BIFURCATING MINIMAL SURFACES IN ELLIPSOIDS OF REVOLUTION

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MIN-MAX THEORY
DOUBLING / DESINGULARIZATION
...
$\rightsquigarrow$ "LOTS" OF MINIMAL SURFACES $\Sigma^2 \hookrightarrow M^3$

THM (MARQUES–NEVES 2017, SONG 2023). CLOSED RIEM. MANIFOLDS (M^n, g), $3 \leq n \leq 7$, HAVE INFINITELY MANY EMBEDDED MINIMAL HYPERSURFACES $\Sigma^{n-1} \hookrightarrow M^n$.

HOWEVER: CONTROL ON THE TOPOLOGY OF Σ IS DIFFICULT!

QUESTION (YAU 1987). ARE THERE ONLY FINITELY MANY EMBEDDED MINIMAL SURFACES $\Sigma^2 \hookrightarrow S^3$ OF EACH GENUS g (UP TO CONGRUENCE)?

(OPEN IF $g \geq 2$)

$g = 0$: ONLY EQUATORS $S^2 \hookrightarrow S^3$ [ALMGREN 1966]

$g = 1$: ONLY CLIFFORD TORI $T^2 \hookrightarrow S^3$ [BRENDLE 2013] "LAWSON CONJECTURE".

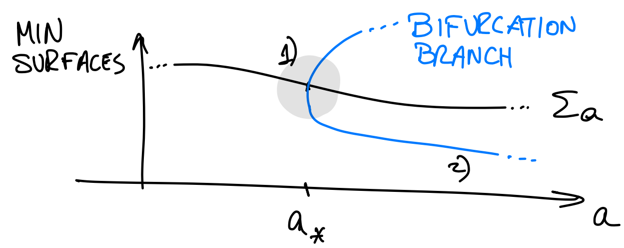
AS $g \nearrow +\infty$, THE NUMBER OF SUCH MIN. SURFACES DIVERGES [KETOVER 2022]

CONJECTURE. EVERY (S^3, g) HAS AT LEAST 4 EMBEDDED MINIMAL 2-SPHERES, AND 5 EMBEDDED MINIMAL TORI. (OPEN UNLESS g IS ALMOST ROUND)

BIFURCATION THEORY: FIND NEW MINIMAL SURFACES WITH GIVEN TOPOLOGY, BY DEFORMING EXISTING ONES

INPUT:

$$\Sigma_a \xhookrightarrow{\text{MIN.}} (M, g_a), \quad a > 0$$



1) $\boxed{\text{JUMP OF MORSE INDEX}} \rightsquigarrow \boxed{\text{(LOCAL) BIFURCATION}}$

OUTPUT: $\Sigma \xhookrightarrow{\text{MIN}} (M, g_a)$ ISOTOPIC BUT NONCONGRUENT TO Σ_a, a NEAR a_*

2) $\boxed{\text{COMPACTNESS (E.G. } M^3, \mathrm{Ric} > 0 \text{ [CHOI–SCHOEN])}} + \boxed{\text{DISCRETE-VALUED INVARIANT FOR BRANCHES}} \rightsquigarrow \boxed{\text{GLOBAL BIFURCATION}}$

OUTPUT: ..., a FAR FROM a_*

LET $M(a,b,c,d) := \left\{ x \in \mathbb{R}^4 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\} \cong S^3$

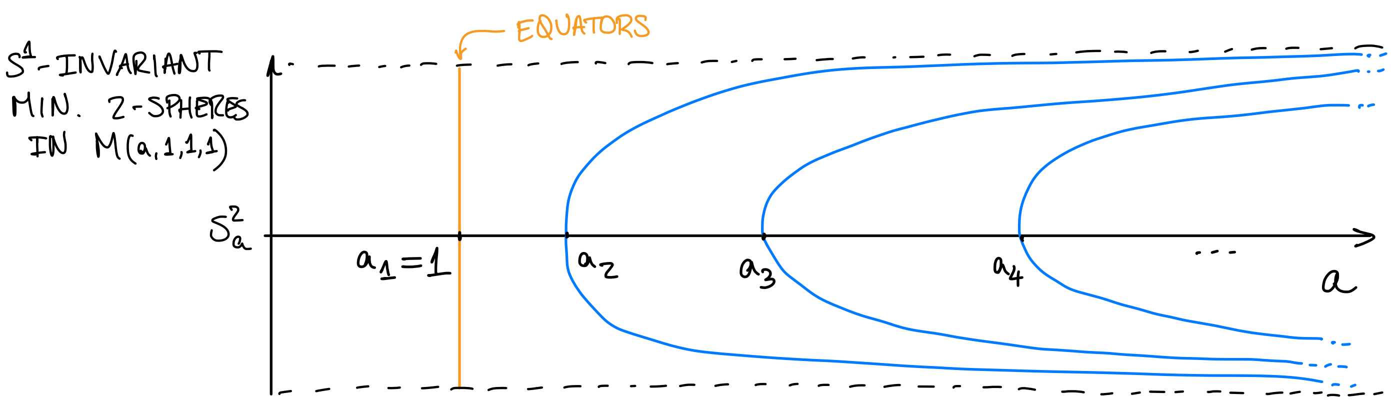
"TRIVIAL" MINIMAL SURFACES (Σ_a)

