CURVATURE OPERATORS AND RATIONAL COBORDISM

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Thm (Lichnerowicz '63). If (M^n, g) is a closed spin Riem. mfld with $\mathrm{scal}>0$, then $\hat{A}(M)=0$.

Pf: ① Dirac operator $D: S \to S$ satisfies $D^2 = \nabla^*\nabla + \frac{\mathrm{scal}}{4}$, so $\mathrm{scal}>0 \Rightarrow \operatorname{Ker} D = \{0\}$.

If $n=4k$, then $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$ w.r.t. $S = S^+ \oplus S^-$, thus $\operatorname{Ker} D^\pm = \{0\}$. (If $n \neq 4k$, then $\hat{A}(M) \equiv 0$.)

② Atiyah-Singer Index Theorem: $\hat{A}(M) = \operatorname{ind}(D^+) = \dim \operatorname{Ker} D^+ - \dim \underbrace{\operatorname{coker} D^+}_{=\operatorname{Ker} D^-} = 0.$ □

Example: K3 surface (Fermat quartic) (Rmk: Need spin, e.g., $\hat{A}(\mathbb{C}P^2) = -\frac{1}{8}$ and has $\mathrm{scal}>0$.)

$M^4 = \{[x_0:x_1:x_2:x_3] \in \mathbb{C}P^3 : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$ is spin and $\hat{A}(M^4) \neq 0$, so does not admit $\mathrm{scal}>0$.

Well-known generalization: twisted spinors Complex vector bundle $E \to M$ ⇝ Twisted Dirac operator $D_E : S\otimes E \to S \otimes E$.

① $D_E^2 = \nabla^*\nabla + R_E$, where R_E depends on connection of E.

If E is "built from" TM, then $\hat{A}(M,E)$ is a rational linear comb. of Pontryagin numbers of M, just like $\hat{A}(M) = \langle \hat{A}(TM), [M]\rangle$ and R_E depends only on R.

② Atiyah-Singer Index Theorem: $\hat{A}(M,E) = \langle \hat{A}(TM)\cdot \mathrm{ch}(E), [M]\rangle = \operatorname{ind}(D_E^+)$.

So it follows that: $\boxed{\underbrace{R_E > 0}_{???} \Rightarrow \underbrace{\hat{A}(M,E) = 0}_{???}}$

E.g., $\hat{A}(M^4) = -\frac{p_1}{24}$, $\hat{A}(M^8) = \frac{7p_1^2 - 4p_2}{5760}$, …

$\hat{A}(M^4, TM) = \frac{5p_1}{6}$, $\hat{A}(M^8, TM) = \frac{37p_1^2 - 124p_2}{720}$, …

$\hat{A}(M^4, \mathrm{Sym}^2 TM) = \frac{67p_1}{12}$, $\hat{A}(M^8, \mathrm{Sym}^2 TM) = \frac{701p_1^2 - 1292p_2}{480}$, …

To meaningfully generalize Lichnerowicz's Theorem:

Need curvature conditions on (M^n, g) that are
- strong enough to imply $\hat{A}(M,E)=0$ for lots of E's (hence restrict the rational cobordism type of M) Thm A ✓
- weak enough to be verifiable and satisfied by lots of M's (e.g., be invariant under surgeries, cobordisms, etc.) Thm B ✓

New family of curvature conditions on (M^n, g): $C_p(R): M \to \mathbb{R}$, $p \geq 1$.

- $C_p(R)$ is a linear combination of eigenvalues $\nu_1 \leq \cdots \leq \nu_{\binom{n}{2}}$ of the curvature operator $R: \Lambda^2 TM \to \Lambda^2 TM$ (and of Ric, if $p=1$); in particular, $\{R \in \mathrm{Sym}_b^2(\Lambda^2\mathbb{R}^n) : C_p(R) > 0\}$ is an $O(n)$-invariant spectrahedral (convex) cone.
  Algorithmically easy to check membership in it!

Thm A. (B–Goodman '22). Let (M^n, g) be a closed Riem. spin mfld, $n = 4k \geq 8$, and $E \subseteq TM^{\otimes p}$ be a parallel subbundle. If (M^n, g) has $C_p(R) > 0$, then $\hat{A}(M, E_{\mathbb{C}}) = 0$.
i.e., preserved by parallel transport.

