COHOMOGENEITY ONE MINIMAL HYPERSURFACES

ROME, NOV 2024

"SPHERICAL BERNSTEIN PROBLEM" (CHERN, ICM 1970):

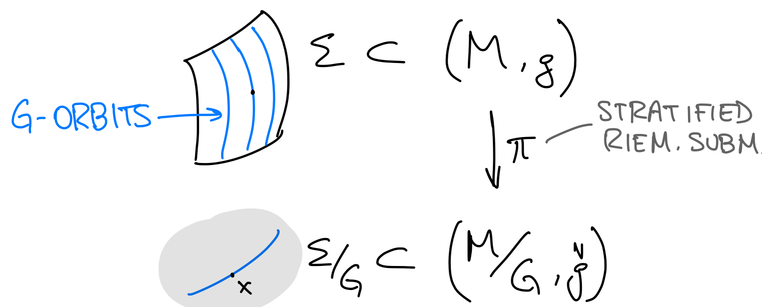
ARE THERE NON-EQUATORIAL EMBEDDED MINIMAL $S^{n-1} \hookrightarrow S^n$?

UNIT ROUND METRIC

- NO, IF $n=3$ [ALMGREN 1966], [CALABI, 1967]
- YES, IF $n=4,5,6$ [HSIANG, 1983]. ← CONSTRUCTED WITH EQUIVARIANT GEOMETRY:

SYMMETRY REDUCTION [HSIANG–LAWSON, 1971]

- $G \curvearrowright (M,g)$ ISOMETRIC ACTION, $(M/G, \check{g})$ ORBIT SPACE, $\pi : M \to M/G$ PROJECTION
- $V : M/G \to \mathbb{R}$ VOLUME OF ORBITS
  $V(x) = \mathrm{Vol}_M(\pi^{-1}(x))$



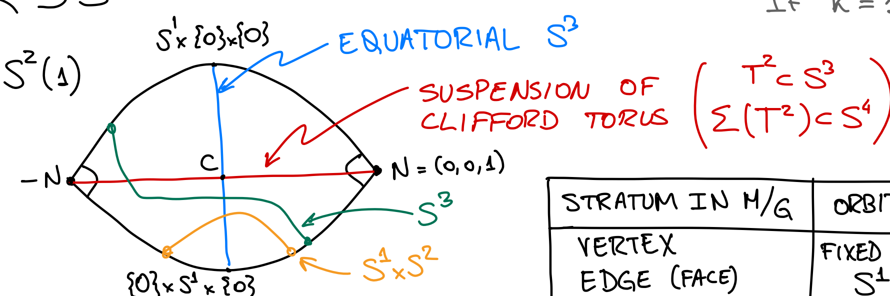
- $\Sigma \subset M$ G-INVARIANT SUBMFLD
- $k = \dim \Sigma/G$ "COHOMOGENEITY" OF Σ.
- SET $\Omega := (M/G, \underbrace{V^{2/k}\check{g}}_{g_\Omega})$, SO $\mathrm{Vol}_M(\Sigma) = \int_{\Sigma/G} V(x)\,dx = \int_{\Sigma/G} d\,\mathrm{vol}_{V^{2/k}\check{g}} = \mathrm{Vol}_\Omega(\Sigma/G)$.

$$\boxed{\Sigma \subset M \text{ MINIMAL}} \overset{\text{"SYMMETRIC CRITICALITY" (PALAIS)}}{\Longleftrightarrow} \boxed{\Sigma/G \subset \Omega \text{ MINIMAL}}$$

I.E., "GEODESIC" IF $k=1$.

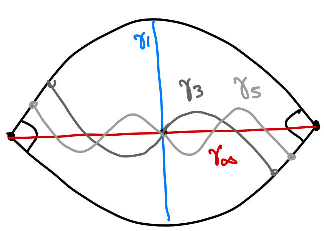
Ex: $O(2) \times O(2) \curvearrowright \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R} \supset S^4$

