Scalar curvature rigidity and extremality in dimension 4

Renato G. Bettiol

THE CITY UNIVERSITY OF NEW YORK

Supported by NSF Award DMS-1904342 and NSF CAREER Award DMS-2142575

Scalar curvature rigidity and extremality

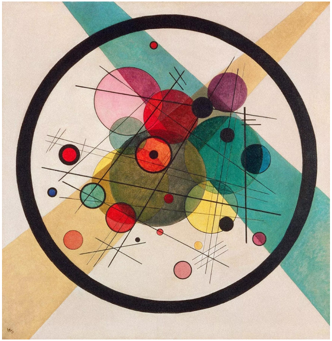
Definition

Let (M, g) be an oriented Riemannian manifold, g' be a *competitor metric* on M,

$$\mathrm{Sc}(g') \geq \mathrm{Sc}(g) \quad \text{and} \quad \wedge^2 g' \succeq \wedge^2 g.$$

Then:

- g is ***extremal*** if $\underline{\ \mathrm{Sc}(g') = \mathrm{Sc}(g)\ }$
- g is ***rigid*** if $\underline{\ g' = g\ }$



Kandinsky, "Circles in a Circle" (1923)

Remark

All results today hold with a *more general definition* allowing some *topologically modified* competitors $f : (N, g') \to (M, g)$.

Dimension 2 (and what one could hope for...)

Let (M^2, g) be a Riemannian manifold with $\pi_1(M) = \{1\}$.

- If $\sec_{\mathrm{g}} \geq 0$, then g is _____extremal_____
- If $\sec_{\mathrm{g}} > 0$, then g is _____rigid_____

Proof.

By Uniformization, competitors are $\mathrm{g}' = e^{2u}\,\mathrm{g}$, $\mathrm{d}A' = e^{2u}\,\mathrm{d}A$.
If $\sec(\mathrm{g}') \geq \sec(\mathrm{g}) \geq 0$ and $\mathrm{d}A' \succeq \mathrm{d}A$, then $\underline{\;e^{2u} \geq 1\;}$ and

$$
\begin{aligned}
0 &= \int_M \sec(\mathrm{g}')\,\mathrm{d}A' - \int_M \sec(\mathrm{g})\,\mathrm{d}A \\
&= \int_M \left(\sec(\mathrm{g}')e^{2u} - \sec(\mathrm{g})\right)\mathrm{d}A \\
&\geq \int_M \underbrace{\left(\sec(\mathrm{g}') - \sec(\mathrm{g})\right)}_{\geq 0}\,\mathrm{d}A
\end{aligned}
$$

If $\sec(\mathrm{g}') = \sec(\mathrm{g}) > 0$, then

$$
\begin{aligned}
\cancel{\sec(\mathrm{g}')}e^{2u} &= \cancel{\sec(\mathrm{g})} \\
e^{2u} &= 1 \\
\Rightarrow \mathrm{g}' &= \mathrm{g}
\end{aligned}
$$

□

Llarull: $(S^n, \mathrm{g}_{\mathrm{round}})$ is rigid.

Min-Oo, Goette–Semmelmann:

For a Riemannian manifold (M, g) with $\chi(M) \neq 0$,

- If $\underline{\; R_{\mathrm{g}} \succeq 0 \;}$, then g is extremal.
- If, in addition, $\underline{\; \frac{\mathrm{Sc}(\mathrm{g})}{2}\mathrm{g} \succ \mathrm{Ric}_{\mathrm{g}} \succ 0 \;}$, then g is rigid.

For a Kähler manifold (M, g),

- If $\underline{\; \mathrm{Ric}_{\mathrm{g}} \succeq 0 \;}$, then g is extremal.
- If $\underline{\; \mathrm{Ric}_{\mathrm{g}} \succ 0 \;}$, then g is rigid.

Other than on S^n, only examples are symmetric or Kähler!

From Gromov's "*A Dozen Problems, Questions and Conjectures about Positive Scalar Curvature*":

> **C.** Problem. Find verifiable criteria for extremality and rigidity, decide which manifolds admit extremal/rigid metrics and describe particular classes of extremal/rigid manifolds.
>
> For instance,
>
> do all closed manifolds which admits metrics with $Sc \geq 0$ also admit (length) extremal metrics?

Dimension 4

Finsler–Thorpe trick

$\boxed{(M^4, \mathrm{g}) \text{ has } \sec_{\mathrm{g}} \geq 0} \iff \boxed{\exists\, \tau\colon M \to \mathbb{R} \text{ with } R_{\mathrm{g}} + \tau * \succeq 0}$

Theorem (B. – Goodman, 2022)

Let (M^4, g) be a Riemannian manifold with $\pi_1(M) = \{1\}$.

- *If* $\sec_{\mathrm{g}} \geq 0$ *and* $\tau\colon M \to \mathbb{R}$ *such that* $R_{\mathrm{g}} + \tau * \succeq 0$ *can be chosen* $\underline{\ \tau \geq 0 \text{ or } \tau \leq 0\ }$*, then* g *is* *extremal*
- *If, in addition,* $\frac{\mathrm{Sc}(\mathrm{g})}{2}\mathrm{g} \succ \mathrm{Ric}_{\mathrm{g}} \succ 0$*, then* g *is* *rigid*

Corollary

(i) $\mathbb{C}P^2 \# \mathbb{C}P^2$ *has* *rigid metrics (Cheeger metrics)*

(ii) $\mathbb{C}P^2$ *has an open set of rigid metrics (generic holonomy)*

Note: $\mathbb{C}P^2 \# \mathbb{C}P^2$ does not admit metrics with $R \succeq 0$ nor Kähler metrics.