$S_a^2 := M(a,\underline{1,1},1) \cap \{x_4 = 0\}$ $\quad$ (or $M(a,b,b,c) \cap \{x_4=0\}$)

$T_a^2 := \left\{ x \in M(\underline{a,a},\underline{1,1}) : \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} = x_3^2 + x_4^2 = \frac{1}{2} \right\}$ $\quad$ (or $M(a,a,b,b)\ldots$)

NOTE: IF $a=1$, THEN S_a^2 IS AN EQUATOR, T_a^2 IS CLIFFORD TORUS.

THM. THERE IS A SEQUENCE $a_n \nearrow +\infty$ AT WHICH A BIFURCATION BRANCH OF S^1-INVARIANT EMBEDDED MINIMAL 2-SPHERES IN $M(a,1,1,1)$ STEMS FROM S_a^2. BRANCHES THAT BIFURCATE AT a_n, $n \geqslant 2$, CONSIST OF NONPLANAR SPHERES, AND PERSIST FOR ALL $a \geqslant a_n$. DIFFERENT BRANCHES CONSIST OF PAIRWISE NONCONGRUENT SPHERES.



COR: THERE ARE ARBITRARILY MANY PAIRWISE NONCONGRUENT <u>NONPLANAR</u> EMBEDDED MINIMAL 2-SPHERES IN $M(a,1,1,1)$ FOR a SUFFICIENTLY LARGE.

<u>QUESTION (YAU, 1987).</u> ARE ALL EMBEDDED MINIMAL 2-SPHERES IN $M(a,b,c,d)$ PLANAR?

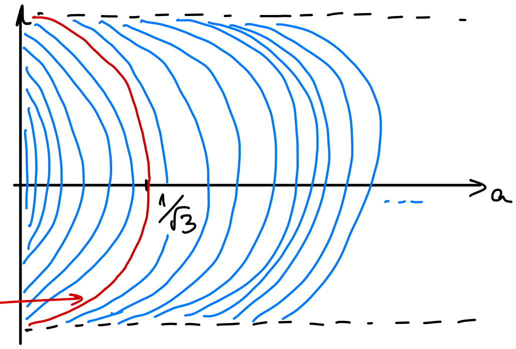
[HASLHOFER-KETOVER' 2019]: IF $a \gg b,c,d$ THEN THERE IS AT LEAST 1 NONPLANAR MINIMAL 2-SPHERE IN $M(a,b,c,d)$.

<u>CONJ:</u> $a_n = n$, $\forall n \in \mathbb{N}$. (STRONG NUMERICAL EVIDENCE; CAN EASILY SHOW $\limsup_{n\to\infty} \frac{a_n}{n} \leq 2$.)

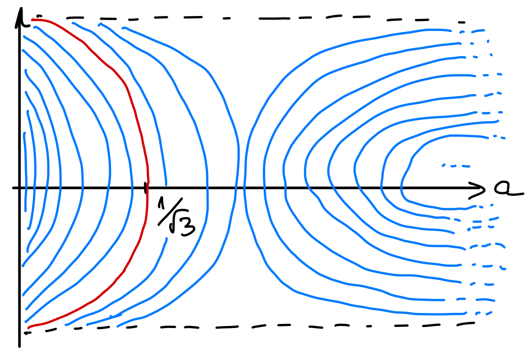
THM. FOR $a \in \{ q/\sqrt{4-q^2} : q \in \mathbb{Q} \cap (0,2) \}$, WHICH IS DENSE IN $(0,+\infty)$, A BIFURCATION BRANCH OF S^1-INVARIANT IMMERSED MINIMAL TORI IN $M(a,a,1,1)$ STEMS FROM T_a^2. BRANCHES PERSIST FOR ALL a UP TO 0 OR $+\infty$; DIFFERENT BRANCHES CONSIST OF PAIRWISE NONCONGRUENT TORI. THE BRANCH THAT BIFURCATES AT $a = \frac{1}{\sqrt{3}}$ CONTAINS ONLY EMBEDDED TORI AND PERSISTS FOR ALL $a \leq \frac{1}{\sqrt{3}}$.

