$①$

COHOMOGENEITY ONE MINIMAL HYPERSURFACES

SPHERICAL BERNSTEIN PROBLEM (CHERN, ICM, 1970)

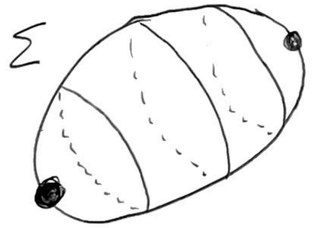
<u>Q</u>: If $S^{n-1} \hookrightarrow (S^n, g_{round})$ is an embedded <u>minimal</u> sphere, is it congruent to an equator?

<u>A</u>: [Almgren, 1966], [Calabi 1967] YES if $n=3$

[Hsiang 1983] NO if $n = 4, 5, 6, \dots$

SYMMETRY REDUCTION ← (<u>Alternative</u> <u>method</u> <u>to</u> <u>construct</u> <u>minimal</u> <u>surfaces</u>, compared to <u>Min-Max</u>; ⊕ control topology and produce C^∞ objects ⊖ requires non-generic ambient space)

Σ

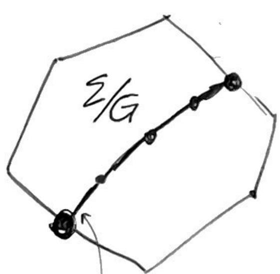


$G \curvearrowright (M, g)$ $\quad V(x) = Vol(\pi^{-1}(x))$ volume

$\downarrow \pi$

$(M/G, \check{g})$

$$Vol_M(\Sigma) = \int_{\Sigma/G} V(x)\, dx = \int_{\Sigma/G} vol_{V^{2/k}\check{g}} = Vol_\Omega(\Sigma/G)$$



$\dim \Sigma/G = k$

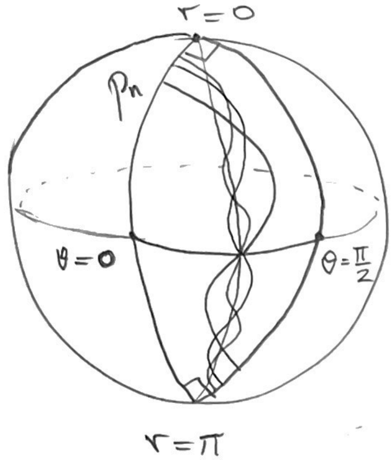
"cohomogeneity of Σ"

$\Omega := (M/G, \underbrace{V^{2/k}\check{g}}_{g_\Omega})$ "conformal orbit space"

$$\boxed{\Sigma \subset M \text{ minimal}} \iff \boxed{\Sigma/G \subset \Omega \text{ minimal}}$$

└ geodesic if $k=1$

HSIANG 1983



$T^2 \curvearrowright \mathbb{C}^2 \oplus \mathbb{R} \supset S^4$

$S^4/T^2 = \left\{ 0 \le r \le \pi, \ 0 \le \theta \le \frac{\pi}{2} \right\}$

spherical sector $\sec \equiv 1$.

$\check{g} = dr^2 + \sin^2 r \, d\theta^2$

$V = 2\pi^2 \sin r \sin 2\theta$

(say: $\boxed{\nexists \text{ Killing fields in } M/G}$)

only pic → (Infinite sequence $\{\gamma_{p_n}\}$ of free boundary geodesics in Ω starting from $p_n = (r_n, 0) \to (0,0)$.)

- $\Sigma_n = \pi^{-1}(\gamma_{p_n}) \subset S^4$ minimal spheres
- $\Sigma_n \longrightarrow \Sigma(T^2)$ suspension of Clifford torus $T^2 \subset S^3$

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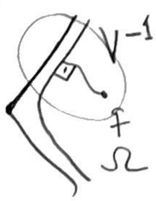
joint work in progress w/ D. Corona, F. Giannoni, P. Piccione

§1. LOCAL EXISTENCE (Any $\dim \Omega$)

> **Theorem A.** For each $p \in \partial\Omega$ in a codimension 1 stratum of M/G, there is a unique maximal g_Ω-geodesic γ_p starting transversely from $\partial\Omega$. In fact, γ_p is $\check{g}$-orthogonal to $\partial\Omega$.

