s,t)$ is a geod.

Let $V = \frac{\partial}{\partial t}\gamma(s,t) = d\gamma_{(s,t)}\left(\frac{\partial}{\partial t}\right)$ be the tangent field to the geodesics

$J = d\gamma_{(s,t)}(w)$ for any given $w\in T_sS$,

which is a Jacobi field along $t\mapsto\gamma(s,t)$.

Let $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be the tensor $A = \nabla V$, ie. $A(X) = \nabla_X V$.
and $R_V : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be the tensor $R_V(X) = R(X,V)V$.

<u>Note</u>: $[J,V] = 0$ hence $\nabla_V J = \nabla_J V = A \cdot J$

Reduce <u>Jacobi equation</u> from 2^{nd} order ODE to system of 1^{st} order ODEs

$$\underbrace{J'' + R(J,V)V}_{=R_V(J)} = 0 \iff \begin{cases} J' = A \cdot J \\ A' + A^2 + R_V = 0 \end{cases}$$

$$\begin{aligned}(\nabla_V A)X &= \nabla_V(AX) - A\nabla_V X \\ &= \nabla_V \nabla_X V - A(\nabla_X V + [V,X]) \\ &= \nabla_X \underbrace{\nabla_V V}_{=0} + R(V,X)V + \nabla_{[V,X]} V - \nabla_{\nabla_X V + [V,X]} V \\ &= -R_V(X) - A(A(X))\end{aligned}$$

So: $A' = -R_V - A^2$ i.e. $\boxed{A' + A^2 + R_V = 0}$ "Riccati equation"

$\nabla V : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the tensor $\nabla V(X) = \nabla_X V$.

This is the case e.g. if $\dim S + 1 = \dim M$, $v(s) \in T_s S^\perp$ and $\gamma(s,t) = \exp_s t v(s)$

<u>Note</u>: This equation can be solved independently of the first $J' = A \cdot J$.

- Suppose ∇V is self-adjoint; i.e. $\forall X, Y$, $\langle \nabla_X V, Y \rangle = \langle X, \nabla_Y V \rangle$

Then $V = \nabla f$ locally, because setting $\xi(X) = \langle X, V \rangle$, we have

$$\begin{aligned} d\xi(X,Y) &= X\xi(Y) - Y\xi(X) - \xi([X,Y]) \\ &= \cancel{\langle \nabla_X Y, V \rangle} + \langle Y, \nabla_X V \rangle - \cancel{\langle \nabla_Y X, V \rangle} - \langle X, \nabla_Y V \rangle - \cancel{\langle [X,Y], V \rangle} \\ &= \langle \nabla_X V, Y \rangle - \langle X, \nabla_Y V \rangle = 0 \end{aligned}$$ so $d\xi$ is closed.

Locally, every closed 1-form ξ is locally exact: $\xi = df$, i.e. $V = \nabla f$.

- Thus $\|V\|^2 = \langle V, V\rangle$ is constant, b/c

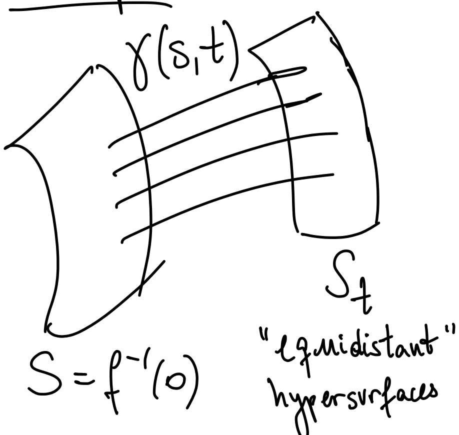
$V = \frac{\partial}{\partial t}\gamma(s,t)$ is velocity field of geodesics.

∇V self adjoint

$$X\langle V,V\rangle = 2\langle \nabla_X V, V\rangle \overset{\downarrow}{=} 2\langle X, \underbrace{\nabla_V V}_{=0}\rangle = 0$$

Assume WLOG $\|V\| = 1$, so, up to an additive constant, $f(x) = $ s-dist(x, S). "signed distance function"

Examples:

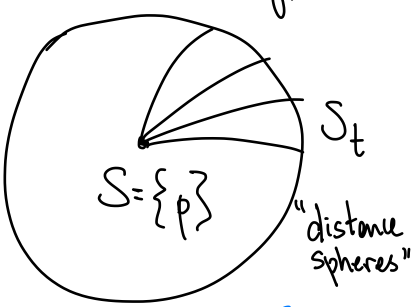


Levelsets are equidistant hypersurfaces: (b/c $\nabla f = V \neq 0$, these are smooth submflds!)

$$S_t = f^{-1}(t) = \{x \in M : \text{dist}(x, S) = t\}$$

with normal vector $\vec{n} = \nabla f = V$; $T_p S_t = \operatorname{Ker} df(p) = V^\perp$,

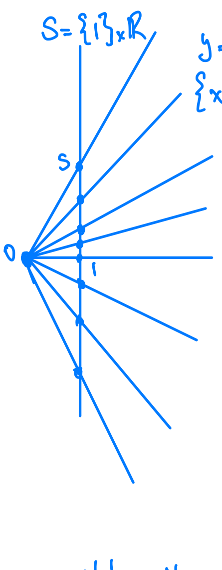
and $\gamma(s,t) = \exp t\, v(s)$, $f(\gamma(s,t)) = t$.



Actual distance function to $S = S_0$, locally, is:

$$\text{dist}(\cdot, S) = |f(\cdot)| \qquad (\text{by Gauss Lemma})$$

Non-example in $\mathbb{R}^2$



$y = sx$

$\{x(1,s) : x \in \mathbb{R}\}$

For each $s \in \mathbb{R}$, the curve $t \mapsto \gamma(s,t) = (t, ts)$ is a geodesic in $\mathbb{R}^2$, with tangent vector

$$V = d\gamma\left(\frac{\partial}{\partial t}\right) = \frac{\partial \gamma}{\partial t} = (1, s) = \frac{\partial}{\partial x} + s\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + \frac{y}{x}\frac{\partial}{\partial y}$$

and $\frac{DV}{dt} = 0$, but $\|V\|^2 = 1 + s^2$ is not constant...

∇V is not self-adjoint:

$$\nabla V = \begin{pmatrix} 1 & -y/x^2 \\ 0 & 1/x \end{pmatrix} \text{ not symm.}$$

Not a Hessian

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y} = \left(\frac{1}{t}\frac{\partial}{\partial s} + \frac{1}{s}\frac{\partial}{\partial t}\right)$$

$$\frac{\partial}{\partial s} = x\frac{\partial}{\partial y} - \frac{x^2}{y}\frac{\partial}{\partial x} = t\frac{\partial}{\partial y} - \frac{t}{s}\frac{\partial}{\partial t}$$

Def: $A = \nabla V$ is called the <u>shape operator</u> of S_t its eigenvalues are called <u>principal curvatures</u>, and its trace $H = \operatorname{tr} A$ is called <u>mean curvature</u>.

Remark: 1) $\frac{\|J(t)\|}{\|J(0)\|}$ measures the <u>length-distortion</u> between S_0 and S_t.

2) The equation $A' + A + R_V = 0$ is <u>nonlinear</u> and may develop singularities, called <u>focal points of S</u>.

Examples:

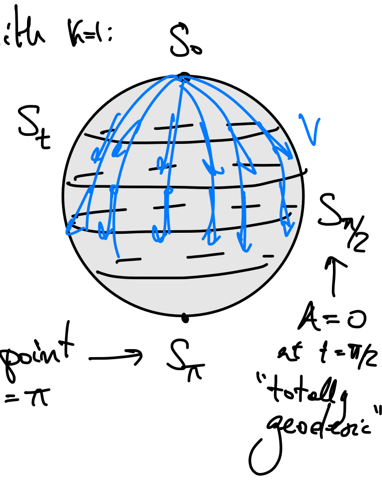
1. On any (M,g), let $S_t = \partial B_t(p) = \{x \in M : \mathrm{dist}(x,p) = t\}$. Then

$$A \sim \frac{1}{t}\,\mathrm{Id} \quad \text{as} \quad t \searrow 0 \quad \text{b/c } M \text{ is infinitesimally Euclidean at } p.$$

Note: If $(M,g) = \mathbb{R}^n$, then $A = \frac{1}{t}\mathrm{Id}$, by next example.

2. If (M,g) has constant curvature $\sec \equiv K$, then $R_V = K\,\mathrm{Id}$ and we can solve the Ricatti equation explicitly when S_t are so-called "umbilical" surfaces, i.e., $A = a\,\mathrm{Id}$.

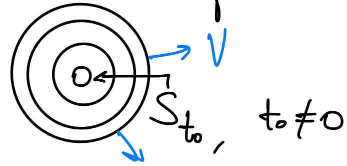
$$A' + A^2 + R_V = 0 \overset{A = a\mathrm{Id}}{\rightsquigarrow} a' + a^2 + K = 0$$



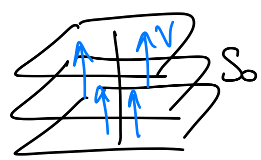
$\boxed{K>0}$: $\quad a(t) = \sqrt{K}\cot(\sqrt{K}\cdot t)$ $\quad$ concentric circles

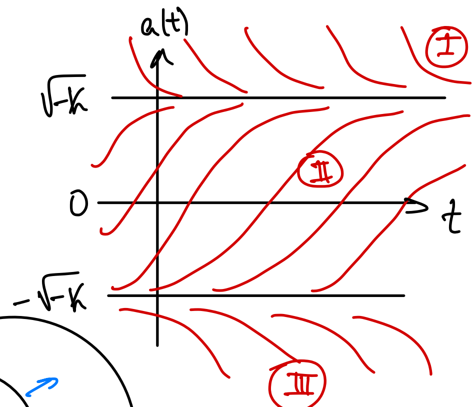
$\approx \frac{1}{t}$

$\boxed{K=0}$: $\quad a(t) = \dfrac{1}{t - t_0}$ $\quad$ concentric spheres



or $\quad a(t) \equiv 0$ $\quad$ parallel planes





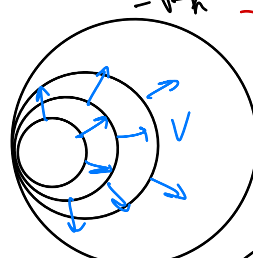
$\boxed{K<0}$:

I, III $\quad a(t) = \sqrt{-K}\coth(\sqrt{-K}\,t)$ $\quad$ concentric spheres

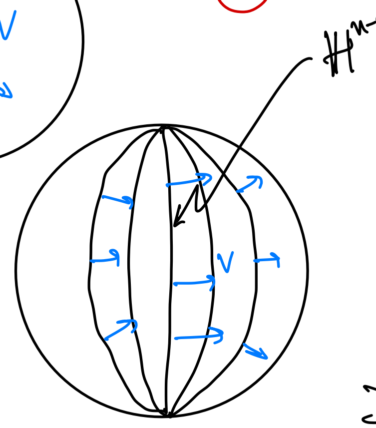
$\approx \frac{1}{t}$ as $t \searrow 0$

or $\quad$ II $\quad a(t) = \sqrt{-K}\tanh(\sqrt{-K}\,t)$ $\quad$ horospheres

$\approx t$ as $t \searrow 0$

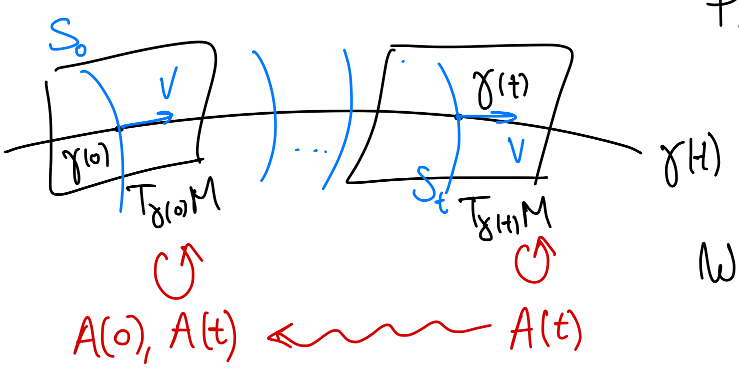


or $\quad a(t) \equiv \pm\sqrt{-K}$ $\quad$ hypersurfaces parallel to $\mathbb{H}^{n-1}$



To facilitate comparison, identify $T_{\gamma(t)}M \cong T_{\gamma(0)}M$ via parallel transport

$$P_{\gamma(0)}^{\gamma(t)} w \overset{\cong}{\longleftrightarrow} w$$



fixed vector space $E = T_{\gamma(0)}M$

With this, $A(t) \in \mathrm{Sym}^2(T_{\gamma(0)}M)$ $\forall t$

Also, recall $\langle A, B\rangle = \operatorname{tr} AB$ in $\mathrm{Sym}^2 E$

and $A \preceq B$ if $B - A \succeq 0$ i.e. $\langle (B-A)x, x\rangle \geq 0$, $\forall x \in E$.

Thm. Let $R_1, R_2 : \mathbb{R} \to \mathrm{Sym}^2 E$ be smooth curves with $R_1(t) \succeq R_2(t)$, $\forall t$. Let $A_i : [t_0, t_i) \to \mathrm{Sym}^2 E$ be the maximal solutions to $A_i' + A_i^2 + R_i = 0$. If $A_1(t_0) \preceq A_2(t_0)$, then $t_1 \leq t_2$ and $A_1(t) \preceq A_2(t)$ for all $t \in [t_0, t_1)$.