COR: THERE ARE INFINITELY MANY PAIRWISE NONCONGRUENT IMMERSED MINIMAL TORI IN $M(a,a,1,1)$ FOR ALL $a \in (0,+\infty)$ EXCEPT POSSIBLY 1 VALUE IN $(\frac{1}{\sqrt{3}},+\infty)$. IF $a \in (0,\frac{1}{\sqrt{3}})$, THEN AT LEAST ONE OF THESE MINIMAL TORI IS EMBEDDED, AND NOT CONGRUENT TO T_a^2.

S^1-INVARIANT MIN. TORI IN $M(a,a,1,1)$



OR

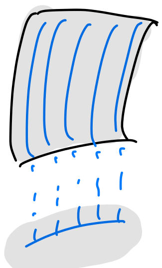


?

METHOD OF PROOF:

0) SYMMETRY REDUCTION [HSIANG–LAWSON' 1971]

$G \curvearrowright M$ ISOM.



$$\Sigma \subset (M,g) \xrightarrow{\pi} \Sigma/G \subset (M/G, \check{g})$$

$k = \dim \Sigma/G$

$$V : M/G \longrightarrow \mathbb{R}, \qquad V(x) = \mathrm{Vol}_M(\pi^{-1}(x))$$

$$\mathrm{Vol}_M(\Sigma) = \int_{\Sigma/G} V \, d\mathrm{vol}_{\check{g}}$$

(push V in as conformal factor)

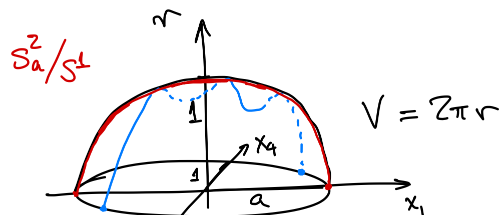
$$= \int_{\Sigma/G} 1 \, d\mathrm{vol}_{V^{2/k}\check{g}}$$

$$= \mathrm{Vol}_\Omega(\Sigma/G), \qquad \Omega = (M/G, V^{2/k}\check{g})$$

THUS:

$\boxed{G\text{-INVARIANT } \Sigma \subset M \text{ IS MINIMAL}}$ $\overset{\text{PALAIS' SYMMETRIC CRITICALITY}}{\Longleftrightarrow}$ $\boxed{\Sigma/G \subset \Omega \text{ IS MINIMAL}}$

$$M(a,1,1,1)/S^1 = \left\{ \frac{x_1^2}{a^2} + r^2 + x_4^2 = 1, \ r \geq 0 \right\}$$



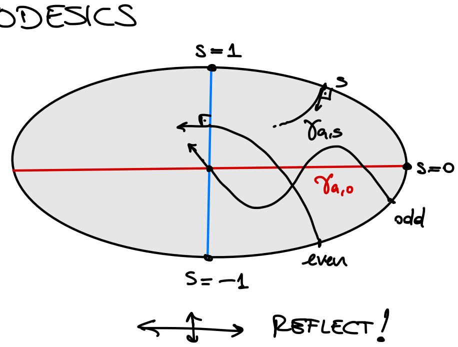
$V = 2\pi r$

ISOMETRIES LEFT: $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ REFLECTIONS ACROSS x_1, x_4

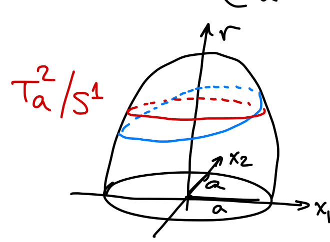
WANT: FREE BOUNDARY GEODESICS

$f_{even}(a,s) = 0$

$f_{odd}(a,s) = 0$



$$M(a,a,1,1)/S^1 = \left\{ \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + r^2 = 1, \ r \geq 0 \right\}$$



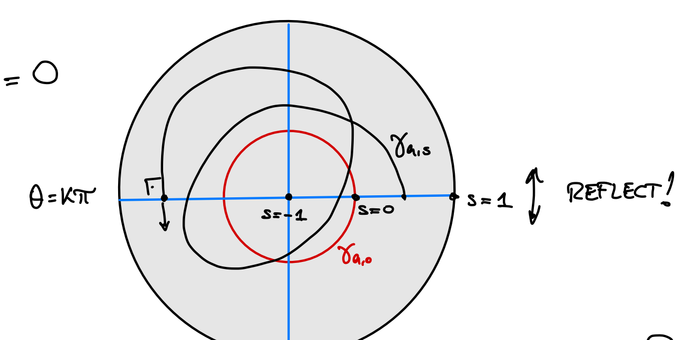
$V = 2\pi r$

$O(2)$ ROTATIONS AND REFLECTIONS

CLOSED GEODESICS

$f_k(a,s) = 0$

$k \in \mathbb{N}$.