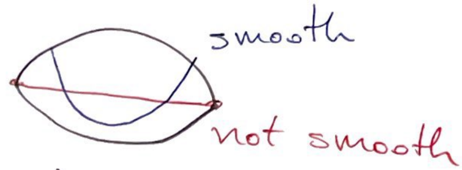
Pf strategy: Let $q \in \Omega$ near $\partial\Omega$, $V = O(\mathrm{dist}_{\check{g}}(\cdot, \partial\Omega)^d)$ ← drop in dimension



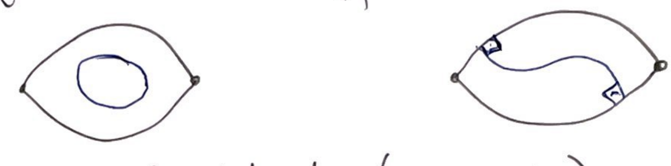
- c_ε minimizing g_Ω-geodesic from q to $V^{-1}(\varepsilon)$
- Key Claim: $c_\varepsilon \longrightarrow c_0$ in $W^{1,s}$, $1 < s < 1 + \frac{1}{d}$, so in $C^{0, 1-\frac{1}{s}}$.

GLOBAL BEHAVIOR OF $M \circlearrowleft \Sigma = \pi^{-1}(\gamma) \subset M/G$

- Σ smooth $\iff$ γ smooth



- Σ closed $\iff$ γ closed or free boundary



- Σ embedded $\iff$ γ embedded (simple)

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§2. COMPACTNESS

CHOI-SCHOEN 1985, M^3, $Ric > 0$

$\{ \Sigma^2 \subset M^3 \text{ min., emb., genus}(\Sigma) = g \}$ is C^k-compact$_{k \geq 2}$

SHARP 2017, M^n, $3 \leq n \leq 7$, $Ric > 0$

$\{ \Sigma^{n-1} \subset M^n \text{ min., emb, } Vol(\Sigma) \leq C_1, \ ind(\Sigma) \leq C_2 \}$ is C^k-compact$_{k \geq 2}$

> **Theorem B.** If $G \curvearrowright M$ is s.t. $\Omega = (M/G, g_\Omega)$ is a smooth 2-mfld w/ smooth $\partial\Omega$ (possibly empty) and $sec_\Omega > 0$, then $\{ \Sigma^{n-1} \subset M^n \text{ min., emb., } G\text{-invariant} \}$ is C^k-compact, $k \geq 2$.

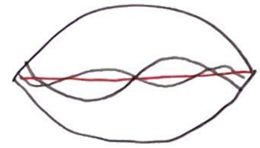
E.g., if $sec_M > 0$ and $V : M/G \to \mathbb{R}$ is log-concave, then

$$sec_\Omega = \big(sec_{\check{g}} - \underbrace{\Delta_{\check{g}} \log V}_{\leq 0} \big) \cdot V^{-2} > 0$$

↑ e.g., if $G \curvearrowright M$ is polar and $sec_M > 0$, then V is log-concave [Wilking]

⚠ $\partial\Omega$ smooth is necessary (Hsiang's example)

$\{S^3 \subset S^4 \text{ min, emb., } T^2\text{-inv.}\}$ is not C^k-compact

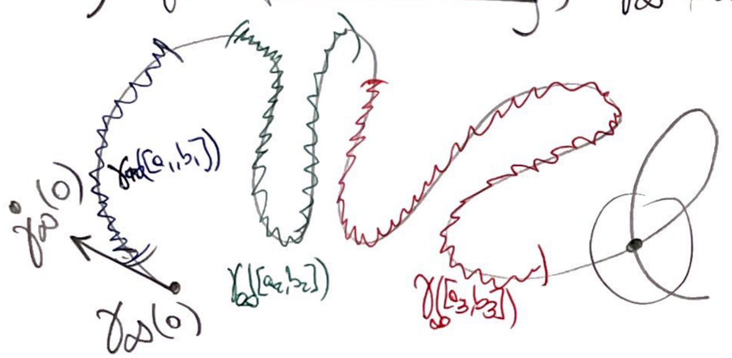


ΣT^2
singular limit
stationary varifold

Pf (Sketch). Suffices to show $\ell(\gamma) \le C$ for all γ emb. geod. in Ω that is free boundary or simple closed.

Suppose γ_n are emb. geod. w/ $\ell(\gamma_n) \nearrow +\infty$. Let $\gamma_\infty = \lim \gamma_n$.