Pf. Let $U = A_2 - A_1$, so $U(t_0) \succeq 0$.

$$U' = A_2' - A_1' = A_2^2 - A_1^2 + \underbrace{R_1 - R_2}_{S}$$

Define $S = R_1 - R_2$ and $X = -\frac{1}{2}(A_1 + A_2)$, so that

$$XU + UX = -\tfrac{1}{2}(A_1 + A_2)(A_2 - A_1) - \tfrac{1}{2}(A_2 - A_1)(A_1 + A_2) = A_1^2 - A_2^2$$

so $U' = XU + UX + S$, an inhomogeneous linear ODE we can solve by "variation of constants". Namely, let $g : (t_0, t') \to \mathrm{Sym}^2 E$ be the solution to the homogeneous linear ODE $g' = Xg$, where $t' = \min\{t_1, t_2\}$. Then $U = g V g^T$ is the desired solution, where V satisfies $V' = g^{-1} S (g^{-1})^T$.

Indeed: $U' = g' V g^T + g V' g^T + g V (g^T)'$

$$= X g V g^T + \cancel{g g^{-1}} S \cancel{(g^{-1})^T g^T} + g V g^T X^T$$

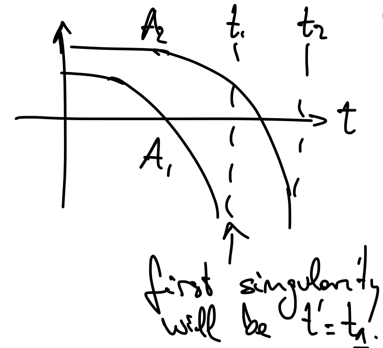
$$= XU + S + UX.$$

Since $S = R_1 - R_2 \geq 0$, we have $V' = g^{-1} S (g^{-1})^T \geq 0$.

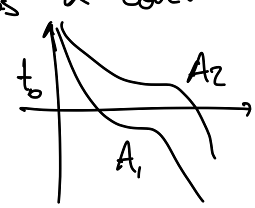
Since $U(t_0) = g(t_0) V(t_0) g(t_0)^T = A_2(t_0) - A_1(t_0) \geq 0$, we have $V(t_0) \geq 0$.

Thus $V(t) \geq 0$ for all $t \in (t_0, t')$ and hence also

$A_2(t) - A_1(t) = U(t) = g(t) V(t) g(t)^T \geq 0$ for all $t \in (t_0, t')$, i.e. $A_1(t) \leq A_2(t)$ for $t \in (t_0, t')$. Since A_i' is bounded from above $\left(A_i' \leq -A_i^2 - R_i \leq -R_i\right)$ the only singularity possible is $-\infty$, so $A_1 \leq A_2$ implies $t' = t_1 \leq t_2$. $\square$



Rmk: The above still holds if A_1, A_2 are singular at t_0, but $U = A_2 - A_1$ has a continuous extension to t_0, with $U(t_0) \geq 0$.



Geometric interpretation: "Principal curvatures of equidistant hypersurfaces decrease faster on the space of larger curvature."

Thm. Let $A_1, A_2 : (t_0, t') \to \mathrm{Sym}^2 E$ be smooth curves with $A_1(t) \leq A_2(t)$. Let $J_i : (t_0, t') \to E$ be nonzero sol. to $J_i' = A_i J_i$. Then $t \mapsto \frac{\|J_1(t)\|}{\|J_2(t)\|}$ is nonincreasing. Moreover, if $\lim_{t \searrow t_0} \frac{\|J_1(t)\|}{\|J_2(t)\|} = 1$, then $\|J_1(t)\| \leq \|J_2(t)\|$ for all $t \in (t_0, t')$. Equality holds for some $t_* \in (t_0, t')$ if and only if $J_i = j \cdot v_i$ on $[t_0, t']$ for some $v_i \in E$ with $A v_i = \lambda v_i$, $j' = \lambda j$, and $A_1 \leq \lambda\, \mathrm{Id} \leq A_2$.

Pf: Since $\|J_i(t)\|$ is smooth, we can differentiate:

$$\frac{\|J_i\|'}{\|J_i\|} = \frac{1}{\|J_i\|}\frac{1}{2\sqrt{\langle J_i, J_i\rangle}}\, 2\langle J_i', J_i\rangle = \frac{\langle J_i', J_i\rangle}{\|J_i\|^2} = \frac{\langle A_i J_i, J_i\rangle}{\|J_i\|^2}$$

$$\in \left[\lambda_{min}(A_i), \lambda_{max}(A_i)\right]$$

min. and max eigenvalues of $A_i \in \mathrm{Sym}^2 E$.

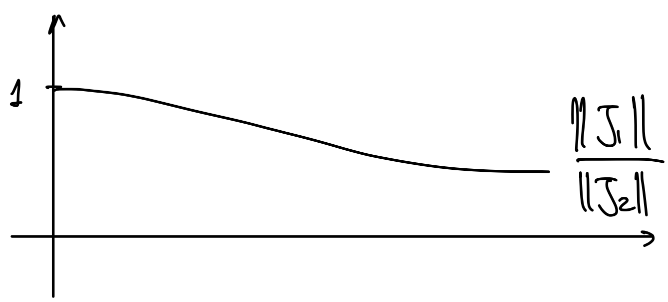
Thus
$$\left(\log \|J_1\|\right)' = \frac{\|J_1\|'}{\|J_1\|} \leq \lambda_{max}(A_1) \leq \lambda_{min}(A_2) \leq \frac{\|J_2\|'}{\|J_2\|} = \left(\log \|J_2\|\right)'$$

$$A_1 \leq A_2$$

i.e. $\left(\log \frac{\|J_1\|}{\|J_2\|}\right)' \leq 0$ so $\frac{\|J_1\|}{\|J_2\|}$ is non-increasing.

By monotonicity, if $\|J_1\| = \|J_2\|$ at $t = t_0$, and $t = t_*$, then $\|J_1\| = \|J_2\|$, $\forall t \in (t_0, t_*)$ and hence $J_i' = A_i J_i = \lambda J_i$, from which the stated conclusions follow. $\square$

from = in the inequalities.



The following corollaries are originally due to Berger and Rauch:

Thm (Rauch I). Suppose J_i are sol. to $J_i'' + R_i J_i = 0$ with $R_1 \geq R_2$ and $J_i(0) = 0$, $\|J_1'(0)\| = \|J_2'(0)\|$. Then $\|J_1\| \leq \|J_2\|$ up to the first zero of J_1.

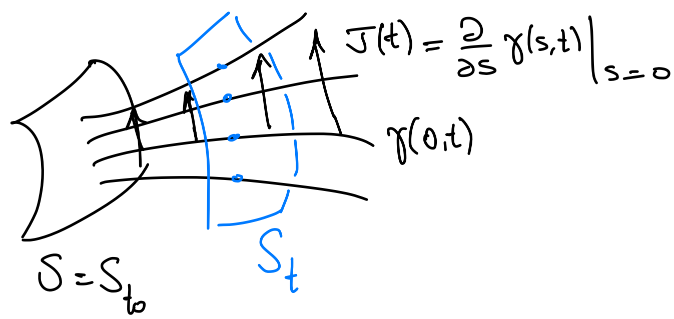
Lecture 5 2/23/2023

Recall:

$R_v(X) = R(X,v)v$

$$J'' + R_v(J) = 0 \iff \begin{cases} J' = A \cdot J \\ A' + A^2 + R_v = 0 \end{cases} \quad \text{"Ricatti equation"}$$

(Jacobi equation)



Family of Jacobi fields corresponding to variations by geodesics starting from S

If ∇A is self-adjoint, then locally $V = \frac{\partial}{\partial t}\gamma(s,t) = \nabla f$ and so

- $S_t = f^{-1}(t)$ are equidistant hypersurfaces
- A is the shape operator
- Eigenvalues of A are principal curvatures
- Eigenvectors of A are principal directions
- $H = \operatorname{tr} A$ is the mean curvature
- Singularities of A are focal points

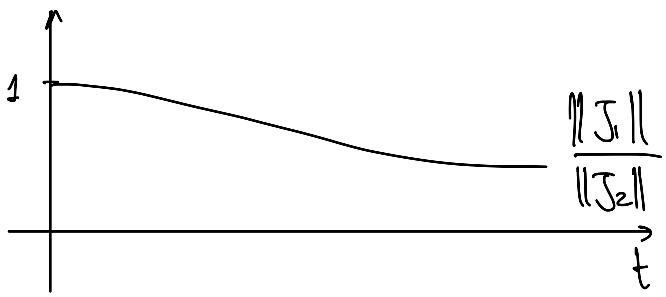
Comparison results:

Recall: $R_1 - R_2 \succeq 0 \iff \langle (R_1 - R_2)v, v\rangle \geq 0, \forall v$

Thm 1. If $R_1, R_2 : \mathbb{R} \to \operatorname{Sym}^2 E$ satisfy $R_1 \succeq R_2$ and $A_i : [t_0, t_i] \to \operatorname{Sym}^2 E$ satisfy $A_i' + A_i^2 + R_i = 0$, then $A_1(t) \preceq A_2(t)$ for all $t_0 \leq t \leq t_1 \leq t_2$.

Thm 2. If $A_1(t) \leq A_2(t)$ and $J_i : [t_0, t'] \to E$ satisfy $J_i' = A_i \cdot J_i$, then $t \mapsto \frac{\|J_1(t)\|}{\|J_2(t)\|}$ is non-increasing. (So if $\lim_{t \searrow t_0} \frac{\|J_1(t)\|}{\|J_2(t)\|} = 1$, then $\|J_1(t)\| \leq \|J_2(t)\|$ for all $t \in (t_0, t')$.) Equality holds for some $t_* \in (t_0, t')$ if and only if $J_i = j \cdot v_i$ on $[t_0, t']$ for some $v_i \in E$ with $Av_i = \lambda v_i$, $j' = \lambda j$, and $A_1 \preceq \lambda \, \mathrm{Id} \preceq A_2$.

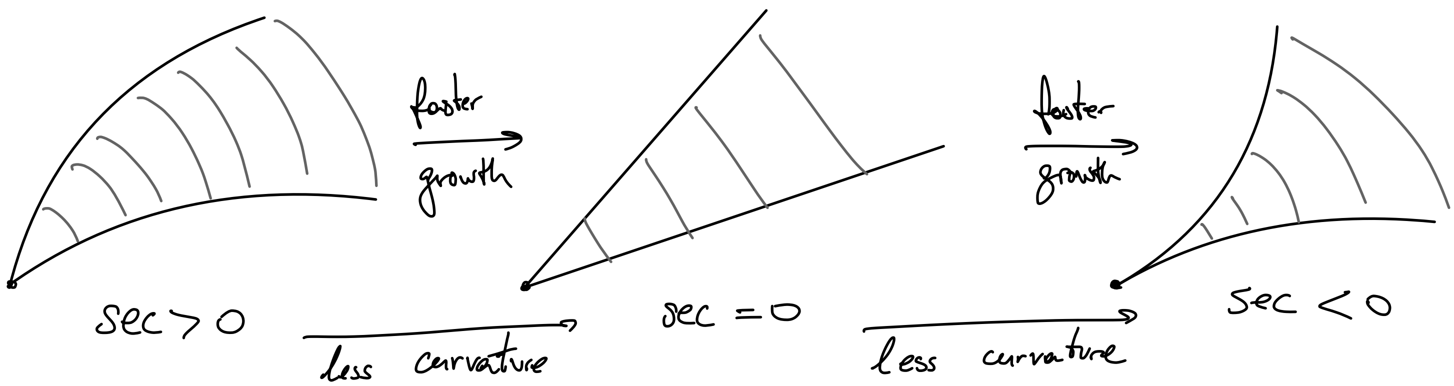
Thm (Rauch I). Suppose J_i are sol. to $J_i'' + R_i J_i = 0$ with $R_1 \geq R_2$ and $J_i(0) = 0$, $\|J_1'(0)\| = \|J_2'(0)\|$. Then $\|J_1\| \leq \|J_2\|$ up to the first zero of J_1.



Rmk: We knew an infinitesimal version:

$$\|J\| = t - \frac{1}{6}\langle R(J), J\rangle t^2 + O(t^3) \qquad \|J'(0)\| = 1$$

so $R_1 \geq R_2 \Rightarrow \|J_1\| \leq \|J_2\|$ for $t \approx 0$.



Thm (Rauch II). Suppose J_i are sol. to $J_i'' + R_i J_i = 0$ with $R_1 \geq R_2$ and $J_i'(0) = 0$, $\|J_1(0)\| = \|J_2(0)\|$. Then $\|J_1\| \leq \|J_2\|$ up to the first zero of J_1.