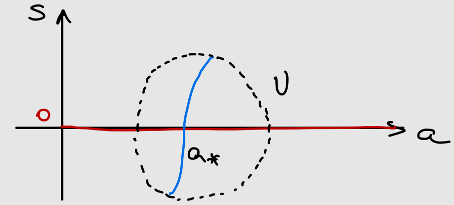
BOTH PROBLEMS REDUCE TO STUDYING ZERO SET OF $f(a,s)=0$, WHERE $s=0$ IS THE "TRIVIAL SOLUTION".

1) LOCAL BIFURCATION

THM (CRANDALL-RABINOWITZ, 1971). IF AT $a=a_*$,

(i) $\frac{\partial f}{\partial s}(a_*,0)=0$ (DEGENERACY)

(ii) $\frac{\partial^2 f}{\partial a\,\partial s}(a_*,0)\neq 0$ (TRANSVERSALITY)



THEN $a=a_*$ IS A BIFURCATION INSTANT: $\exists\, U\ni(a_*,0)$ OPEN NEIGHBORHOOD S.T.

$$f^{-1}(0)\cap U=\{(a,0)\in U\}\cup\{(a(t),s(t)) : t\in(-\varepsilon,\varepsilon)\},\quad a(0)=a_*,\ s(0)=0,\ s'(0)>0.$$

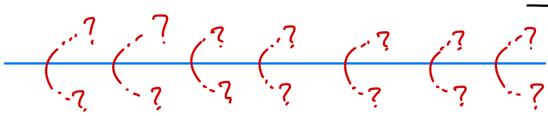
ANALYZING JACOBI EQUATION OF $\gamma_{a,0}$ AS a VARIES, WE FIND (ALL) THE BIFURCATION INSTANTS; AS STATED IN THE THEOREMS.

NOTE: CASE OF SPHERES IS HARDER, NEEDS SINGULAR STURM-LIOUVILLE THEORY

NO KILLING FIELDS LEFT TO USE!

TORI IS EASIER, CAN FIND SOLUTIONS EXPLICITLY B/C $O(2)$ ACTS.

HSIANG'S WORK FROM 1980'S ALSO HAS "LEFT OVER" KILLING FIELD.



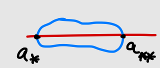
DO LOCAL BRANCHES PERSIST?

2) GLOBAL BIFURCATION

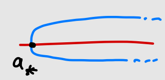
THM (RABINOWITZ' 1971) IF $a=a_*$ IS AS ABOVE, $a(t)$ NOT CONSTANT, AND RESTRICTION OF $(a,s)\mapsto a$ TO $f^{-1}(0)$ IS PROPER, THEN CAN EXTEND $(-\infty,\infty)\ni t\mapsto(a(t),s(t))$,

(CHOI-SCHOEN OR AD HOC...)

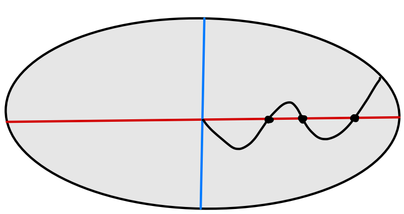
(I) BRANCH REATTACHES TO TRIVIAL BRANCH: $\lim_{t\to+\infty}(a(t),s(t))=(a_{**},0)$;



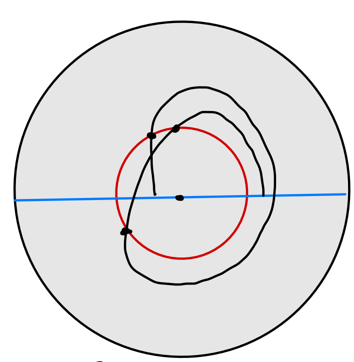
(II) BRANCH IS NONCOMPACT: $\lim_{t\to+\infty} a(t)=0$ OR $+\infty$.



TO AVOID (I), USE DISCRETE-VALUED INVARIANT TO PROVE BRANCHES ARE DISJOINT, SO THERE'S NO REATTACHMENT BECAUSE WE FOUND ALL BRANCHES



$\#\{\gamma_{a,s}\cap\gamma_{a,0}\}$



$\#\{\gamma_{a,s}\cap\gamma_{a,0}\}$ AND $\Delta\theta$ WINDING NUMBER