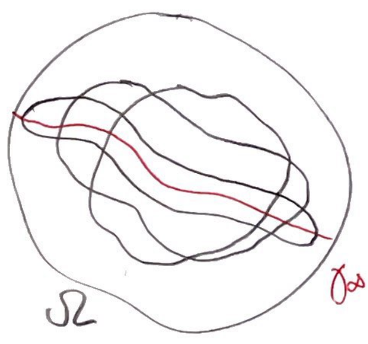
a) γ_n free boundary, $\gamma_\infty = \lim \gamma_n$, $\Rightarrow$ $\gamma_\infty([a_n, b_n])$ are disjoint geod. segments, $|b_n - a_n| \nearrow +\infty$



$(\sec_\Omega > 0)$ $\Longrightarrow$ eventually have conjugate points

$(\dim\Omega = 2)$ $\Longrightarrow$ self-intersection. ✗ □

b) γ_n simple closed, then either $\gamma_\infty = \lim \gamma_n$

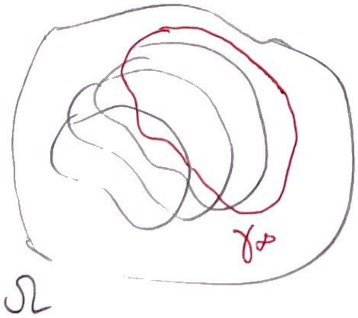


- γ_∞ is multiplicity 2 free boundary geodesic (use a))

or

- γ_∞ is away from $\partial\Omega$, so can use Toponogov's bound

$$\ell(\gamma) \le 2\pi r \text{ if } \sec \ge \frac{1}{r^2}$$



§3. GEOMETRIC APPLICATION

Q. (YAU, 1987). Are all emb. min. 2-spheres in

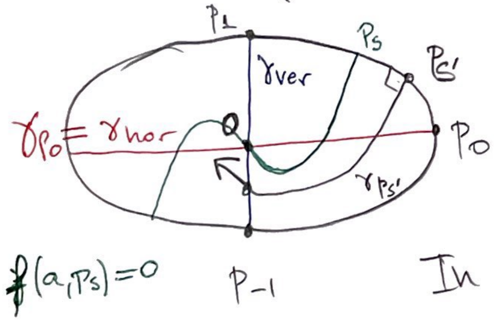
$$E^3 = \left\{ x \in \mathbb{R}^4 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\}$$

planar, i.e., of the form $\Sigma^2 = E^3 \cap \Pi^3$, $\Pi^3 \subset \mathbb{R}^4$ linear hyperplane

HASLHOFER–KETOVER 2019: No: at least one nonplanar sol. if $a \gg b, c, d$

> **Theorem C.** For all $n \geq 3$, if a is sufficiently large, then
>
> $$E^n = \left\{ x \in \mathbb{R}^{n+1} : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \dots + \frac{x_n^2}{b^2} + \frac{x_{n+1}^2}{c^2} = 1 \right\}$$
>
> has arbitrarily many pairwise noncongruent nonplanar $O(n-1)$-invariant embedded minimal $(n-1)$-spheres.

Pf. $E^n / O(n-1) = \left\{ (x, r, y) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{r^2}{b^2} + \frac{y^2}{c^2} = 1,\ r \geq 0 \right\}$, $V = \omega_{n-2} r^{n-2}$



Thm A $\Rightarrow \exists\, f_{(odd)} : \underset{a}{(0, +\infty)} \times \underset{s}{[-1, 1]} \to \mathbb{R}$ s.t.

$f(a, s) = 0 \iff \gamma_{p_s}$ crosses γ_{ver} at O

In part., $f(a, s) = 0$, $\forall a > 0$ "trivial branch"

LOCAL BIFURCATION (CRANDALL–RABINOWITZ)

$\mathrm{ind}(\gamma_{p_0}) \nearrow +\infty$ as $a \nearrow +\infty \implies \exists \{a_m^{(odd)}\}$ sequence of bifurcat. instants

GLOBAL BIFURCATION (RABINOWITZ)

By Thm B, restriction of $(a, p_s) \mapsto a$ to $f^{-1}(0)$ is proper, so branches in $f^{-1}(0)$ either:

- reattach ✗ ← (counting nodal domains argument)
- are noncompact ✓