- If $C_p(R) > 0$ and $1 \leq q < p$, then $\frac{\mathrm{scal}}{4} \geq C_q(R) \geq C_p(R) > 0$.

Example 1: If $n=8$, then $C_1(R) = \min\left\{3(\nu_1 + \cdots + \nu_5), \frac{\mathrm{scal}}{8}\right\} + \underbrace{\frac{\mathrm{scal}}{8} - \mu}$.
largest eigenvalue of Ric.
vanishes if $\mathrm{Ric}_g = \frac{\mathrm{scal}}{8} g$ (Einstein)

- $M^4 = \mathbb{H}P^2$ has $\hat{A}(M, TM_{\mathbb{C}}) \neq 0$, so does not admit $C_1(R) > 0$. In particular, no Einstein metric w/ 5-positive curvature operator!
  is spin; same works for $\#^\ell \mathbb{H}P^2$, $\ell \geq 1$.
  $R \geq 0$ of Fubini-Study is 19-positive

Example 2: If $n=16$, then $C_2(R) = \min\left\{ 8\underbrace{(\nu_1+\dots+\nu_8)}, \frac{15}{4}\underbrace{(\nu_1+\dots+\nu_8)} \right\} + \frac{\mathrm{scal}}{8} - 4\underbrace{\left(\nu_{\binom{16}{2}-7}+\dots+\nu_{\binom{16}{2}}\right)}$

sum of smallest 8 eigenvalues of R — sum of largest 8 eigenvalues of R

Note: $\dim \Lambda^2\mathbb{R}^{16} = \binom{16}{2} = 120$.

- $M^{16} = \mathbb{C}a P^2$ has $\hat{A}(M, \Lambda^2 TM_{\mathbb{C}}) \neq 0$, so does not admit $C_2(R) > 0$.

So what manifolds have $C_p(R)>0$?

Thm B. (B. – Goodman '22).

(i) Every nontorsion cobordism class in Ω_n^{SO}, $n \geq 10$, contains a manifold with $C_1(R) > 0$.

i.e., without spin condition, there is no restriction on rational cobordism class.

(ii) If M^n is spin, $n \geq 10$, and $\hat{A}(M) = \hat{A}(M, TM_{\mathbb{C}}) = 0$, then $\#^\ell M^n$ is spin-cobordant to a mfld with $C_1(R)>0$.

i.e., with spin condition, these are the only restrictions on the rational cobordism class.

(iii) $C_p(R)>0$ is preserved under surgeries* of codimension d if $(d-1)(d-3) > 8p(p+n-2)$.

Cor: $C_1(R)>0$ does not restrict any Betti number b_i nor any Pontryagin number p_i in suff. high dimension

Note: Round spheres S^n have $C_p(\mathrm{Id})>0$ if $n \gg p$

$$C_p(\mathrm{Id}) = \frac{1}{4}n^2 - \left(p+\frac{1}{4}\right)n - p(p-2)$$

but, of course, $M = S^n = \partial B^{n+1}$ so $\hat{A}(M, E_{\mathbb{C}}) = 0$ for any E.

*Recall: Surgery of codimension d is to
- Remove $S^{n-d} \times D^d \subset M^n$
- Glue in $D^{n-d+1} \times S^{d-1}$

Result is cobordant but has
- smaller b_{n-d} if $0 \neq [S^{n-d}] \in H_{n-d}(M, \mathbb{Q})$
- larger b_{n-d+1} if $0 = [S^{n-d}] \in H_{n-d}(M, \mathbb{Q})$

cf. Gromov–Lawson / Schoen–Yau: scal>0 is preserved if $d \geq 3$.

Remainder of the talk: §2. Sketch proof of Thm A
§3. Applications

§2. Bochner technique and Theorem A

Let (M^n, g) be a closed/spin oriented Riem. mfld.