Both Rauch I and II follow from comparison theorems above:

Rauch I: $A_i(t) \sim \frac{1}{t}\,\mathrm{Id}$ as $t \downarrow 0$, i.e. use initial condition "$A_i(0) = \infty$"

$$J_i' = A_i J_i \Rightarrow t J_i' \sim J_i \text{ as } t \downarrow 0 \rightarrow J_i(0) = 0$$

$$\|J_1'(0)\| = \|J_2'(0)\| \Rightarrow \lim_{t \downarrow 0} \frac{\|J_1(t)\|}{\|J_2(t)\|} = \lim_{t \downarrow 0} \frac{t\|J_1'(t)\|}{t\|J_2'(t)\|} = 1$$

Rauch II: use initial condition $A_i(0) = 0$

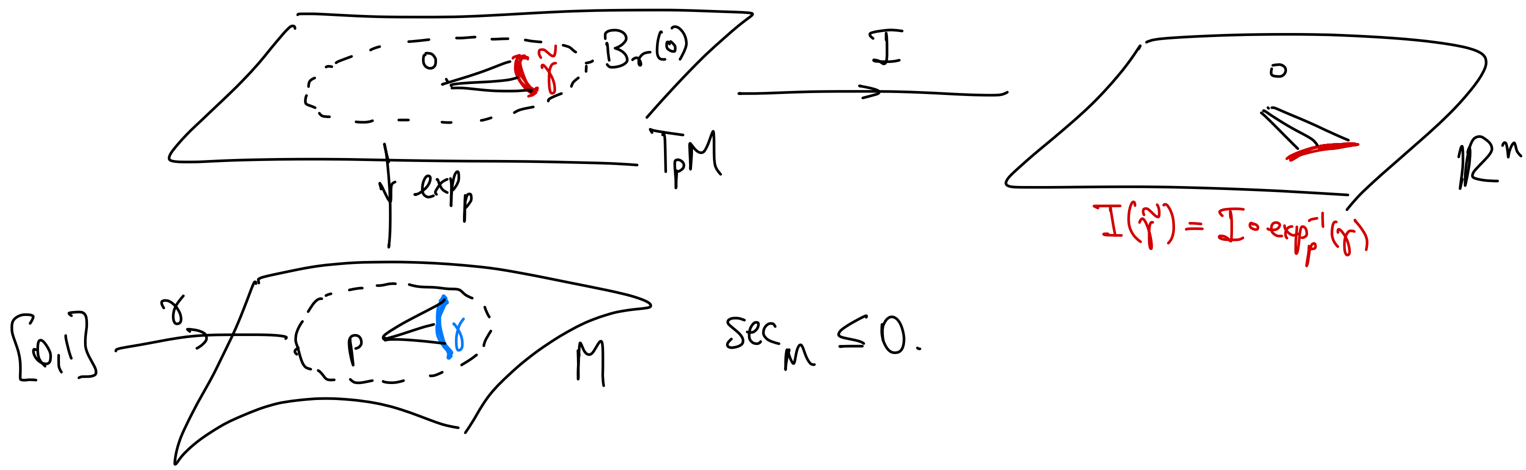
$$J_i' = A_i \cdot J_i \Rightarrow J_i'(0) = 0.$$

$$\|J_1(0)\| = \|J_2(0)\| \Rightarrow \lim_{t \downarrow 0} \frac{\|J_1(t)\|}{\|J_2(t)\|} = 1$$

Application of Rauch I:

Cor: Let (M^n, g) be a complete Riem. mfld with $\sec \leq 0$, and $r > 0$ s.t. $\exp_p : B_r(0) \longrightarrow M$ is a diffeom. onto its image. Fix a linear isometry $I : T_pM \to \mathbb{R}^n$. Given $\gamma : [0,1] \to \exp_p(B_r(0))$, we have

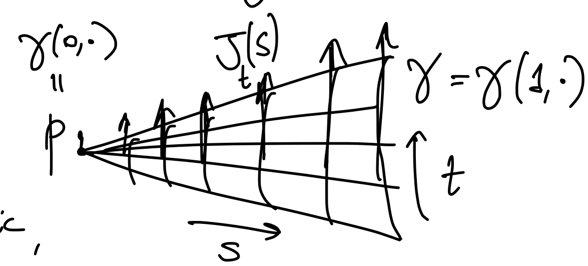
$$\mathrm{length}_g(\gamma) \geq \mathrm{length}_{\mathbb{R}^n}\left(I \circ \exp_p^{-1}(\gamma)\right).$$



$\sec_M \leq 0.$

Pf: Let $\tilde{\gamma} = \exp_p^{-1} \gamma$, and consider the "rectangle"

$$\gamma(s,t) = \exp_p s\tilde{\gamma}(t)$$



For fixed t, $s \mapsto \gamma(s,t)$ is a geodesic, and $J_t(s) = \frac{\partial}{\partial t}\gamma(s,t)$ is a Jacobi field along $s \mapsto \gamma(s,t)$, with $J_t(0) = 0$ and $J_t(1) = \dot{\gamma}(t)$. Since $\sec_M \leq 0$, by Rauch I,

$$\|J_t(s)\| \geq \underbrace{s\,\|J_t'(0)\|}_{\text{length of comparison Jacobi field in } \mathbb{R}^n} \text{ so } \mathrm{length}_g(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|\,dt = \int_0^1 \|J_t(1)\|\,dt$$

$$\overset{s=1}{\geq} \int_0^1 \|J_t'(0)\|\,dt = \mathrm{length}_{\mathbb{R}^n}\left(I \circ \exp_p^{-1}\gamma\right)$$

Indeed, $J_t'(0) = \frac{D}{ds} J_t(s)\Big|_{s=0} = \frac{D}{ds}\frac{\partial}{\partial t} \exp_p s\tilde{\gamma}(t)\Big|_{s=0}$

$$= \frac{D}{dt}\frac{\partial}{\partial s} \exp_p s\tilde{\gamma}(t)\Big|_{s=0} = \frac{D}{dt} \underbrace{d(\exp_p)_0}_{id} \tilde{\gamma}(t) = \tilde{\gamma}'(t)$$

and so $\operatorname{length}_{\mathbb{R}^n}(I \circ \exp_p^{-1} \gamma) = \int_0^1 \left\| \frac{\partial}{\partial t} \underbrace{I \circ \exp_p^{-1}(\gamma)}_{\tilde{\gamma}} \right\| dt = \int_0^1 \| J_t'(0) \| \, dt.$ $\square$

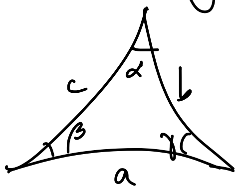
(*) In $\mathbb{R}^n$, the Jacobi equation $J'' = 0$ has solutions $J(s) = J(0) + sJ'(0)$, so Jacobi fields with $J(0) = 0$ are given by $J(s) = s\,J'(0)$.

Thm (Cartan-Hadamard). Let (M^n, g) be a complete Riem. mfld with $\sec \leq 0$. Then for any $p \in M$, $\exp_p : T_pM \to M$ is a covering map; so $\pi_k M = \{1\}$ for all $k \geq 2$. In particular, if $\pi_1 M = \{1\}$, then $M^n \underset{\text{diff.}}{\cong} \mathbb{R}^n$.

Pf. By Rauch I, given any geodesic $\gamma : \mathbb{R} \to M$ and a Jacobi field $J : \mathbb{R} \to M$ along γ with $J(0) = 0$, we have $\|J(t)\| \geq t\|J'(0)\| > 0$ so there are no conjugate points along γ. Thus, $\exp_p : T_pM \to M$ has non singular differential everywhere, i.e. $d(\exp_p)_v : T_v T_pM \to T_{\exp_p v} M$ is invertible for all $v \in T_pM$ (because $0 \neq J(t) = d(\exp_p)_{\underbrace{t\dot{\gamma}(0)}_{v}} t J'(0),\ \forall t \neq 0$).

Since $\exp_p : T_pM \to M$ is a local diffeom., it is a covering map. If $\pi_1 M = \{1\}$, then $\exp_p$ is a homeomorphism (by Topology), and since it is smooth and nonsingular, it is a diffeomorphism. $\square$

Cor: A geodesic triangle on a complete manifold with $\sec \leq 0$ satisfies

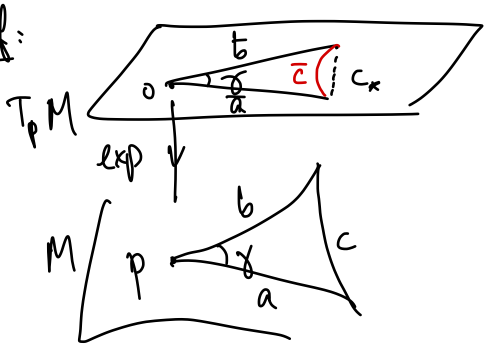


(i) $\ell(c)^2 \geq \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma$ $\quad$ (ℓ = length)

(ii) $\alpha + \beta + \gamma \leq \pi$

If $\sec < 0$, then get strict inequalities.

Pf:



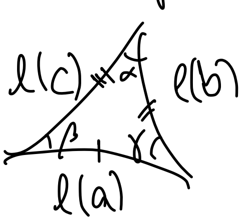
Let $\bar{a}, \bar{b}, \bar{c}$ in T_pM be such that
$a = \exp_p \bar{a}$, $\quad b = \exp_p \bar{b}$, $\quad c = \exp_p \bar{c}$

Note that $\bar{a}$ and $\bar{b}$ are straight line segments ($\exp_p$ is radial isometry); with $\ell(\bar{a}) = \ell(a)$ and $\ell(\bar{b}) = \ell(b)$. Let c_* be the straight line segment with same endpoints as $\bar{c}$, so $\ell(\bar{c}) \geq \ell(c_*)$. By the Application of Rauch I, $\ell(c) \geq \ell(\bar{c}) \geq \ell(c_*)$. Thus, altogether:

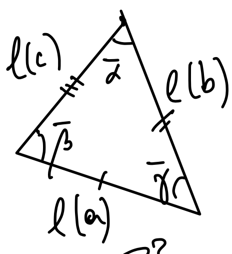
$$\ell(c)^2 \geq \ell(c_*)^2 \overset{\text{Law of cosines in } T_pM \cong \mathbb{R}^n}{=} \ell(\bar{a})^2 + \ell(\bar{b})^2 - 2\ell(\bar{a})\ell(\bar{b})\cos\gamma$$

$$\overset{\text{Gauss Lemma}}{=} \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma.$$

To compare angles, since $\ell(a), \ell(b), \ell(c)$ satisfy the triangle inequalities (b/c every geodesic is minimizing in $\sec \leq 0$, i.e., $\ell(a), \ell(b), \ell(c)$ achieve distances) we can build a comparison triangle in $\mathbb{R}^2$, with same side angles, but possibly different angles, say $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$. Then from $\ell($



in M



in $\mathbb{R}^2$

$$\ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma \leq \ell(c)^2$$
$$= \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\bar{\gamma}$$

$$\Rightarrow \cos\gamma \geq \cos\bar{\gamma} \Rightarrow \gamma \leq \bar{\gamma}$$

Same for α, β and get $\alpha + \beta + \gamma \leq \bar{\alpha} + \bar{\beta} + \bar{\gamma} = \pi$. $\square$

Rmk: By Preismann's Theorem, if M is closed, $\Gamma < \pi_1 M$ is Abelian, and $\sec_M < 0$, then $\Gamma \cong \mathbb{Z}$.

Nice topic for a Student Talk!

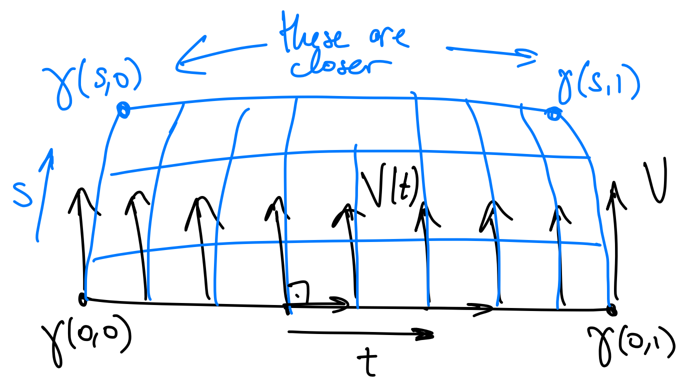
Application of Rauch II:

Corollary: Let (M,g) be a complete Riem. mfld with $\sec \geq 0$, $\gamma:[0,1]\to M$ min. geodesic, $V:[0,1]\to M$ a parallel vector field along γ, with $g(\dot\gamma, V)=0$. Consider the rectangle $\gamma(s,t)=\exp_{\gamma(t)} sV(t)$. Then

$$\mathrm{dist}_g(\gamma(s,0),\gamma(s,1)) \leq \mathrm{dist}_g(\gamma(0,0),\gamma(0,1))$$

If $\sec>0$, then get strict inequality.

with equality if and only if $\gamma([0,1]\times[0,1])\subset M$ is a totally geodesic flat rectangle.



Pf: Since both $t\mapsto\gamma(0,t)$ and $t\mapsto\gamma(s,t)$ are parametrized with $t\in[0,1]$, it suffices to show that $\left\|\frac{\partial}{\partial t}\gamma(s,t)\right\| \leq \left\|\frac{\partial}{\partial t}\gamma(0,t)\right\|$. The Jacobi field $J_t(s)=\frac{\partial}{\partial t}\gamma(s,t)$ along the geodesic $s\mapsto\gamma(s,t)$ satisfies

$$J_t(0)=\frac{\partial}{\partial t}\gamma(0,t)=\dot\gamma(t) \text{ and } J_t'(0)=\frac{D}{ds}\frac{\partial}{\partial t}\exp_{\gamma(t)}sV(t)\Big|_{s=0}=\frac{D}{dt}\frac{\partial}{\partial s}\exp_{\gamma(t)}sV(t)\Big|_{s=0}$$

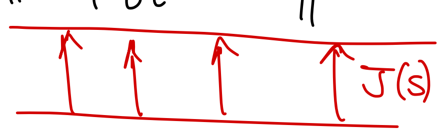
$$=\frac{D}{dt}\underbrace{d(\exp_{\gamma(t)})_0}_{\mathrm{id}}V(t)=\frac{D}{dt}V\overset{V\text{ parallel}}{=}0.$$

By Rauch II, $\|J_t(s)\|\leq\underbrace{\|J_t(0)\|}_{(*)}$

$(*)$ length of comparison Jacobi field in $\mathbb{R}^n$

So $\left\|\frac{\partial}{\partial t}\gamma(s,t)\right\|=\|J_t(s)\|\leq\|J_t(0)\|=\left\|\frac{\partial}{\partial t}\gamma(0,t)\right\|$. $\square$

$(*)$ $J(s)=J(0)$ is constant length.



Rigidity statement: Exercise (using rigidity in comparison thm).

Lecture 6 (Recall Rauch I, II and Appl. of Rauch II) 3/2/2023

Cor. Let (M^n, g) be a complete Riem. mfld with $0 < \kappa \leq \sec \leq K$. Then the distance d between consecutive conjugate points along geodesics in (M^n, g) is $\frac{\pi}{\sqrt{K}} \leq d \leq \frac{\pi}{\sqrt{\kappa}}$.

