Curvature operators and rational cobordism

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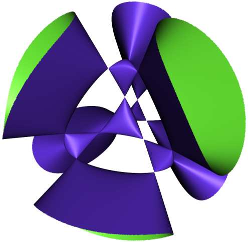
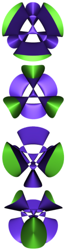
Example (Fermat quartic / Kummer surface)

$M^4 = \left\{x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\right\} \subset \mathbb{C}P^3$ is spin, and $\hat{A}(M^4) \neq 0$.





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- **weak enough** to be satisfied by lots of manifolds;
- **strong enough** to restrict their rational cobordism type.

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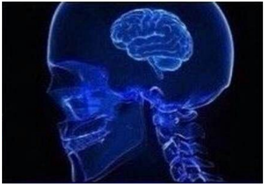

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$$\hat{A}(M^4, TM_{\mathbb{C}}) = \tfrac{5p_1}{6}$$

$$\hat{A}(M^8, TM_{\mathbb{C}}) = \frac{37p_1^2 - 124p_2}{720}$$

$$\hat{A}(M^{12}, TM_{\mathbb{C}}) = \frac{11p_1^3 - 124p_1p_2 + 656p_3}{80640}$$

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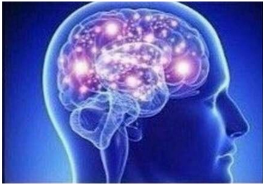

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$$\hat{A}(M^4, \wedge^2 TM_{\mathbb{C}}) = \frac{7p_1}{4}$$

$$\hat{A}(M^8, \wedge^2 TM_{\mathbb{C}}) = \frac{409p_1^2 - 28p_2}{1440}$$

$$\hat{A}(M^{12}, \wedge^2 TM_{\mathbb{C}}) = \frac{499p_1^3 + 3844p_1p_2 - 27056p_3}{161280}$$

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$$\hat{A}(M^4, \mathrm{Sym}^2\, TM_{\mathbb{C}}) = \frac{67p_1}{12}$$

$$\hat{A}(M^8, \mathrm{Sym}^2\, TM_{\mathbb{C}}) = \frac{701p_1^2 - 1292p_2}{480}$$

$$\hat{A}(M^{12}, \mathrm{Sym}^2\, TM_{\mathbb{C}}) = \frac{20933p_1^3 - 64612p_1p_2 + 58928p_3}{161280}$$

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Challenge

Given $E \to M$, find “reasonable” sufficient conditions for $\mathcal{R}_E \succ 0$.

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Extreme cases: $\Sigma(1, R) = \nu_1$, and $\Sigma\big(\binom{n}{2}, R\big) = \frac{\mathrm{scal}}{2}$.

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Thus, $C_1(R) > 0$ does not restrict any Betti numbers b_i nor individual Pontryagin numbers p_i in sufficiently large dimension.

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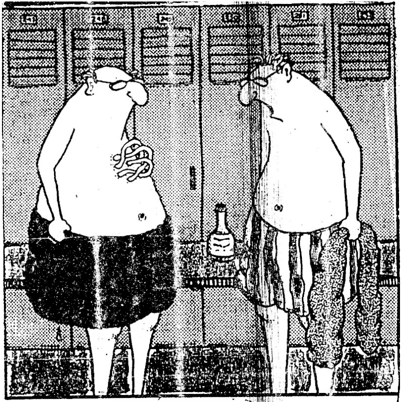

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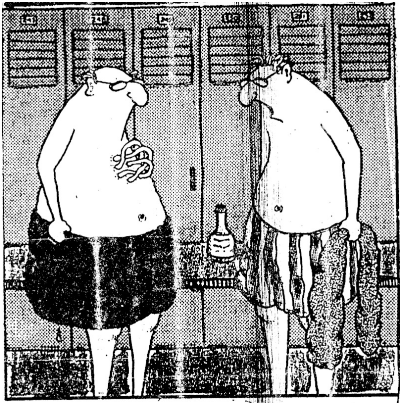