$\mathrm{Fr}(TM) = \{\text{orthonormal/spinor frames on } M\}$ is a principal G-bundle, $G = SO(n), Spin(n), \dots$ (special holonomy)

- Associated bundle construction: $\boxed{\pi: G \to \mathrm{Aut}(E) \text{ unitary representation}} \rightsquigarrow \boxed{E \to E_\pi = \mathrm{Fr}(TM) \times_\pi E \to M}$

- Functorial: $E_{F(\pi)} = F(E_\pi)$ for $F = \Lambda^p, \mathrm{Sym}^p, \otimes, (\cdot)^*, \dots$ e.g., $E_{\pi_1 \oplus \pi_2} = E_{\pi_1} \oplus E_{\pi_2}$

- Laplacian on sections of E_π:

$$\Delta = \nabla^*\nabla + t\, K(R,\pi) \text{ where } t \in \mathbb{R}, \text{ and } K(R,\pi) = -\sum_a d\pi(RX_a) \circ d\pi(X_a), \quad \{X_a\} \text{ o.n.b. of } \Lambda^2\mathbb{R}^n = \mathfrak{so}(n)$$

- Bochner technique: $\int_M \langle \Delta\phi, \phi\rangle = \int_M \underbrace{|\nabla\phi|^2}_{\geq 0} + t\langle K(R,\pi)\phi, \phi\rangle$, so $tK(R,\pi) > 0 \Rightarrow \underbrace{\operatorname{Ker} \Delta = \{0\}}$.

this can have topological or geometric significance!

Example:

Representation	Curvature term	Vanishing result
$SO(n) \curvearrowright \mathbb{R}^n$	$K(R,\pi) = \mathrm{Ric}_R$	$E_\pi = TM$, Lichnerowicz Laplacian $(t=-2)$ $\mathrm{Ric}<0 \Rightarrow$ No nonzero Killing fields $\Rightarrow \lvert \mathrm{Iso}(M,g)\rvert < +\infty$
$SO(n) \curvearrowright (\mathbb{R}^n)^*$		$E_\pi = TM^*$ Hodge Laplacian $(t=2)$ $\mathrm{Ric}>0 \Rightarrow$ No nonzero harmonic 1-forms $\Rightarrow b_1(M,\mathbb{R})=0$
$SO(n) \curvearrowright \Lambda^p(\mathbb{R}^n)^*$	...	$E_\pi = \Lambda^p TM^*$ Hodge Laplacian $(t=2)$ $K(R, \Lambda^p TM) \Rightarrow$ No nonzero harmonic p-forms $\Rightarrow b_p(M,\mathbb{R})=0$
$Spin(n) \curvearrowright S$	$K(R,\pi_S) = \frac{\mathrm{scal}}{8}$	$E_\pi = S$ $D^2 =$ Spinor Laplacian $(t=2)$, $\mathrm{scal}>0 \Rightarrow \hat{A}(M)=0$.

And the obvious versions for ≥ 0 instead of >0...

Inspired by recent works of Petersen–Wink:

$\Sigma(1,R)=\nu_1 \underset{\text{small} \quad r \quad \text{large}}{\overset{\text{strong} \qquad \text{weak}}{\longleftrightarrow}} \Sigma(\binom{n}{2},R)=\frac{\mathrm{scal}}{2}$

Let $\Sigma(r,R)=\nu_1+\dots+\nu_{\lfloor r\rfloor}+(r-\lfloor r\rfloor)\nu_{\lfloor r\rfloor+1}$, where $\nu_1\leq\dots\leq\nu_{\binom{n}{2}}$ are the eigenvalues of R, $1\leq r\leq\binom{n}{2}$

Prop: $K(R,\pi)\geq\|\lambda\|^2\cdot\Sigma(r,R)\,\mathrm{Id}$, where λ = highest weight of π, $r=\dfrac{\langle\lambda,\lambda+2\rho\rangle}{\|\lambda\|^2}$, ρ = half sum of positive roots.

irreducible

Def ("Petersen–Wink invariant"). $PW(\pi)=\min\left\{\dfrac{\langle\lambda,\lambda+2\rho\rangle}{\|\lambda\|^2} : \lambda\neq 0 \text{ highest weight of irreducible factors of } \pi\right\}\leq\binom{n}{2}$.

Example: $PW(\Lambda^p\mathbb{R}^n)=n-p$, for any $1\leq p<\frac{n}{2}$. so $K(R,\Lambda^p\mathbb{R}^n)\geq c^2\cdot\Sigma(n-p,R)\,\mathrm{Id}$.