Pf. Let $\gamma : [0,L] \to M$ be a geodesic, $J : [0,L] \to M$ a Jacobi field with $J(0) = 0$. Let $\tilde{J}$ be a Jacobi field on $S^n\left(\frac{1}{\sqrt{K}}\right)$ with $\tilde{J}(0) = 0$ and $\|\tilde{J}'(0)\| = \|J'(0)\|$. Then, by Rauch II, $\|J(t)\| \geq \|\tilde{J}(t)\| > 0$ for all $t \in \left(0, \frac{\pi}{\sqrt{K}}\right)$, b/c $\tilde{J}(t) = \tilde{J}'(0) \cdot \frac{\sin(t\sqrt{K})}{\sqrt{K}}$, so $d \geq \frac{\pi}{\sqrt{K}}$.

Similarly, if $d > \frac{\pi}{\sqrt{\kappa}}$, then by Rauch II, the round sphere $S^n\left(\frac{1}{\sqrt{\kappa}}\right)$ would only have conjugate points after distance $\frac{\pi}{\sqrt{\kappa}}$, a contradiction. $\square$

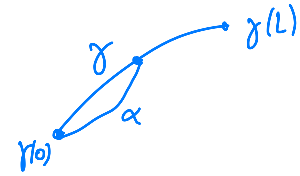
Later will relax this to Ric

Thm (Myers, 1941). If (M, g) is a complete Riem. mfld with $\underline{\sec \geq \kappa > 0}$, then $\operatorname{diam}(M, g) \leq \frac{\pi}{\sqrt{\kappa}}$. In particular, M is compact and $\pi_1 M$ is finite.

Pf. Let $p, q \in M$ and $\gamma : [0,L] \to M$ be a minimizing geodesic with $\gamma(0) = p$ and $\gamma(L) = q$. It suffices to show $\operatorname{length}(\gamma) = L \leq \frac{\pi}{\sqrt{\kappa}}$. Suppose $L > \frac{\pi}{\sqrt{\kappa}}$, and let $J(t)$ be a Jacobi field along $\gamma(t)$ with $J(0) = 0$. Then by Rauch I, $\|J(t)\| \leq \|J'(0)\| \cdot \frac{\sin t\sqrt{\kappa}}{\sqrt{\kappa}}$ for all $t \in \left(0, \frac{\pi}{\sqrt{\kappa}}\right)$, and the first conjugate point along γ happens before distance $\frac{\pi}{\sqrt{\kappa}}$. Therefore the geodesic γ is <u>not minimizing</u>$^{(*)}$ from $\gamma(0) = p$ to $\gamma(L) = q$, which is the desired contradiction. Thus $\operatorname{diam}(M, g) = \sup_{p,q} \operatorname{dist}_g(p,q) \leq \frac{\pi}{\sqrt{\kappa}}$; and M is compact. Applying the same argument on $(\tilde{M}, \tilde{g})$, we find that $\tilde{M}$ is also compact, so $\pi_1 M$ is finite $\pi_1 M \to \tilde{M} \to M$. $\square$

⊛ γ is <u>not minimizing</u> from $\gamma(0)$ to $\gamma(L)$, i.e., $L > \text{dist}(\gamma(0), \gamma(L))$, if and only if $\exists\, t_* \in (0,L)$ s.t. $\gamma(t_*) \in \text{Cut}(\gamma(0))$: so either

- $\gamma(0)$ is conjugate to $\gamma(t_*)$; or
- $\exists\, \alpha \neq \gamma$ geodesic with $\alpha(0) = \gamma(0)$ and $\alpha(t_*) = \gamma(t_*)$.



i.e., M is compact and $\partial M = \emptyset$.

> <u>Thm</u> (Synge, 1936). Let (M^n, g) be a <u>closed</u> Riem. mfld with $\sec > 0$.
>
> If n is even, then M orientable $\Rightarrow \pi_1 M \cong \{1\}$
>
> M non-orientable $\Rightarrow \pi_1 M \cong \mathbb{Z}_2$
>
> If n is odd, then M is orientable.

<u>Pf</u>. Let $\gamma : [0,L] \to M$ be a closed geodesic, ie. $\gamma(0) = \gamma(L) = p$, $\dot\gamma(0) = \dot\gamma(L)$.

Parallel transport along γ gives a linear isometry P of T_pM, with $P\dot\gamma(0) = \dot\gamma(L) = \dot\gamma(0)$. Moreover, P restricts to a linear isometry $P : E \to E$, where $E = \text{span}(\dot\gamma(0))^\perp \subset T_pM$.

If n is even and M^n is orientable, then $\dim E$ is odd and parallel transport along any loop preserves orientation, so $\det P|_E > 0$. Thus, $P|_E$ must have an <u>eigenvector with eigenvalue 1</u>*, say $w \in E$, $\|w\| = 1$. Then let $w(t)$ be the parallel vector field along $\gamma(t)$ with $w(0) = w(L) = w$; and set

$$\gamma(s,t) = \exp_{\gamma(t)} s\,w(t)$$

By Application of Rauch II, we know that

$$\text{length}_g(\gamma(s,t)) < \text{length}_g(\gamma(0,t))$$

b/c $\sec > 0$.

*In the appropriate basis of E, the orthogonal matrix representing P is:

$$P|_E = \left(\begin{array}{ccc|ccc} R_{\theta_1} & & & & & \\ & \ddots & & & & \\ & & R_{\theta_k} & & & \\ \hline & & & \pm 1 & & \\ & & & & \ddots & \\ & & & & & \pm 1 \end{array}\right)$$

2x2 rotation blocks

eigenvalues ± 1.

Thus, if M^n is not simply-connected, let γ be a shortest curve in a nontrivial free homotopy class. Then γ is a closed geodesic of least length among curves in that homotopy class $[\gamma]$. However $\gamma_s = \gamma(s,\cdot)$ has smaller length and $\gamma_s \in [\gamma]$, contradiction.

If n is even and M^n is non-orientable, apply the above argument to its orientable double-cover $(\tilde{M}^n, \tilde{g})$ and find that $\tilde{M}$ is simply-connected, so $\mathbb{Z}_2 \to \tilde{M} \to M$ is the universal cover of M, i.e., $\pi_1 M \cong \mathbb{Z}_2$.

If n is odd and non-orientable, then $\dim E$ is even and there exists a closed geodesic γ (minimizing length among non contractible loops) such that $P: E \to E$ has $\det P|_E < 0$. Thus $P|_E$ must have an eigenvector with eigenvalue $+1$, say $w \in E$, $\|w\| = 1$. Reasoning as before, get a contradiction with γ being of shortest length among such loops, since $\gamma(s,t) = \exp_{\gamma(t)} s\,w(t)$ would be even shorter for $s \neq 0$ small. $\square$

<u>Remark</u>: Closed manifolds (M^n, g) with n odd and $\sec > 0$ may have a large $\pi_1 M$, e.g., consider $\mathbb{Z}_p \circlearrowright S^3$ and Lens space $S^3/\mathbb{Z}_p$, which has $\sec \equiv 1$ and $\pi_1 M \cong \mathbb{Z}_p$.

<u>Chern Problem (1965)</u>: If (M^n, g) is a closed Riem. mfld with $\sec > 0$ and $\Gamma < \pi_1 M$ is an Abelian subgroup, is it true that Γ is cyclic?

<u>A</u>: (K. Shankar 1997) No, there exist examples with $n = 7$ and $\Gamma \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

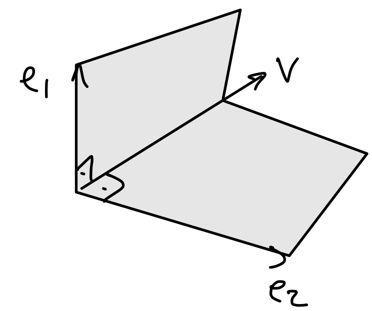
<u>Note</u>: So far, we have shown T^n does not admit $\sec < 0$ (Preismann)
nor $\sec > 0$ (Myers/Synge).

Average comparison Theorems

Def. The Ricci tensor of (M^n, g) is the bilinear symmetric tensor $Ric: \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ given by $Ric(X,Y)_p = \underbrace{\sum_{i=1}^{n} \langle R(e_i, X)Y, e_i\rangle}_{\operatorname{tr} R(\cdot, X)Y}$ where $\{e_i\}$ is an orthonormal basis of T_pM.

In particular, $Ric(V) = Ric(V,V) = \operatorname{tr} R_V$, since $R_V: \mathfrak{X}(M) \to \mathfrak{X}(M)$, $R_V(X) = R(X,V)V$

Geometrically, $Ric(V) = \sum_{i=1}^{n-1} sec(V, e_i)$ is an "average" of sectional curvatures of planes that contain V.



Def. (M^n, g) is Einstein if $Ric_g = \lambda g$. ← "Constant Ricci curvature"

λ: "Einstein constant" (cosmological const...)

Trace the Ricatti eqn. for self-adjoint A:

$$A' + A^2 + R_V = 0 \quad (\text{in } Sym^2 E) \quad \overset{\operatorname{tr}}{\rightsquigarrow} \quad \operatorname{tr} A' + \underbrace{\operatorname{tr}(A^2)}_{\text{this is not a function of } \operatorname{tr} A \dots} + Ric(V) \overset{(*)}{=} 0 \quad (\text{in } \mathbb{R})$$

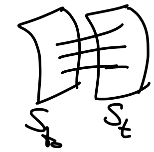
Since $A(V) = \nabla_V V = 0$, can restrict A to $V^\perp$, $A: V^\perp \to V^\perp$, and $A \in Sym^2 V^\perp$. Let $a = \dfrac{\operatorname{tr} A}{n-1}$, and note that

$$A = a \, Id + A_0, \quad \text{where } \operatorname{tr} A_0 = 0. \quad \text{"trace-free part"}$$

So $\langle A_0, I\rangle = 0$. ← recall $\langle A, B\rangle = \operatorname{tr} AB$

Then $\operatorname{tr}(A^2) = \|A\|^2 = a^2 \|Id\|^2 + \|A_0\|^2 = (n-1)a^2 + \|A_0\|^2$ so $(*)$ gives $a' + a^2 + r = 0$, where $r = \dfrac{1}{n-1}\left(\|A_0\|^2 + Ric(V)\right) \geq \dfrac{Ric(V)}{n-1}$

Geometrically, $a(t) = \dfrac{H}{n-1}$ where $H = \operatorname{tr} A$ is the mean curvature of S_t.



Thm. Suppose $A : [t_0, t_1) \to \operatorname{Sym}^2 V^\perp$ is the maximal solution to $A' + A^2 + R = 0$, where $R : \mathbb{R} \to \operatorname{Sym}^2 V^\perp$ is given. Suppose $\exists K \in \mathbb{R}$ s.t.

(i) $\operatorname{tr} R \geq (n-1) K$

(ii) $\operatorname{tr} A(t_0) \leq (n-1) \bar{a}(t_0)$

where $\bar{a} : [t_0, t_2) \to \mathbb{R}$ is the maximal solution to $\bar{a}' + \bar{a}^2 + K = 0$. Let $a = \dfrac{\operatorname{tr} A}{n-1}$

Then $t_1 \leq t_2$ and $a(t) \leq \bar{a}(t)$ for all $t \in [t_0, t_1)$.

Pf: Apply ODE comparison from previous class! $\underset{\substack{\| \\ A}}{A_1} \lesssim \underset{\substack{\| \\ \bar{A}}}{A_2} \Rightarrow \underset{\substack{\| \\ (n-1)a}}{\operatorname{tr} A_1} \leq \underset{\substack{\| \\ (n-1)\bar{a}}}{\operatorname{tr} A_2}$

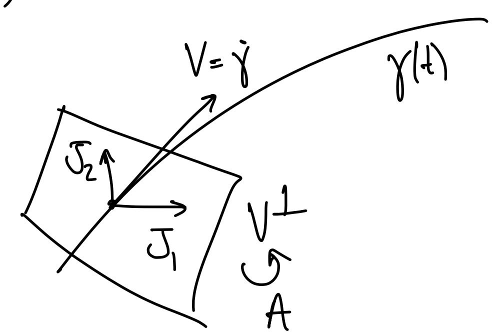
Rmk: Above result remains true if $\bar{a}$ has a pole at t_0; namely

$$A(t) \sim \frac{1}{t - t_0} \operatorname{Id}, \qquad \bar{a} = \frac{\operatorname{sn}_K'}{\operatorname{sn}_K} \quad \text{where} \quad \begin{cases} \operatorname{sn}_K'' + K \operatorname{sn}_K = 0 \\ \operatorname{sn}_K(t_0) = 0 \\ \operatorname{sn}_K'(t_0) = 1. \end{cases}$$

Let $J_1, \dots, J_{n-1}$ be Jacobi fields along γ that form a basis of solutions to

$$J' = A J \qquad (A : V^\perp \to V^\perp)$$

all identified via parallel transport



and set $j = \det(J_1, J_2, \dots, J_{n-1})$.

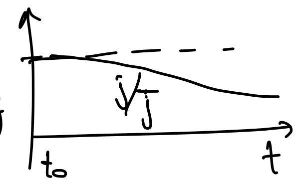
Since
$$(J_1 \wedge \dots \wedge J_{n-1})' = \sum_{k=1}^{n-1} J_1 \wedge \dots \wedge J_k' \wedge \dots \wedge J_{n-1}$$
$$= \sum_{k=1}^{n-1} J_1 \wedge \dots \wedge A J_k \wedge \dots \wedge J_{n-1}$$

we have $j' = (n-1) a j$, because $j = \langle J_1 \wedge \dots \wedge J_{n-1}, \mathrm{vol} \rangle$.

Thm. Let $A : [t_0, t_1) \to \mathrm{Sym}^2 V^\perp$ and $a = \frac{1}{n-1} \operatorname{tr} A$ be s.t. $a \le \overline{a}$, and $j' = (n-1) a j$. Choose $\overline{j}$ s.t. $\overline{j}' = (n-1)\overline{a}\,\overline{j}$. Then $j/\overline{j}$ is nonincreasing.