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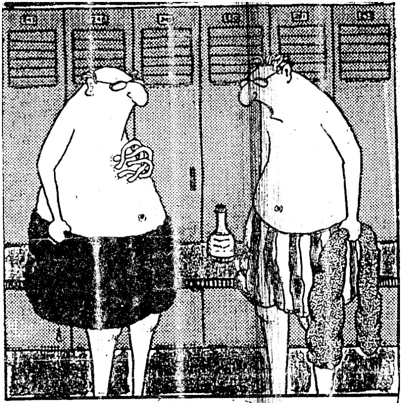

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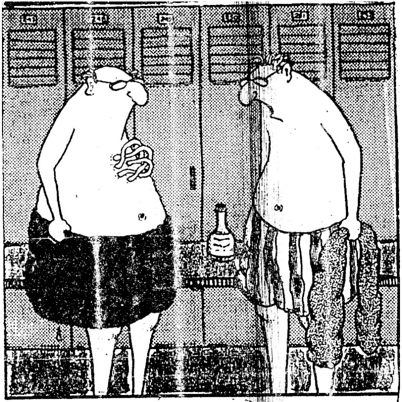

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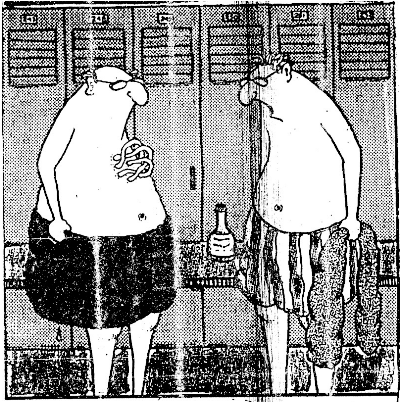

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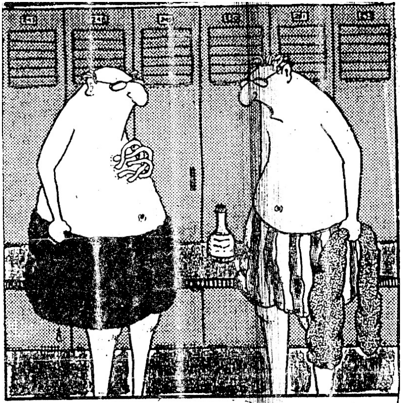

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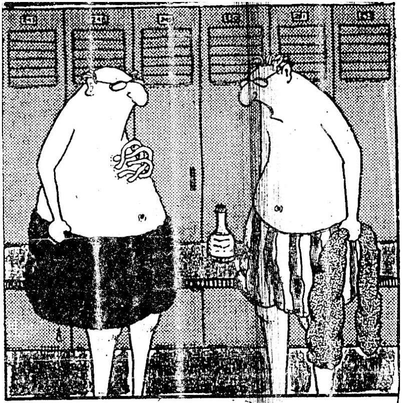

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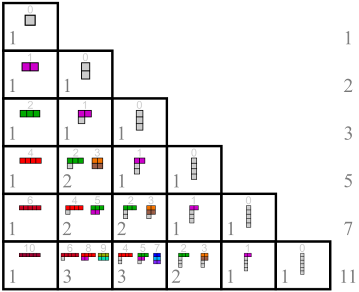

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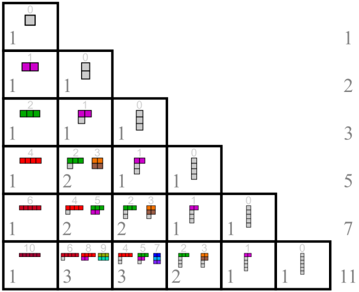

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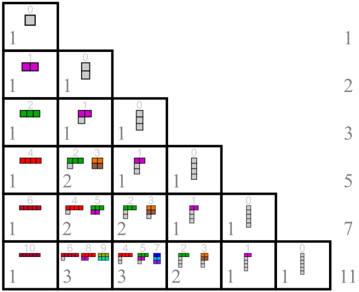