Thm (Petersen–Wink '21). $\Sigma(n-p,R)>0 \Rightarrow b_p(M,\mathbb{R})=b_{n-p}(M,\mathbb{R})=0$.
In particular, $\Sigma(\lceil\frac{n}{2}\rceil,R)>0\Rightarrow M^n$ is a rational homology sphere.

cf. Böhm–Wilking '08
$\Sigma(2,R)>0\Rightarrow M\cong S^n/\Gamma$

Prop: $\mathcal{R}_{E_\pi}=K(R,\pi_S\otimes\pi)+\frac{\mathrm{scal}}{8}\mathrm{Id}-K(R,\pi)$, where $D^2_{E_\pi}=\nabla^*\nabla+\mathcal{R}_{E_\pi}$

Pf of Theorem A: Convex optimization arguments show that, for any subrepresentation π of $(\mathbb{R}^n)^{\otimes p}$:

$$PW(\pi_S\otimes\pi)\geq PW(\pi_S\otimes \mathrm{Sym}^p\mathbb{R}^n)=:r_p=\frac{n^2+(8p-1)n+8p(p-1)}{n+8p(p+1)}$$

$$PW(\pi)\geq PW(\mathrm{Sym}^p\mathbb{R}^n)=:r_p'=\frac{n+p-2}{p}$$

i.e. $\mathrm{Sym}^p\mathbb{R}^n\subseteq TM^{\otimes p}$ is always the "worst case scenario".

Combine the above Propositions and $r\longmapsto\Sigma(r,R)/r$ is nondecreasing to find that

$$C_1(R)=\min\left\{\left(\frac{n}{8}+2\right)\Sigma(r_1,R),\ \frac{\mathrm{scal}}{8}\right\}+\frac{\mathrm{scal}}{8}-\mu$$

μ: largest eigenvalue of $\mathrm{Ric}_R=K(R,id)$.

$$p>1:\quad C_p(R)=\min\left\{\left(\frac{n}{8}+p^2+p\right)\Sigma(r_p,R),\ \frac{n(n-1)}{8r_p}\Sigma(r_p,R)\right\}+\frac{\mathrm{scal}}{8}+p^2\Sigma(r_p',-R),$$

(These coeff. are related to $\|\lambda\|^2$...)

i.e., sum of largest r_p' eigenvalues of R.

are such that $C_p(R)>0\Rightarrow\mathcal{R}_{E_\pi}>0$ for any $E_\pi\subseteq TM^{\otimes p}\Rightarrow\hat{A}(M,(E_\pi)_{\mathbb{C}})=0$. □

§3. Applications

Thm C (B.–Goodman '22). Let (M^{4k},g) be a closed Riem. spin manifold, with $\frac{\mathrm{scal}}{8}\mathrm{Id}-\mathrm{Ric}>0$ and
$\Sigma(5,R)>0$ if $k=2$,
$\Sigma(2k+4,R)>0$ if $k\geq 6$ even,
$\Sigma(2k+6)>0$ if $k\geq 9$ odd,
then M is rationally null-cobordant, i.e., $\#^\ell M^n=\partial W^{n+1}$ for some $\ell\geq 1$.

Note: these conditions are always weaker than $\Sigma(\lceil\frac{n}{2}\rceil,R)=\Sigma(2k,R)>0$; cf. Petersen–Wink.

Method of proof: [Thom '1954]: Pontryagin numbers $(p_{I_1},\dots,p_{I_{p(k)}})$: $\Omega^{\mathrm{Spin}}_{4k}\otimes\mathbb{Q}\cong\Omega^{SO}_{4k}\otimes\mathbb{Q}\xrightarrow{\cong}\mathbb{Q}^{p(k)}$ ← partitions of k
define an isomorphism, so M is rationally null-cobordant iff all its Pontryagin numbers vanish.

- So it suffices to show that $\hat{A}(M,E_{\mathbb{C}})=\begin{pmatrix}\mathbb{Q}\text{-linear combination}\\ \text{of Pontryagin numbers}\end{pmatrix}=0$ for sufficiently many E's

coefficients depend on choice of E

- Achieve this applying Thm A and Petersen–Wink's result.