Pf: Once again, apply ODE comparison from before!

$$(n-1)a \le (n-1)\overline{a} \implies \left(\log \frac{j}{\overline{j}}\right)' \le 0 \implies j/\overline{j} \text{ nonincreasing}$$



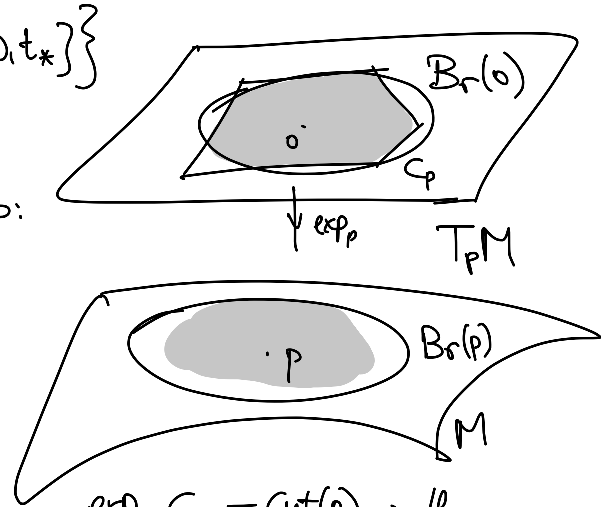
Lecture 7

3/9/2023

Thm (Bishop Volume Comparison). Let (M^n, g) be a Riem. mfld with $\mathrm{Ric} \geq (n-1)K$ and $\overline{M}$ be the simply-connected Riem. mfld with $\sec_{\overline{M}} \equiv K$. Then $\forall p \in M$, $\mathrm{Vol}(B_r(p)) \le \mathrm{Vol}(\overline{B_r})$, where $B_r(p) \subset M$ and $\overline{B_r} \subset \overline{M}$ are balls of radius r. Moreover, equality holds if and only if $B_r(p) \underset{\text{isom}}{\cong} \overline{B_r}$.

Pf: We will show that $r \mapsto \dfrac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(\overline{B_r})}$ is nonincreasing; the conclusion follows since $\displaystyle\lim_{r \downarrow 0} \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(\overline{B_r})} = 1$ because both approach Euclidean balls as $r \downarrow 0$.

Let $\mathrm{cut}(v) = \max\{t_* > 0 : \exp_p tv \text{ is min. geod. on } [0, t_*]\}$ and $C_p = \{tv : t \le \mathrm{cut}(v), \|v\| = 1\} \subset T_pM$. Then $\exp_p : C_p \to M$ is a diffeom. onto its image, so:



$\exp_p C_p = \mathrm{Cut}(p)$ is the cut locus of p

$$\mathrm{Vol}(B_r(p)) = \int_{B_r(p)} 1 \; d\mathrm{vol} = \int_{\exp_p(B_r(0) \cap C_p)} 1 \; d\mathrm{vol}$$

Change of variables formula

$$\overset{\text{}}{=} \int_{B_r(0) \cap C_p} \det\left(d(\exp_p)_u\right) du$$

Polar coord.

$$= \int_{S^{n-1}(1)} \int_0^{r(v)} \det\left(d(\exp_p)_{tv}\right) t^{n-1} \, dt \, dv$$

Recall:
$B_r(p) = \exp_p(B_r(0)) = \exp_p(B_r(0) \cap C_p)$

where $r(v) = \min\{r, \mathrm{cut}(v)\}$ for $v \in T_pM$, $\|v\| = 1$, i.e. $v \in S^{n-1}(1) \subset T_pM$.

Since $d(\exp_p)_{tv}\, e_i = \frac{1}{t}\left(d(\exp_p)_{tv}\, te_i\right) = \frac{1}{t} J_i(t)$ is the Jacobi field along $t \mapsto \exp_p tv$ with $J_i(0)=0$ and $J_i'(0)=e_i$, it follows that

$$\det\left(d(\exp_p)_{tv}\right) = \frac{1}{t^{n-1}} \det\left(J_1(t), \dots, J_{n-1}(t)\right) \quad \text{and hence:}$$

$$\mathrm{Vol}(B_r(p)) = \int_{S^{n-1}(1)} \int_0^{r(v)} \underbrace{\det(J_1(t), \dots, J_{n-1}(t))}_{j_v(t)}\, dt\, dv$$

if needed, extend $j_v(t)$ as $j_v(t)=0$ for $t > \mathrm{cut}(v)$.

By previous result, $j_v(t)/\bar{j}(t)$ is non increasing on $[0,r]$, where $\bar{j}(t) = \det(\bar{J}_1, \dots, \bar{J}_{n-1})$, for corresponding Jacobi fields $\bar{J}_i$ on $\bar{M}$.

Set $$q(t) = \frac{1}{\mathrm{Vol}(S^{n-1}(1))} \int_{S^{n-1}(1)} \frac{j_v(t)}{\bar{j}(t)}\, dv,$$ which is also non-increasing (because it is an average of nonincreasing quantities). As before,

$$\mathrm{Vol}(\bar{B}_r) = \int_{S^{n-1}(1)} \int_0^r \bar{j}(t)\, dt\, dv \overset{\substack{\text{(space w/ sec} \equiv k\\ \text{is isotropic)}}}{=} \mathrm{Vol}(S^{n-1}) \int_0^r \bar{j}(t)\, dt$$

Thus,

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(\bar{B}_r)} = \frac{\int_{S^{n-1}(1)} \int_0^r j_v(t)\, dt\, dv}{\mathrm{Vol}(S^{n-1}(1)) \cdot \int_0^r \bar{j}(t)\, dt} \overset{\text{Fubini}}{=} \frac{\int_0^r q(t) \cdot \bar{j}(t)\, dt}{\int_0^r \bar{j}(t)\, dt}$$

is non increasing, because RHS is the ($\bar{j}$ – weighted) average(*) of the nonincreasing function $q(t)$ over growing intervals.

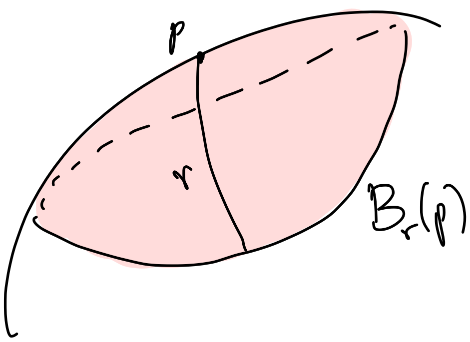
(*) More explicitly: if $\phi, \psi > 0$, and $t \mapsto \dfrac{\phi(t)}{\psi(t)}$ is non increasing, then

$$r \mapsto \frac{\int_0^r \phi(t)\,dt}{\int_0^r \psi(t)\,dt} = \frac{\int_0^{\bar r} \frac{\phi(s)}{\psi(s)}\,ds}{\int_0^{\bar r} ds} \text{ is non increasing, where } \begin{cases} ds = \psi(t)\,dt \\ \bar r = s(r). \end{cases}$$

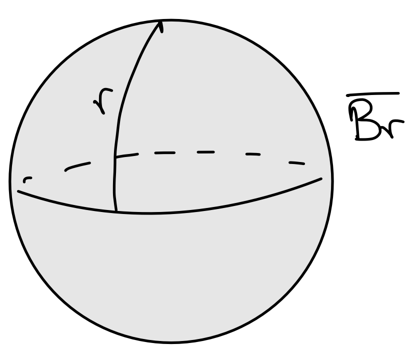
Rigidity statement follows from rigidity statements in ODE comparison: if $\forall v \in S^{n-1}(1)$, $\forall\, 0 \le t \le r$, $j_v(t) = \bar{j}(t)$, then $a(t) = \bar a(t)$, for all $0 \le t \le r$, so $R(t) = \bar R(t) = \kappa\, \mathrm{Id}$. Thus $B_r(p)$ has constant curvature $\sec \equiv \kappa$ and is hence isometric to $\overline{B_r}$. $\square$

Remark: Similarly, one can prove $r \mapsto \dfrac{\mathrm{Vol}(\partial B_r(p))}{\mathrm{Vol}(\partial \overline{B_r})}$ is non increasing.

Geometrically:



$\mathrm{Ric} \geq (n-1)\kappa$

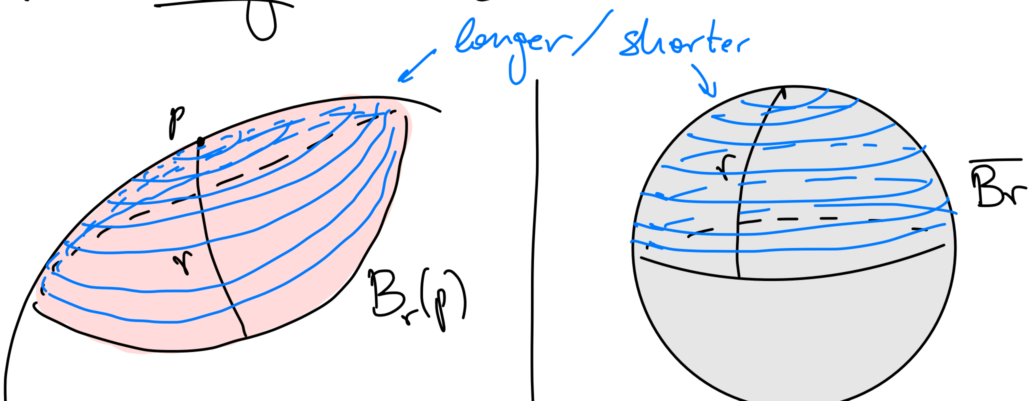


$\sec \equiv \kappa$

(so $\mathrm{Ric} = (n-1)\kappa$)

$$\mathrm{Vol}(B_r(p)) \le \mathrm{Vol}(\overline{B_r})$$
$$=$$
$$\Updownarrow$$
$$B_r(p) \underset{\text{isom}}{\cong} \overline{B_r}$$

With stronger control on curvature $\sec \geq \kappa$ we know that:



so "integrating" get the above.

BUT

$\mathrm{Ric} \geq \kappa(n-1)$ is enough for this "integral" control.

Another situation in which "integral"/"average" control is enough:

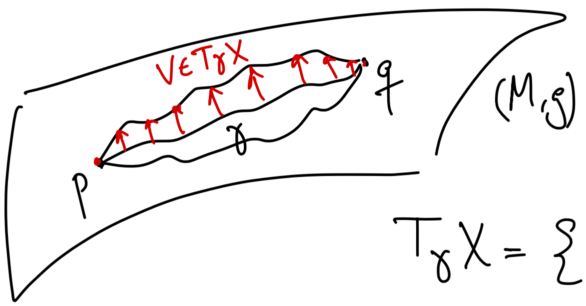
Did this before w/ $\sec \geq \kappa$

Thm (Myers, 1941). If (M^n, g) is a complete Riem. mfld w/ $\underline{Ric \geq \kappa(n-1)}$, with $\kappa > 0$, then $\mathrm{diam}(M^n, g) \leq \frac{\pi}{\sqrt{\kappa}}$. In particular, (M^n, g) is compact and $\pi_1 M$ is finite.

To prove this, need more about <u>variational structure</u> of geodesics:

Fix $p, q \in M$ and $X = \{ \gamma \in W^{1,2}([0,\ell], M) : \gamma(0) = p, \ \gamma(\ell) = q \}$

This is a Hilbert manifold locally modeled on Hilbert space $W^{1,2}([0,1], \mathbb{R}^n)$



Given $\gamma \in X$, can identify

$$T_\gamma X = \left\{ V \in W^{1,2}([0,\ell], TM) : \begin{array}{l} \text{vector field along } \gamma \text{ with} \\ V(0) = 0, \ V(\ell) = 0 \end{array} \right\}$$

Define the energy functional $E : X \to \mathbb{R}$

$$E(\gamma) = \frac{1}{2} \int_0^\ell g(\dot\gamma, \dot\gamma)\, dt$$

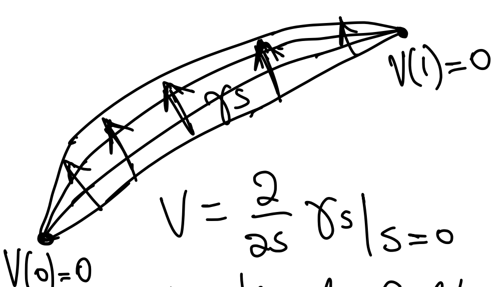
Alternatively, can consider the length functional

$$L(\gamma) = \int_0^\ell \|\dot\gamma\|\, dt$$

on curves $\gamma \in W^{1,1}$ (more on this later!)

Then $\gamma \in X$ is a critical point, i.e. $\underbrace{\delta E(\gamma)}_{\delta E(\gamma) : T_\gamma X \to \mathbb{R}} = 0$, iff γ is a geodesic. Indeed:

<u>First Variation</u>: $\delta E(\gamma)(V) = \frac{d}{ds} E(\gamma_s)\Big|_{s=0} = \frac{1}{2} \int_0^\ell \frac{d}{ds} g(\dot\gamma_s, \dot\gamma_s)\Big|_{s=0} dt$



$$= \int_0^\ell g\left(\frac{D}{ds} \dot\gamma_s\Big|_{s=0}, \dot\gamma\right) dt = \int_0^\ell g\left(\frac{DV}{dt}, \dot\gamma\right) dt$$

$V = \frac{\partial}{\partial s} \gamma_s \Big|_{s=0}$ variational field

$$\overset{\text{int. by parts}}{=} \underbrace{g(V, \dot\gamma)\Big|_0^\ell}_{\substack{=0 \text{ b/c boundary conditions are} \\ V(0)=0,\ V(\ell)=0}} - \int_0^\ell g\left(V, \frac{D\dot\gamma}{dt}\right) dt$$

(e.g. $\gamma_s(t) = \exp_{\gamma_0(t)} sV(t)$.)

so $\delta E(\gamma)(V) = \frac{d}{ds} E(\gamma_s)\Big|_{s=0} = 0$ for all variations γ_s if and only if

Fundamental Lemma of the Calculus of Variations:

$$\int \phi \psi = 0, \ \forall \psi \iff \phi = 0$$

i.e. $\langle \phi, \psi \rangle_{L^2} = 0, \forall \psi \iff \phi = 0$

(being sloppy about regularity... use test function $\psi \in C^\infty_{cc}$ etc..)