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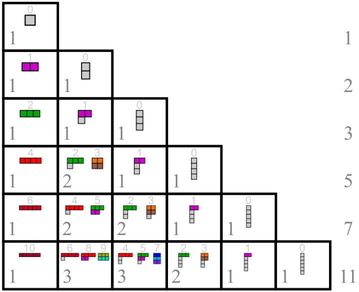

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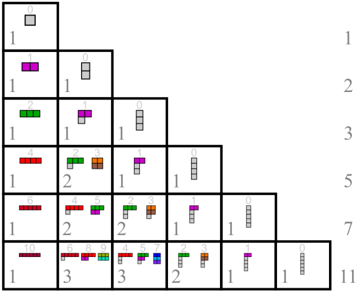

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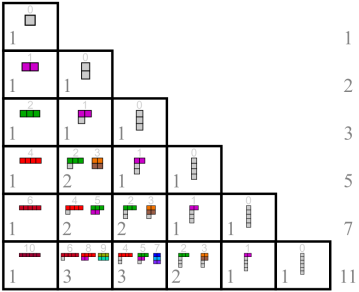

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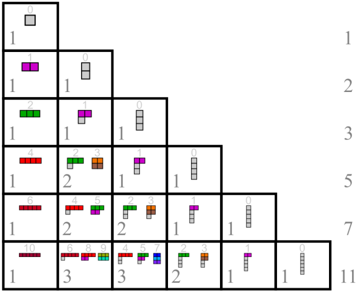

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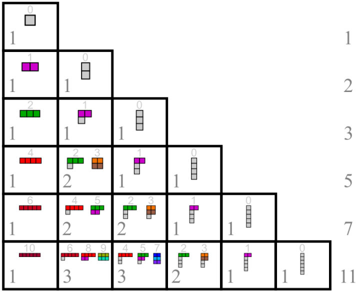

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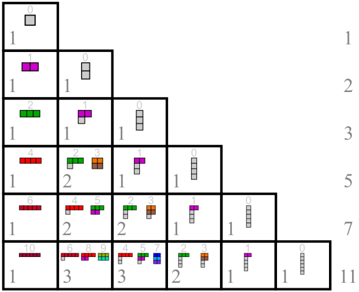

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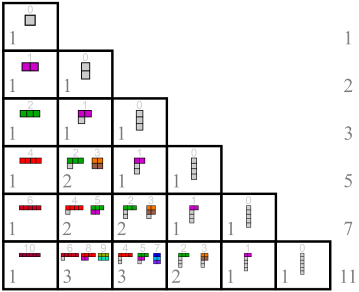

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Fubini–Study metric has $\Sigma(r, R) > 0$ only for $r \geq 19$.

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The Witten genus of M^{4k} is the formal power series

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Conjecture (Stolz, 1996)

If (M, g) has $\mathrm{Ric} \succ 0$ and $\frac{1}{2} p_1(TM) = 0$, then $\varphi_W(M) = 0$.

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Ric $\succ 0$ does not imply $C_p(R) > 0$ for p as above.

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Remark 2

If $24 \leq n < 48$ or $n = 52$ and $p_1(TM) = 0$, then $\varphi_W(M) = 0$ if and only if $\#^\ell M$ is cobordant to a manifold with $C_1(R) > 0$.

Applications to elliptic genus, signature, ...

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About the proof of:

Theorem (B.–Goodman, 2022)

Let (M^n, g) be a closed Riemannian spin manifold, $n \geq 8$, and $E \subseteq TM^{\otimes p}$ a parallel subbundle. If $C_p(R) > 0$, then $\hat{A}(M, E_{\mathbb{C}}) = 0$.

Bochner strategy via Representation Theory

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Theorem (B.–Goodman, 2022)

If the highest weight of π is λ, then $K(R,\pi) \succeq \|\lambda\|^2\,\Sigma(r,R)\,\mathrm{Id}$ where $r = \frac{\langle \lambda, \lambda + 2\rho\rangle}{\|\lambda\|^2}$ and ρ is the half-sum of positive roots.

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Thank you for your attention!