- $(M/G, \check{g}) \overset{\text{isom}}{=} \left\{ \begin{array}{l} \text{SPHERICAL SECTOR} \\ \text{W/ RIGHT ANGLE} \end{array} \right\} \subset S^2(1)$
- $\check{g} = dr^2 + \sin^2 r \, d\theta^2$, $\sec_{\check{g}} \equiv 1$
  $0 \leq r \leq \pi$, $0 \leq \theta \leq \pi/2$
- $V(r,\theta) = 2\pi^2 \sin r \sin 2\theta$, $k=1$
- TWO ISOMETRIC INVOLUTIONS, (LOCAL) KILLING FIELDS

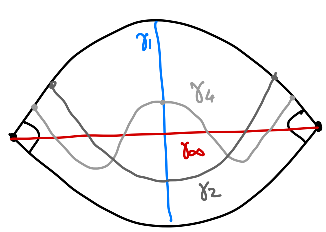


STRATUM IN M/G	ORBIT
VERTEX	FIXED PT
EDGE (FACE)	S^1
INTERIOR	T^2

HSIANG: THERE IS AN INFINITE SEQUENCE $\{\gamma_j\}$ OF "FREE BOUNDARY" GEODESICS ON Ω:



ODD



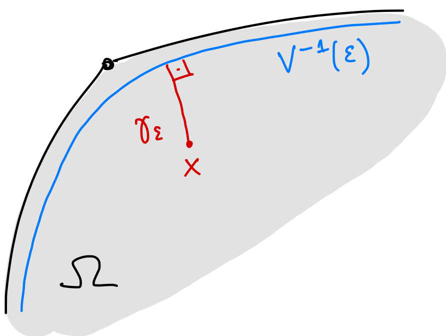
EVEN

- AD HOC SINGULAR ODE INITIAL VALUE PROB., USING THAT M/G HAS KILLING FIELDS
- IDEAS OF [BOMBIERI, DE GIORGI, GIUSTI, 1969], AFTER BLOW UP AT FIXED POINT
- ODE ANALYSIS OF PHASE PORTRAIT

JOINT WORK WITH: DARIO CORONA, FABIO GIANNONI, PAOLO PICCIONE

HENCEFORTH: COHOMOGENEITY $K=1$, I.E., SEEK GEODESICS IN Ω. dim ≥ 2, but possibly no Killing fields

§1. LOCAL EXISTENCE (WITHOUT KILLING FIELDS)



- $V=0$ ON $\partial\Omega$, SO METRIC $g_\Omega = V^2 \check{g}$ DEGENERATES
- NEAR EACH "FACE", $V = O\left(\mathrm{dist}_{\check{g}}(\cdot,\partial\Omega)^d\right)$ FOR SOME $d\geq 1$ ← DIMENSION DROP IN ORBIT TYPE
- GIVEN $\varepsilon>0$, $x\in \mathrm{int}\,\Omega$, LET γ_ε BE UNIT SPEED MIN. g_Ω-GEOD FROM x TO $V^{-1}(\varepsilon)\subset \mathrm{int}\,\Omega$.

KEY CONTRIBUTION BUT RATHER TECHNICAL, SO I'LL SKIP DETAILS

CLAIM: AS $\varepsilon\searrow 0$, $\gamma_\varepsilon \longrightarrow \gamma$ IN $W^{1,p}([0,1],\Omega)$, $1<p<1+\frac{1}{d}$

⚠ NONTRIVIAL B/C $\underbrace{V^2(\gamma_\varepsilon)}_{\to 0}\cdot\|\dot\gamma_\varepsilon\|^2_{\check{g}}\equiv 1$, SO $\|\dot\gamma_\varepsilon\|\to+\infty$ AS $\varepsilon\searrow 0$.

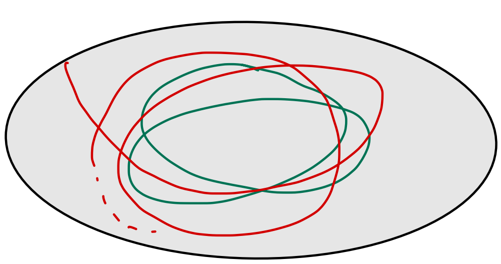
> **THM A.** FOR EACH $p\in\partial\Omega$ IN THE INTERIOR OF A FACE, THERE IS A UNIQUE MAXIMAL g_Ω-GEODESIC γ_p IN Ω, THAT STARTS AT p, TRANSVERSELY TO $\partial\Omega$. IN FACT, γ_p IS $\check{g}$-ORTHOGONAL TO $\partial\Omega$.