$\frac{D\dot\gamma}{dt} = 0$ i.e. γ is a geodesic.

(and hence $\|\dot\gamma\| = \text{const.}$)

<u>Second Variation:</u> Suppose γ is a geodesic. Then the "Hessian" of E at γ is

$$\underbrace{\delta^2 E(\gamma)}(V, V) = \frac{d^2}{ds^2} E(\gamma_s)\Big|_{s=0} = \frac{1}{2}\int_0^\ell \frac{d^2}{ds^2} g(\dot\gamma_s, \dot\gamma_s)\Big|_{s=0} dt$$

$\delta^2 E(\gamma) : T_\gamma X \times T_\gamma X \to \mathbb{R}$ symmetric bilinear form called the "Index Form", or $\delta^2 E(\gamma) : T_\gamma X \to T_\gamma X$ symmetric endomorphism

$$= \int_0^\ell \frac{d}{ds} g\left(\frac{D}{ds}\dot\gamma_s, \dot\gamma_s\right)\Big|_{s=0} dt$$

$$= \int_0^\ell g\left(\frac{D^2}{ds^2}\dot\gamma_s\Big|_{s=0}, \dot\gamma\right) + g\left(\frac{D}{ds}\dot\gamma_s, \frac{D}{ds}\dot\gamma_s\right)\Big|_{s=0} dt$$

$V = \frac{\partial}{\partial s}\gamma_s$

$$\overset{\downarrow}{=} \int_0^\ell g\left(\frac{D}{ds}V', \dot\gamma\right) + g(V', V')\ dt$$

$$V' = \frac{DV}{dt} = \frac{D}{dt}\frac{\partial}{\partial s}\gamma_s = \frac{D}{ds}\frac{\partial}{\partial t}\gamma_s = \frac{D}{ds}\dot\gamma_s$$

$$= \int_0^\ell g\left(\frac{D}{dt}\frac{D}{ds}V + R(V, \dot\gamma)V, \dot\gamma\right) + g(V', V')\,dt$$

$$= \int_0^\ell g\left(\frac{D}{dt}\frac{D}{ds}V, \dot\gamma\right) - g(R(V,\dot\gamma)\dot\gamma, V) + g(V', V')\,dt$$

int. by parts (x2)

$$\overset{\searrow}{=} \underbrace{g\left(\frac{D}{ds}V, \dot\gamma\right)\Big|_0^\ell}_{=0 \text{ b/c } V(0)=0,\, V(\ell)=0} - \int_0^\ell g\left(\frac{D}{ds}V, \cancelto{0}{\frac{D}{dt}\dot\gamma}\right) dt$$

$$+ \underbrace{g(V', V)\Big|_0^\ell}_{=0 \text{ b/c } V(0)=0,\, V(\ell)=0} - \int_0^\ell g(V'', V) + g(R(V,\dot\gamma)\dot\gamma, V)\,dt$$

$$= -\int_0^\ell g(V'', V) + g(R(V,\dot\gamma)\dot\gamma, V)\, dt$$

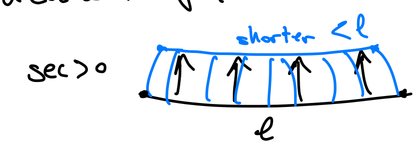
$$= -\int_0^\ell g(\underbrace{V'' + R(V,\dot\gamma)\dot\gamma}, V)\, dt$$

This vanishes iff V is a Jacobi field:
$V'' + R(V,\dot\gamma)\dot\gamma = 0$.

Note: If $\sec_M > 0$, then $g(R(V,\dot\gamma)\dot\gamma, V) > 0$, so using a parallel vector field V along a geodesic γ, get

$$\delta^2 E(\gamma)(V,V) = -\int_0^\ell \underset{\overset{\|}{0}}{g(V'', V)} + \underbrace{g(R(V,\dot\gamma)\dot\gamma, V)}_{>0} < 0$$

i.e. γ is unstable; small variations of γ decrease its energy (and its length). Recall/cf. application of Rauch II.



Remarks about Energy v. Length of curves:

- Critical points of E come parametrized w/ constant speed, i.e. $\delta E(\gamma) = 0$ implies $\|\dot\gamma\| = \text{const.}$, while the length functional is invariant under reparametrizations of γ; in particular critical points need not have constant speed.
- Apply Cauchy–Schwartz inequality $\left(\int_0^\ell \phi \cdot \psi\right)^2 \leq \int_0^\ell \phi^2 \cdot \int_0^\ell \psi^2$ with $\phi \equiv 1$ to get

$$L(\gamma)^2 = \left(\int_0^\ell \|\dot\gamma\|\, dt\right)^2 \leq \ell \cdot \int_0^\ell \|\dot\gamma\|^2 dt = 2\ell\, E(\gamma) \quad \text{and "$=$" iff } \|\dot\gamma\| \equiv 1.$$

So if γ is a unit speed min. geod. from p to q, and β is a curve from p to q, then $E(\gamma) = \frac{1}{2\ell} L(\gamma)^2 \leq \frac{1}{2\ell} L(\beta)^2 \leq E(\beta)$, with $E(\gamma) = E(\beta)$ if and only if β is unit speed and hence $L(\beta) = L(\gamma)$ so β is a unit speed min. geod. from p to q.

Upshot: γ is a critical point of $E \iff \gamma$ is a unit speed geodesic. (maybe not min...)

$\underbrace{\gamma \text{ is a minimizer of } E}_{\text{w/ boundary conditions: } E: X \to \mathbb{R},\ X = \{\gamma \in W^{1,2}([0,\ell], M),\ \gamma(0) = p,\ \gamma(\ell) = q\}.} \iff \gamma$ is a unit speed min. geodesic (realizes distance)

Useful for later: if $\gamma:[0,\ell]\to M$ is unit speed, then given any variation V, i.e. a vector field V along γ, we have:

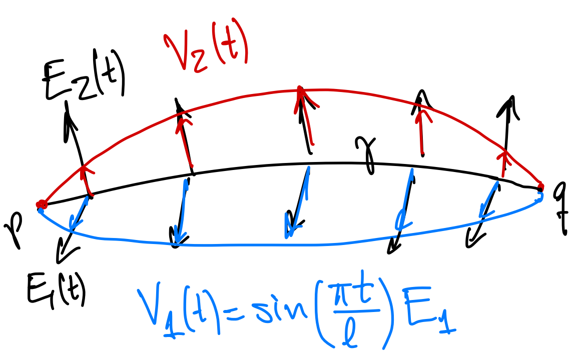
$$\boxed{\delta L(\gamma)(V)=\frac{1}{\ell}\,\delta E(\gamma)(V)=\frac{1}{\ell}\left(g(V,\dot\gamma)\Big|_0^\ell-\int_0^\ell g\left(V,\frac{D\dot\gamma}{dt}\right)dt\right)}$$

If $\delta L(\gamma)=0$ (equivalently $\delta E(\gamma)=0$), then

$$\boxed{\begin{aligned}\delta^2 L(\gamma)(V,V)&=\frac{1}{\ell}\,\delta^2E(\gamma)(V,V)\\&=\frac{1}{\ell}\left(g(\nabla_V V,\dot\gamma)\Big|_0^\ell+\int_0^\ell g(V',V')-g(R(V,\dot\gamma)\dot\gamma,V)\,dt\right)\end{aligned}}$$

<u>Pf of Myers Thm:</u> Suppose M has $\mathrm{Ric}\geqslant K(n-1)>0$, and let $\gamma:[0,\ell]\to M$ be a unit speed geodesic, i.e. $\delta E(\gamma)=0$. If γ is min., i.e. $\mathrm{dist}_g(\gamma(0),\gamma(\ell))=\ell$, then $\delta^2E(\gamma)(V,V)\geqslant 0$ for all V along γ with $V(0)=0$ and $V(\ell)=0$. Let $\{E_i\}$ be a parallel o.n.b. of vector fields along γ, i.e., $g(E_i,\dot\gamma)=0$, $g(E_i,E_j)=\delta_{ij}$; and set

$$V_i(t)=\sin\left(\frac{\pi t}{\ell}\right)E_i(t),\quad \text{so } V_i(0)=0 \text{ and } V_i(\ell)=0$$



$$\delta^2E(\gamma)(V_i,V_i)=-\int_0^\ell g(V_i'',V_i)+g(R(V_i,\dot\gamma)\dot\gamma,V_i)\,dt$$

$$=\int_0^\ell \sin\left(\frac{\pi t}{\ell}\right)^2\left(\frac{\pi^2}{\ell^2}-g(R(E_i,\dot\gamma)\dot\gamma,E_i)\right)dt$$

$$V_i'(t)=\frac{\pi}{\ell}\cos\left(\frac{\pi t}{\ell}\right)E_i(t)+\sin\left(\frac{\pi t}{\ell}\right)\underbrace{E_i'(t)}_{=0}$$

$$V_i''(t)=-\frac{\pi^2}{\ell^2}\sin\left(\frac{\pi t}{\ell}\right)E_i(t)+\frac{\pi}{\ell}\cos\left(\frac{\pi t}{\ell}\right)\underbrace{E_i'(t)}_{=0}$$

Thus, adding from $i=1$ to $i=n-1$:

$$0 \leq \sum_{i=1}^{n-1} \delta^2 E(\gamma)(V_i, V_i) = \sum_{i=1}^{n-1} \int_0^\ell \sin\left(\frac{\pi t}{\ell}\right)^2 \left(\frac{\pi^2}{\ell^2} - g(R(E_i,\dot\gamma)\dot\gamma, E_i)\right) dt$$

$$= \int_0^\ell \sin\left(\frac{\pi t}{\ell}\right)^2 \left((n-1)\frac{\pi^2}{\ell^2} - \underbrace{\sum_{i=1}^{n-1} g(R(E_i,\dot\gamma)\dot\gamma, E_i)}_{Ric(\dot\gamma,\dot\gamma) \geq K(n-1)}\right) dt$$

$$\leq \int_0^\ell \sin\left(\frac{\pi t}{\ell}\right)^2 (n-1) \underbrace{\left(\frac{\pi^2}{\ell^2} - K\right)}_{<0 \text{ if } \ell > \frac{\pi}{\sqrt{K}} \ldots} dt$$

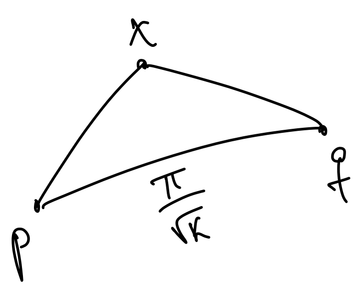
So such minimizing unit speed geod. $\gamma:[0,\ell]\to M$ must have length $\ell \leq \frac{\pi}{\sqrt{K}}$, for otherwise we get a contradiction above. $\square$

Rigidity in Myers Theorem

(Originally by Shiu-Yuen Cheng, with different proof)
↳ student of S.S. Chern

Thm. Let (M^n, g) be a complete Riem. mfld with $Ric \geq K\cdot(n-1) > 0$ and $\mathrm{diam}(M^n,g) = \mathrm{diam}(S^n(1/\sqrt{K})) = \frac{\pi}{\sqrt{K}}$. Then $(M^n,g) \underset{\text{isom.}}{\cong} S^n(1/\sqrt{K})$.

Pf: Let $p, q \in M$ be points at maximal distance, i.e. $\mathrm{dist}(p,q) = \frac{\pi}{\sqrt{K}}$. Then, for all $r>0$, the balls $B_r(p)$ and $B_{\frac{\pi}{\sqrt{K}}-r}(q)$ are disjoint: if $d(p,x) < r$ and $d(x,q) < \frac{\pi}{\sqrt{K}} - r$, then



$$\frac{\pi}{\sqrt{K}} = d(p,q) \leq d(p,x) + d(x,q) < \frac{\pi}{\sqrt{K}}$$

so no such x can exist. Thus,

$$M \supseteq B_r(p) \,\dot\cup\, B_{\frac{\pi}{\sqrt{K}}-r}(q) \quad \text{(disjoint union)}$$

hence $\mathrm{Vol}(M) \overset{\circledast}{\geq} \mathrm{Vol}(B_r(p)) + \mathrm{Vol}(B_{\frac{\pi}{\sqrt{\kappa}}-r}(q))$. From Bishop Vol. Comp.,

$$r \longmapsto \frac{\mathrm{Vol}(B_r(x))}{\mathrm{Vol}(\overline{B_r})}$$ is non increasing; in particular,

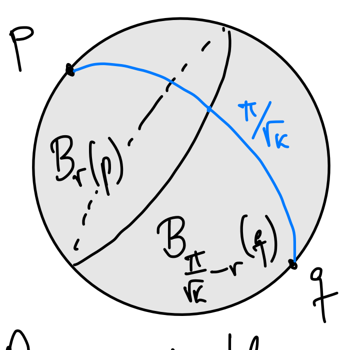
$$\frac{\mathrm{Vol}(B_r(x))}{\mathrm{Vol}(\overline{B_r})} \geq \frac{\mathrm{Vol}(B_{\frac{\pi}{\sqrt{\kappa}}}(x))}{\mathrm{Vol}(\overline{B_{\frac{\pi}{\sqrt{\kappa}}}})} = \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S^n(1/\sqrt{\kappa}))} \quad \text{b/c} \begin{cases} \overline{B_{\frac{\pi}{\sqrt{\kappa}}}} = S^n(1/\sqrt{\kappa}) \\ B_{\frac{\pi}{\sqrt{\kappa}}}(x) = M \end{cases}$$

i.e. $\mathrm{Vol}(B_r(x)) \geq \dfrac{\mathrm{Vol}(M)}{\mathrm{Vol}(S^n(1/\sqrt{\kappa}))}\mathrm{Vol}(\overline{B_r})$. Thus, applying this in $\circledast$:

$$\mathrm{Vol}(M) \geq \frac{\mathrm{Vol}(M)}{\mathrm{Vol}(S^n(1/\sqrt{\kappa}))}\Big(\underbrace{\mathrm{Vol}(\overline{B_r}) + \mathrm{Vol}(\overline{B_{\frac{\pi}{\sqrt{\kappa}}-r}})}_{\mathrm{Vol}(S^n(1/\sqrt{\kappa}))}\Big) = \mathrm{Vol}(M),$$ so all

the inequalities using Bishop Vol. Comp. above are <u>equalities</u>. Thus, from rigidity in the equality case of Bishop Vol. Comp., we have

$B_r(p) \underset{\text{isom}}{\cong} \overline{B_r}$ and $B_{\frac{\pi}{\sqrt{\kappa}}-r}(q) \underset{\text{isom}}{\cong} \overline{B_{\frac{\pi}{\sqrt{\kappa}}-r}}$, thus $M \underset{\text{isom}}{\cong} S^n(1/\sqrt{\kappa})$.