$\Omega^{SO}_*\otimes\mathbb{Q}\cong\mathbb{Q}[\mathbb{C}P^2,\mathbb{C}P^4,\dots]$
$\Omega^{SO}_*/\text{torsion}\cong\mathbb{Q}[\mathbb{C}P^{2m}, H_{ij} : m\geq 1,\ i\geq 2,\ j\geq 6]$
$\Omega^{\mathrm{Spin}}_*\otimes\mathbb{Q}\cong\mathbb{Q}[K3,\mathbb{H}P^2,\mathbb{H}P^3,\dots]$

Witten genus: $\varphi_W(M) = \hat{A}\left(M, \bigotimes_{\ell=1}^{\infty} \mathrm{Sym}_{q^\ell} TM_{\mathbb{C}}\right) \prod_{\ell=1}^{\infty} (1-q^\ell)^{4k}$, where $\mathrm{Sym}_t TM_{\mathbb{C}} = \mathbb{C} + TM_{\mathbb{C}} t + \mathrm{Sym}^2 TM_{\mathbb{C}} t^2 + \cdots$

is a formal power series $\varphi_W(M) \in \mathbb{Q}[[q]]$ with coefficients $\hat{A}(M)$, $\hat{A}(M, TM_{\mathbb{C}})$, $\hat{A}(M, TM_{\mathbb{C}} \oplus \mathrm{Sym}^2 TM_{\mathbb{C}})$...

Witten: $\varphi_M(M)$ "is" the S^1-equivariant index of Dirac operator on loop space $\mathcal{L}M$. ← not rigorously defined

> **Thm D.** (B.–Goodman '22) Let (M^{4k}, g) be a closed Riem. spin manifold, and set
>
> $p = \lfloor \frac{k}{6} \rfloor - 1$ if $k \equiv 1 \mod 6$, $\quad p = \lfloor \frac{k}{6} \rfloor$ otherwise $\quad \left(\text{so } p \approx \frac{\dim M}{24}.\right)$
>
> If $p \geq 1$, $C_p(R) > 0$, and $p_1(TM) = 0$, then $\varphi_W(M) = 0$.

> **Conjecture** (Stolz, 1996). If (M^n, g) has $\mathrm{Ric} > 0$ and $\underbrace{\tfrac{1}{2} p_1(TM) = 0}_{\text{"string"}}$, then $\varphi_W(M) = 0$. ← if true, would yield first examples of simply-connected manifolds with scal>0 that do not admit Ric>0.

Note: $\mathrm{Ric} > 0$ does not imply $C_p(R) > 0$!
There exist manifolds with $C_p(R) > 0$ with p as above and $|\pi_1 M| = +\infty$, so no $\mathrm{Ric} > 0$.

Method of proof: • $p_1(TM) = 0 \Rightarrow \varphi_W(M)$ is a modular form of weight $2k$

$\Rightarrow \varphi_W(M) = 0$ if first $p \approx \frac{k}{6}$ coeff. vanish.

• Apply Theorem A to show these coeff. vanish, since they involve $\hat{A}(M, E_{\mathbb{C}})$, $E \subseteq TM^{\otimes p}$. ☐

Bonus application:

Elliptic genus: $\phi(M) = \left(2 \prod_{\ell=1}^{\infty} \frac{(1-q^\ell)^2}{(1+q^\ell)^2}\right)^{2k} \left\langle L(TM) . \mathrm{ch}\left(\psi_2\left(\bigotimes_{\ell=1}^{\infty} \mathrm{Sym}_{q^\ell} TM_{\mathbb{C}} \otimes \Lambda_{q^\ell} TM_{\mathbb{C}}\right)\right), [M] \right\rangle$

So first term is signature $\sigma(M) = \langle L(TM), [M] \rangle$ — Adams operation — analogous to Sym_{q^ℓ} for Λ^p.

> **Thm E.** (B.–Goodman '22). Let (M^{4k}, g) be a closed Riem. spin mfld. If $k \geq 2$ and $C_{\lfloor k/2 \rfloor}(R) > 0$, then the elliptic genus $\phi(M)$ vanishes (and hence the signature $\sigma(M)$ vanishes).