§2. GLOBAL BEHAVIOR OF $\Sigma=\pi^{-1}(\gamma)$

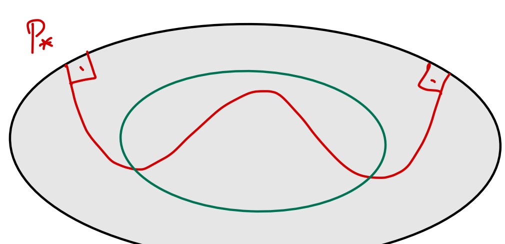
E.G., $\Sigma(T^2)\subset S^4$ IS NOT SMOOTH.

- Σ IS SMOOTH $\iff$ γ IS SMOOTH
- Σ IS CLOSED $\iff$ γ IS CLOSED OR FREE BOUNDARY
- Σ IS EMBEDDED $\iff$ γ IS EMBEDDED (SIMPLE)

IMMERSED MIN. SUBMFLD.



EMBEDDED MIN. SUBMFLD.



C^k – COMPACTNESS: ← I.E., UNIFORM BOUNDS ON $\int_\Sigma |A|^2$ AND $\mathrm{Vol}(\Sigma)$.

- CHOI–SCHOEN, 1985: IF (M^3,g) IS CLOSED RIEM. MFLD., $\mathrm{Ric}>0$, THEN $\{\underbrace{\Sigma^2}_{\text{FIXED TOPOLOGICAL TYPE}}\subset M^3$ MINIMAL, EMBEDDED$\}$ IS C^k-COMPACT, $\forall k\geq 2$.

- SHARP, 2017: IF (M^n,g), $3\leq n\leq 7$ IS CLOSED RIEM. MFLD, $\mathrm{Ric}>0$, THEN $\{\Sigma^{n-1}\subset M^n$ MINIMAL, EMBEDDED, $\mathrm{Vol}(\Sigma)\leq C_1$, $\mathrm{index}(\Sigma)\leq C_2\}$ IS C^k-COMPACT, $\forall k\geq 2$.

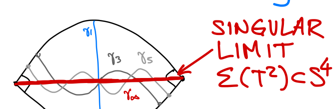
> **THM B.** IF $G\curvearrowright M$ IS SUCH THAT $\Omega=(M/G, g_\Omega)$ IS A SMOOTH* 2-MANIFOLD WITH SMOOTH* BOUNDARY AND $\sec_{g_\Omega}>0$, THEN $\{\Sigma^{n-1}\subset M^n$ MIN., EMBEDDED, G-INVARIANT$\}$ IS C^k-COMPACT, $\forall k\geq 2$.

E.G., IF $\sec_M>0$ AND $V: M/G\to\mathbb{R}$ LOG-CONCAVE, THEN

$$\sec_{g_\Omega}=\Big(\underbrace{\sec_{\check g}}_{\substack{>0\\ \text{[O'NEILL]}}}-\underbrace{\Delta_{\check g}\log V}_{\geq 0}\Big)\cdot V^{-2}>0.$$

← E.G., IF $G\curvearrowright M$ IS POLAR AND $\sec_M>0$, THEN V IS LOG-CONCAVE [WILKING]

*SMOOTHNESS IS NECESSARY, BY HSIANG'S EXAMPLE:

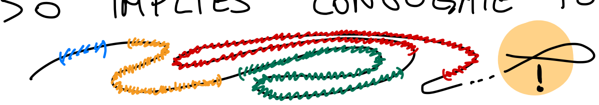


$\{S^3\subset S^4$ MIN., EMBEDDED, $O(2)\times O(2)$-INV.$\}$ IS NOT C^k-COMPACT!

PF (SKETCH):

- SUFFICES TO SHOW $\ell(\gamma)\leq C$ FOR ALL EMBEDDED GEODESICS γ IN Ω THAT ARE FREE BOUNDARY OR SIMPLE CLOSED. ⚠ $\partial\Omega$ NEED NOT BE CONVEX!