$M \underset{\text{isom}}{\cong} S^n(1/\sqrt{\kappa})$

Indeed, there is no room for any $M \setminus \overline{(B_r(p) \cup B_{\frac{\pi}{\sqrt{\kappa}}-r}(q))}$ because that would increase the diameter. $\square$

<u>Open problem:</u> If (M^n, g) has $\mathrm{Ric} \geq (n-1)\kappa > 0$ and $\mathrm{Vol}(M,g) > \frac{1}{2}\mathrm{Vol}(S^n(1/\sqrt{\kappa}))$, then $M \underset{\substack{\text{homeo?} \\ \text{diffeo?}}}{\cong} S^n$.

<u>Exercise:</u> a) Find counter-example with $\mathrm{Vol}(M,g) = \frac{1}{2}\mathrm{Vol}(S^n(1/\sqrt{\kappa}))$.

b) Prove that (M^n, g) as above is simply-connected.

<u>Hint:</u> if M is not simply connected, take its universal cover.

Lecture 8 3/16/2023

Sol to Exercise

a) $\mathbb{RP}^n(\frac{1}{\sqrt{K}}) = \mathbb{S}^n(\frac{1}{\sqrt{K}})/\mathbb{Z}_2$, where $\mathbb{Z}_2 \curvearrowright \mathbb{S}^n(\frac{1}{\sqrt{K}})$ has a metric
$\pm 1 \cdot x = \pm x$
with $\sec \equiv K$, hence $Ric = (n-1)K$, and
$Vol(\mathbb{RP}^n(1/\sqrt{K})) = \frac{1}{2} Vol(\mathbb{S}^n(1/\sqrt{K}))$. Clearly, $\pi_1 \mathbb{RP}^n = \mathbb{Z}_2$.

b) If (M^n, g) has $Ric \geq (n-1)K$, and $[\gamma] \in \pi_1 M$, let $\Gamma = \langle [\gamma] \rangle < \pi_1 M$ be the subgroup generated by $[\gamma]$, set $d = |\Gamma|$. Then let $\tilde{M}^n \longrightarrow M^n$ be the covering space corresponding to Γ, recall it is a degree d covering. In particular, with the pullback metric, by Fubini, and our assumption,

$$Vol(\tilde{M}, \tilde{g}) \overset{!}{=} d \cdot Vol(M, g) > \frac{d}{2} Vol(\mathbb{S}^n(1/\sqrt{K})).$$

Since $(\tilde{M}^n, \tilde{g})$ also has $Ric \geq (n-1)K$, by Bishop Volume Comparison,

$$Vol(\tilde{M}, \tilde{g}) \leq Vol(\mathbb{S}^n(1/\sqrt{K})), \text{ so } d < 2, \text{ i.e. } |\Gamma| = d = 1 \text{ so}$$

$[\gamma]$ is trivial, hence $\pi_1 M = \{1\}$. $\square$

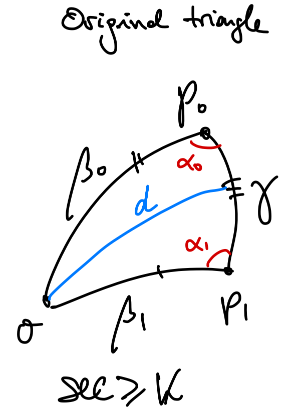
Toponogov Triangle Comparison

Here and throughout: if $K > 0$, then assume all lengths are $< \frac{\pi}{\sqrt{K}}$.

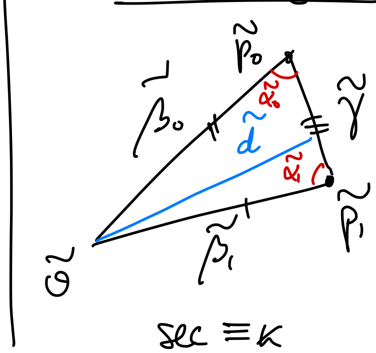
Triangle Version

If (M^n, g) has $\sec \geq K$, $o, p_0, p_1 \in M$, $\gamma : [0, L] \to M$ geod from p_0 to p_1, β_i min. geod from o to p_i, then $d = dist_g(o, \gamma(t)) \geq \tilde{d} = dist_{\tilde{g}}(\tilde{o}, \tilde{\gamma}(t))$ for all $t \in [0, L]$ and $\alpha_i \geq \tilde{\alpha}_i$.

Original triangle



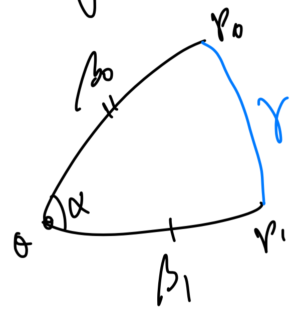
Comp. triangle w/ same side lengths



Hinge Version

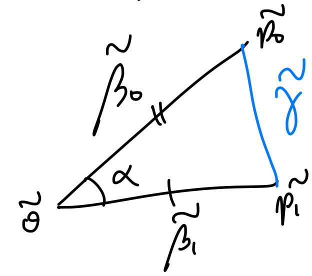
If (M^n, g) has $\sec \geq K$, $o, p_0, p_1 \in M$, β_i min. geod. from o to p_i, Then $\ell(\gamma) \leq \ell(\tilde{\gamma})$; where $\gamma, \tilde{\gamma}$ are the min. geod. that close the hinge: $\ell(\gamma) = \mathrm{dist}_g(p_0, p_1)$
$\ell(\tilde\gamma) = \mathrm{dist}_{\tilde g}(\tilde p_0, \tilde p_1)$.

Original triangle



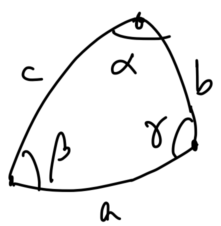
$\sec \geq K$

Comp. triangle w/ same hinge: $\ell(\beta_i) = \ell(\tilde\beta_i)$, $\alpha = \tilde\alpha$



$\sec \equiv K$

Corollary: A geodesic triangle on a manifold with $\sec \geq 0$ satisfies



(i) $\ell(c)^2 \leq \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma$ $\qquad$ ℓ = length

(ii) $\alpha + \beta + \gamma \geq \pi$ $\qquad$ If $\sec > 0$, then get strict inequalities.

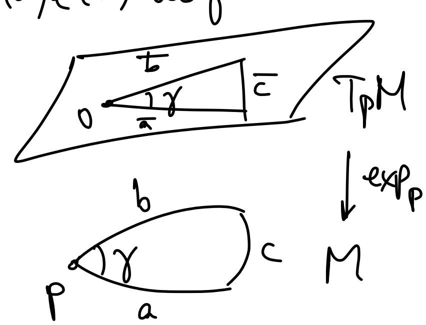
Pf: (i) is immediate:

$$\ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma \overset{\text{Gauss Lemma}}{=} \ell(\bar a)^2 + \ell(\bar b)^2 - 2\ell(\bar a)\ell(\bar b)\cos\gamma$$

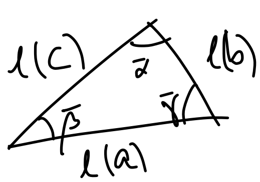
$$\overset{\text{Law of Cosines in } \mathbb{R}^2}{=} \ell(\bar c)^2$$

$$\overset{\text{Toponogov (Hinge)}}{\geq} \ell(c)^2.$$

where $a = \exp_p \bar a$, $b = \exp_p \bar b$, $c = \exp_p \bar c$.



(ii) Follows from (i) as in the $\sec \leq 0$ case: build comparison triangle in $\mathbb{R}^2$ with side lengths $\ell(a), \ell(b), \ell(c)$, and angles $\bar\alpha, \bar\beta, \bar\gamma$. Then $\ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma \geq \ell(c)^2$
$= \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\bar\gamma$



So $\cos\gamma \leq \cos\bar\gamma$ hence $\gamma \geq \bar\gamma$. Similarly for α, β and get

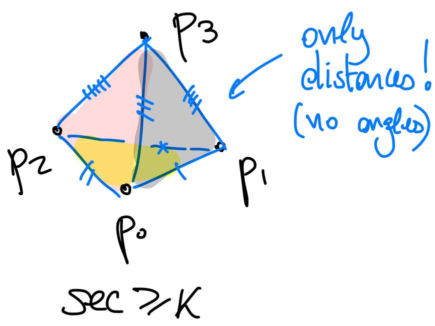
$$\alpha + \beta + \gamma \geq \bar\alpha + \bar\beta + \bar\gamma = \pi. \qquad \square$$

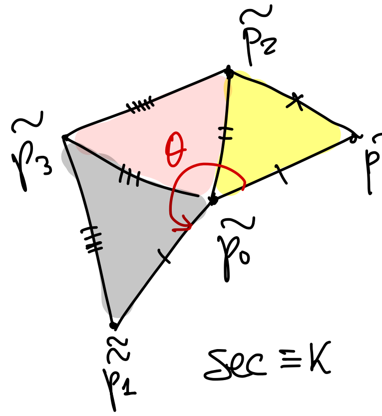
Combining above w/ earlier work on $\sec \leq 0$

As before...

Cor: (M^n, g) has $\sec \geq 0$ (≤ 0) iff $\forall p \in M$, $\exp_p : C_p \subset T_pM \xrightarrow{\cong} M$ is distance non-increasing (non-decreasing).

Four point Version





In $\mathbb{R}^2$, build comparison triangles for each of the 3 triangles containing p_0; then sum of angles at $\tilde{p}_0$ is:

$$\theta \leq 2\pi$$

Thm. On a Riem mfld, each of the above characterizes sec $\geq K$:

$$\sec \geq K \iff \text{Triangle Version} \iff \text{Hinge Version} \iff \text{Four point Version.}$$

Metric space where distances are realized by lengths of curves

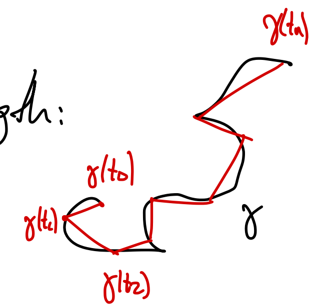
Note: The four point Version makes sense on any length space (M, d), and is sometimes written curv $\geq K$, if M is not a manifold. If M is a locally compact complete length space with curv $\geq K$ then it is called an Alexandrov space.

Very useful to study Riem. mflds w/ sec $\geq K$
cf. "distributions" to study solutions to PDEs...

An aside about length spaces:

Def: $\gamma : [0, L] \to M$ is rectifiable if it has finite length:

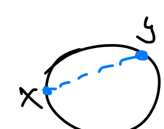
$$\ell(\gamma) = \sup_{0 = t_0 < t_1 < \dots < t_n = L} \underbrace{\sum_{k=1}^{n} d(\gamma(t_k), \gamma(t_{k-1}))}_{\text{polygonal length.}}$$



Def: (M, d) is a length space if $\forall x, y \in M$,

$$d(x, y) = \inf \{ \ell(\gamma) : \gamma \text{ rectifiable curve joining } x \text{ and } y \}.$$

It is complete if $\forall x, y \in M$ there exists a minimizing rectifiable curve realizing $d(x, y)$ i.e. achieving the above inf.

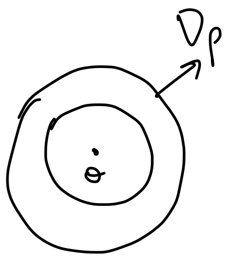


Example: Any Riem. mfld. / **Non-example:** $S^1 \subset \mathbb{R}^2$ w/ chordal metric.

Proof of Toponogov Triangle Comparison (Triangle Version):

Proof by Karcher, not the original Toponogov proof. Other proofs possible using Rauch.

<u>Preliminaries:</u> $\rho(x) = \mathrm{dist}_g(x, o)$ is C^∞ on $M \setminus (o \cup \mathrm{Cut}(o))$



$\nabla\rho$ is the unit radial vector field, $\rho(\exp_o(v)) = \|v\|$.

$S_r = \rho^{-1}(r) = \partial B_r(o)$ are equidistant hypersurfaces.

If $\sec \geq K$, then Ricatti comparison gives $A \leq \tilde{A} = \frac{sn_K'}{sn_K} \mathrm{Id}$, where

$A = \nabla(\nabla\rho) = \mathrm{Hess}\,\rho$ and $sn_K'' + K\, sn_K = 0$, $sn_K(0) = 0$, $sn_K'(0) = 1$.