- IF γ_n ARE FREE BOUNDARY AND $\ell(\gamma_n)\to\infty$, THERE ARE DISJOINT SEGMENTS $\gamma_n([a_n,b_n])$ WITH $|b_n-a_n|\to\infty$, BUT $\sec_\Omega>0$ IMPLIES CONJUGATE POINTS HENCE SELF-INTERSECTIONS (CONTRADICTION).



- IF γ_n ARE SIMPLE CLOSED AND $\ell(\gamma_n)\to+\infty$, THEN $\gamma_n\to\gamma_\infty$ WHERE γ_∞ IS EITHER (MULT. 2) FREE BOUNDARY WITH $\ell(\gamma_\infty)=\infty$ (APPLY PREVIOUS ARGUMENT); OR γ_∞ IS AWAY FROM $\partial\Omega$, AND CAN USE TOPONOGOV'S BOUND $\left(\ell(\gamma)\leq 2\pi r \text{ IF } \sec\geq\frac{1}{r^2}\right)$

§3. GEOMETRIC APPLICATION

- **Q** (YAU, 1987). ARE ALL EMBEDDED MIN. 2-SPHERES IN
$$E^3 = \left\{ x \in \mathbb{R}^4 ;\ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\} \cong S^3$$
PLANAR, I.E., OF THE FORM $\Sigma = E^3 \cap \{x_i = 0\}$?

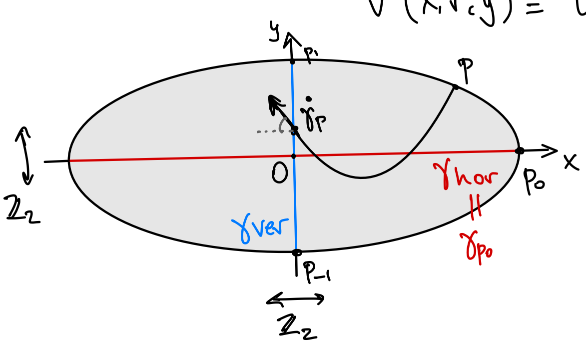
- HASLHOFER–KETOVER, 2019: NO; IF $a \gg b, c, d$, THEN THERE IS AT LEAST ONE NONPLANAR EMBEDDED MIN. 2-SPHERE IN E^3.

> **THM C.** FOR ALL $n \geq 3$, IF a IS CHOSEN SUFF. LARGE IN
> $$E^n(a,b,c) = \left\{ x \in \mathbb{R}^{n+1} ;\ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \dots + \frac{x_n^2}{b^2} + \frac{x_{n+1}^2}{c^2} = 1 \right\}$$
> THERE ARE ARBITRARILY MANY (PAIRWISE NONCONGRUENT) NONPLANAR EMBEDDED $O(n-1)$-INVARIANT MINIMAL n-SPHERES IN $E^n(a,b,c)$.

PF (SKETCH). $E^n(a,b,c)/O(n-1) = \left\{ (x,r,y) \in \mathbb{R}^3 ;\ \frac{x^2}{a^2} + \frac{r^2}{b^2} + \frac{y^2}{c^2} = 1,\ r \geq 0 \right\}$,

$V(x,r,y) = \omega_{n-2} \cdot r^{n-2}$.

DEFINE f_{odd}, f_{even} S.T.: ← USE THM A HERE!



$f_{odd}(a,p) = 0 \iff \gamma_p$ CROSSES γ_{ver} AT O

$f_{even}(a,p) = 0 \iff \gamma_p$ MEETS γ_{ver} ORTHOGONALLY

"TRIVIAL BRANCH": $f_{odd}(a,p_0) = f_{even}(a,p_0) = 0,\ \forall a$

LOCAL BIFURCATION (CRANDALL-RABINOWITZ).

$i_{Morse}(\gamma_{p_0}) \nearrow +\infty$ AS $a \nearrow +\infty \Rightarrow \exists \{a_m\}$ SEQUENCE OF BIFURCATION INSTANTS, FOR f_{odd} IF m ODD, f_{even} IF m EVEN

GLOBAL BIFURCATION (RABINOWITZ). ← USE THM B HERE!

BRANCHES EITHER:
- REATTACH ✗ ← COUNTING NODAL DOMAINS ARGUMENT
- ARE NONCOMPACT ✓