So $A|_{\nabla\rho^\perp} \leq \frac{sn_K'}{sn_K} \mathrm{Id}$ and $A|_{\nabla\rho} = 0$ b/c ρ grows linearly along integral curves of $\nabla\rho$. Moreover, $\tilde{A}|_{\nabla\tilde{\rho}^\perp} = \frac{sn_K'}{sn_K} \mathrm{Id}$ and

$\tilde{A}|_{\nabla\tilde{\rho}} = 0$.

Consider $f \circ \rho$, where <u>f is to be chosen later.</u> Then by Chain Rule,

Very common trick in Geometric Analysis

$$\nabla(f\circ\rho) = f'(\rho)\nabla\rho \implies \mathrm{Hess}(f\circ\rho)(X,Y) = \langle \nabla_X \nabla(f\circ\rho), Y\rangle$$
$$= \langle X(f'(\rho))\nabla\rho + f'(\rho)\nabla_X\nabla\rho, Y\rangle$$
$$= f''(\rho)\, d\rho(X)\, d\rho(Y) + f'(\rho)\, \mathrm{Hess}\,\rho(X,Y)$$

So, on $\nabla\rho^\perp$, since first term vanishes, $\mathrm{Hess}(f\circ\rho) = f'(\rho)\,\mathrm{Hess}\,\rho$
$$\leq f'(\rho) \frac{sn_K'}{sn_K} \mathrm{Id}$$

and on $\mathrm{span}\{\nabla\rho\}$, $\mathrm{Hess}(f\circ\rho)(\nabla\rho,\nabla\rho) = f''(\rho)\underbrace{\|\nabla\rho\|^4}_{=1} + f'(\rho)\underbrace{\mathrm{Hess}\,\rho(\nabla\rho,\nabla\rho)}_{=0} = f''(\rho)$.

Choose f so that $\boxed{f' = sn_K}$, so $sn_K'' + K sn_K = 0 \implies f''' + K f' = 0 \overset{\text{integrate}}{\implies}$ $f'' + Kf = C$, so $f''(\rho) = -K f(\rho) + C$. Thus,

This choice makes both bounds on $\text{span}\{\nabla\rho\} \oplus \nabla\rho^\perp$ become the same, so get a bound on the whole space.

$$\begin{cases} \text{Hess}(f\circ\rho) \leq \dfrac{f' sn_K'}{sn_K} \text{Id} = sn_K' \,\text{Id} = \boxed{f''(\rho)}\, \text{Id} = -K f(\rho) + C & \text{on } \nabla\rho^\perp \\ \text{Hess}(f\circ\rho) \leq \boxed{f''(\rho)} = -K f(\rho) + C & \text{on } \text{span}\{\nabla\rho\}. \end{cases}$$

Upshot: $\text{Hess}(f\circ\rho) \leq -K(f\circ\rho)\,\text{Id} + C$; analogously, $\underbrace{\text{Hess}(f\circ\tilde\rho) \leq -K(f\circ\tilde\rho)\,\text{Id} + C}_{\text{on comparison space with } sec \equiv K.}$

Let $\delta(t) := f(\text{dist}_g(\sigma, \gamma(t))) - f(\widetilde{\text{dist}}(\tilde\sigma, \tilde\gamma(t))) = (f\circ\rho\circ\gamma)(t) - (f\circ\tilde\rho\circ\tilde\gamma)(t)$ (b/c $sn > 0$)

We want to show that $\delta(t) \geq 0$ for all $t \in [0, L]$. Note: f is increasing, so $\delta \geq 0$ is equiv. to $d \geq \tilde d$.

If not, $\exists t_* \in [0, L]$ with $\delta(t_*) < 0$, and $m = \min_{t\in[0,L]} \delta(t) < 0$.

If $K > 0$: Let $K' > K$ be suff. close and $\tau > 0$ s.t.

$$L < \frac{\pi}{\sqrt{K'}} - \tau$$

so that the comp. triangle still w/ length $L + \tau$ exists in $sec \equiv K'$. If $K \leq 0$, ignore.

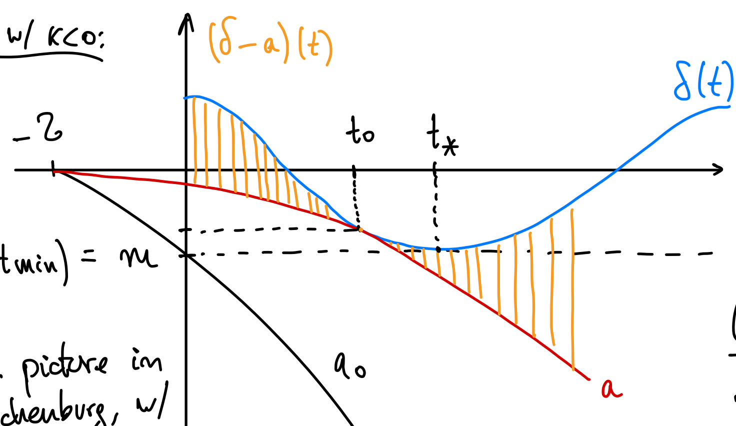
Let a_0 be sol. to $a_0'' + K' a_0 = 0$ s.t. $a_0(-\tau) = 0$, $a_0|_{[0,L]} \leq m$;

i.e. $a_0(t) = -\mu \cdot sn_{K'}(t + \tau)$. (suff. large constant $\mu > 0$) Then $\exists \lambda > 0$ s.t. $a = \lambda a_0$ satisfies

$$a \leq \delta \quad \text{and} \quad \boxed{a(t_0) = \delta(t_0) \quad \text{for some} \quad t_0 \in (0, L).}$$

Note: we do not need $t_0 \in (0,L)$ to be s.t. $m = \delta(t_0)$. These are unrelated.

Pic w/ $K<0$:

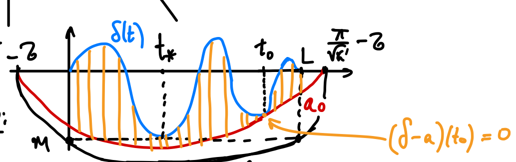


Since $m = \delta(t_*) < 0$, we must have that $\delta(t_0) < 0$, otherwise $\lambda a_0 < \delta$ for arbitrarily small $\lambda \geq 0$ i.e. $\delta \geq 0$.

Case 1: $\gamma(t_0) \notin \text{Cut}(\sigma)$, so δ is C^∞ near t_0, hence

cf. picture in Eschenburg, w/ a, a_0 of opposite concavity!

K>0:



"Upshot" above (a.k.a. Hessian comparison)

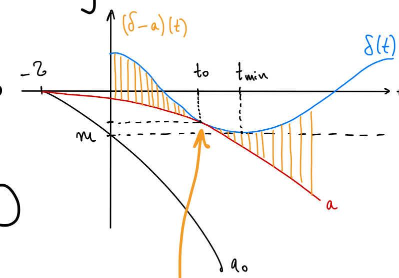
$$\frac{d^2}{dt^2} f(\text{dist}(\sigma, \gamma(t))) = \text{Hess}(f\circ\rho\circ\gamma)(t)\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) \leq -K f(\rho(\gamma(t))) + C$$

Similarly, on $\tilde{M}$ with $\sec \equiv K$, we have

"Upshot" above

$$\frac{d^2}{dt^2} f(\widetilde{dist}(\tilde{o},\tilde{\gamma}(t))) = \mathrm{Hess}(f\circ\tilde{\rho}\circ\tilde{\gamma})\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right) \le -K(f\circ\tilde{\rho}\circ\tilde{\gamma})(t) + C$$

Thus, $\delta'' = \frac{d^2}{dt^2}(f\circ\rho\circ\gamma - f\circ\tilde{\rho}\circ\tilde{\gamma}) \le -K\left[f\circ\rho\circ\gamma - f\circ\tilde{\rho}\circ\tilde{\gamma}\right] = -K\delta.$

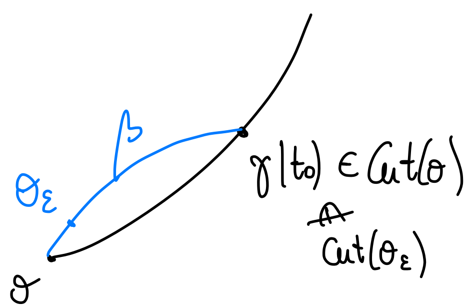


On the other hand, $a'' = -K'a$ and $\delta(t_0) = a(t_0) < 0$ so

$$(\delta - a)''(t_0) \le -K\delta(t_0) + K'\underbrace{a(t_0)}_{=\delta(t_0)} = \underbrace{\delta(t_0)}_{<0}\cdot\underbrace{(K'-K)}_{>0} < 0$$

which contradicts the fact that t_0 is a minimum for $\boxed{\delta(t) - a(t)}$

<u>Case 2:</u> $\gamma(t_0) \in \mathrm{Cut}(o)$. Let β be a min. geod. from o to $\gamma(t_0)$



Choose $o_\varepsilon = \beta(\varepsilon)$, replace $\rho = dist(\cdot, o)$ with $\rho_\varepsilon = dist(\cdot, o_\varepsilon) + dist(o_\varepsilon, o)$. By the triangle inequality, $\rho_\varepsilon(\gamma(t_0)) \ge \rho(\gamma(t_0))$, and $\rho_\varepsilon(\gamma(t_0)) = \rho(\gamma(t_0))$

i.e: ρ_ε is "upper support function" for ρ at $\gamma(t_0)$

Since β is min. from o_ε to $\gamma(t_0)$, we have $\gamma(t_0) \notin \mathrm{Cut}(o_\varepsilon)$. Reasoning as before with $f\circ\rho_\varepsilon$, and sending $\varepsilon \downarrow 0$, get

$$(f\circ\rho_\varepsilon\circ\gamma)'' \le -K(f\circ\rho_\varepsilon\circ\gamma) + C + \text{error}$$

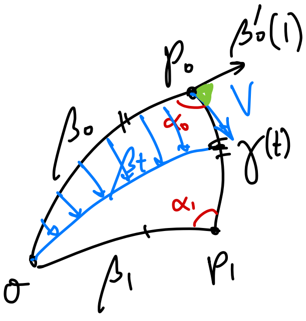
Then $f\circ\rho_\varepsilon$ is upper support function for $f\circ\rho$ at $\gamma(t_0)$, and hence $\delta_\varepsilon = f\circ\rho_\varepsilon\circ\gamma - f\circ\tilde{\rho}\circ\tilde{\gamma}$ is s.t. $\delta_\varepsilon - a$ is upper support function for δ at t_0. Thus, it also attains a minimum at t_0, contradicting $(\delta_\varepsilon - a)''(t_0) < 0$.

Finally, in order to prove that $\alpha_i \geq \tilde{\alpha}_i$, we again argue by contradiction. Suppose

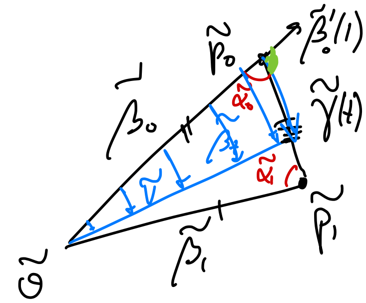
$$\alpha_0 = \text{angle between } \beta_0' \text{ and } \gamma'(0)$$
$$\tilde{\alpha}_0 = \text{———} \,''\, \text{———} \ \tilde{\beta}_0' \text{ and } \tilde{\gamma}'(0)$$

satisfy $\alpha_0 < \tilde{\alpha}_0$. Assume $p_0 \notin Cut(o)$, so $\exp_o$ is invertible near p_0.

Let β_t be the shortest curve joining o to $\gamma(t)$, and similarly $\tilde{\beta}$.
Assume $\beta_t : [0,1] \to M$ for each t;



$\sec \geq K$



$\sec \equiv K$

Then, Taylor series expansions give:

$$\mathrm{dist}_g(o, \gamma(t)) = L(\beta_t) = \mathrm{dist}_g(o, p_0) + t \frac{d}{dt} L(\beta_t)\Big|_{t=0} + O(t^2)$$

$$\widetilde{\mathrm{dist}}(\tilde{o}, \tilde{\gamma}(t)) = L(\tilde{\beta}_t) = \widetilde{\mathrm{dist}}(\tilde{o}, \tilde{p}_0) + t \frac{d}{dt} L(\tilde{\beta}_t)\Big|_{t=0} + O(t^2)$$

While, by first variation of length,

No contribution from o nor $\tilde{o}$ since endpoint is fixed there: $V(1)=0$. $\tilde{V}(1)=0$

$$\delta L(\beta_0)(V) = \frac{d}{dt} L(\beta_t)\Big|_{t=0} = -\underbrace{g(\gamma'(0), \beta_0'(1))}_{\pi - \alpha_0}$$

where V is a variational field along β_t w/ $V(1) = \gamma'(0)$

$$\delta L(\tilde{\beta}_t)(\tilde{V}) = \frac{d}{dt} L(\tilde{\beta}_t)\Big|_{t=0} = -\underbrace{\tilde{g}(\tilde{\gamma}'(0), \tilde{\beta}_0'(1))}_{\pi - \tilde{\alpha}_0}$$

where $\tilde{V}$ is a variational field along $\tilde{\beta}_t$ w/ $\tilde{V}(1) = \tilde{\gamma}'(0)$.

Since $\alpha_0 < \tilde{\alpha}_0$, for small t, we get from the above Taylor expansion that

$L(\beta_t) < L(\tilde{\beta}_t)$, contradicting the previous conclusion that

$L(\beta_t) \geq L(\tilde{\beta}_t)$ i.e. $\delta(t) \geq 0$. If $p_0 \in Cut(o)$, use $o_\varepsilon = \beta_0(\varepsilon)$ instead...

similar to the above perturbation argument.

$\square$