$\mathbb{C}P^2$

$R_{\mathrm{FS}} = \mathrm{diag}(0,0,6,2,2,2)$

$* = \mathrm{diag}(1,1,1,-1,-1,-1)$

so $R + \tau * \succ 0$ if $\tau \equiv 1$

$\mathbb{C}P^2 \# \mathbb{C}P^2$

[picture of τ for Cheeger metric]

$\sec_{\mathrm{g}} \geq 0$: S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $\mathbb{C}P^2 \# \mathbb{C}P^2$

all have rigid metrics

$\sec_{\mathrm{g}} > 0$: S^4, $\mathbb{C}P^2$

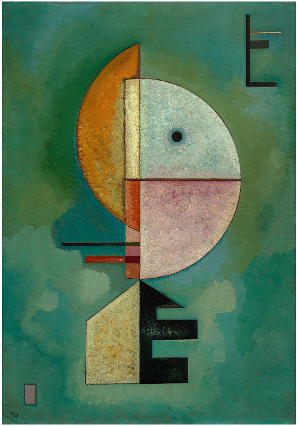
all have *open sets* of rigid metrics

Theorem (B. – Mendes, 2017)

Let (M^4, g) be a Riemannian manifold with $\pi_1(M) = \{1\}$. If $\sec_{\mathrm{g}} \geq 0$ and $\tau \colon M \to \mathbb{R}$ such that $R_{\mathrm{g}} + \tau * \succeq 0$ can be chosen $\tau \geq 0$ or $\tau \leq 0$, then either:

$M \cong_{homeo} \#^k \mathbb{C}P^2$ *or* $M \cong_{isom} (S^2 \times S^2, \mathrm{g}_{prod})$

Local version



Kandinsky, "Upward" (1929)

Definition

Let (M, g) be a Riemannian manifold with boundary, g' be a *competitor metric*,

$$\mathrm{Sc}(g') \geq \mathrm{Sc}(g), \qquad \wedge^2 g' \succeq \wedge^2 g,$$
$$H_{\partial M}(g') \geq H_{\partial M}(g), \qquad g'|_{\partial M} = g|_{\partial M}.$$

- g is ***extremal*** if $\mathrm{Sc}(g') = \mathrm{Sc}(g)$, $H_{\partial M}(g') = H_{\partial M}(g)$
- g is ***rigid*** if $g' = g$

Theorem (B. – Goodman, 2022)

Let (M^4, g) be a Riemannian manifold with convex boundary.

- *If* $\sec_g \geq 0$ *and* $\tau : M \to \mathbb{R}$ *such that* $R_g + \tau * \succeq 0$ *can be chosen* $\tau \geq 0$ *or* $\tau \leq 0$, *then* g *is extremal*
- *If, in addition,* $\frac{\mathrm{Sc}(g)}{2} g \succ \mathrm{Ric}_g \succ 0$, *then* g *is rigid*

Example

$\mathbb{C}P^2 \setminus B \cong \nu(\mathbb{C}P^1)$ has rigid metrics (Cheeger metrics)

Note: $\mathbb{C}P^2 \setminus B$ does not admit metrics with $R \succeq 0$ and convex boundary.

Corollary

If (X^4, g) *has* sec > 0 *at* $p \in X$*, then* g *is extremal on all sufficiently small convex neighborhoods of* p*.*

- sec > 0 on $p \in M \subset X \Rightarrow \exists\, \tau \colon M \to \mathbb{R}$ with $R + \tau * \succ 0$
- Shrink $M \ni p$ so that $\tau|_M$ does not change sign

Upshot:

Cannot increase Sc nor $H_{\partial M}$ in convex neighborhoods M of points with sec > 0 without decreasing areas or changing ∂M.

Outline of Proofs

Fix orientation of (M^4, g) so that $R_{\mathrm{g}} + \tau * \succeq 0$ with $\tau \leq 0$.

Part I: Index Theory

- Globally defined twisted spinor bundle $S_{\mathrm{g}'} \otimes S_{\mathrm{g}}^+ \to M$.
- Twisted Dirac operator $D_{\mathrm{g}',\mathrm{g}} \colon \Gamma(S_{\mathrm{g}'} \otimes S_{\mathrm{g}}^+) \to \Gamma(S_{\mathrm{g}'} \otimes S_{\mathrm{g}}^+)$

$$D_{\mathrm{g}',\mathrm{g}}(\phi \otimes \psi) = \sum_{i=1}^{4} \left(e_i \nabla_{e_i}^{S_{\mathrm{g}}'} \phi\right) \otimes \psi + (e_i \phi) \otimes \left(\nabla_{e_i}^{S_{\mathrm{g}}} \psi\right), \quad D_{\mathrm{g}',\mathrm{g}} = \begin{pmatrix} 0 & D_{\mathrm{g}',\mathrm{g}}^- \\ D_{\mathrm{g}',\mathrm{g}}^+ & 0 \end{pmatrix}$$

- By the Atiyah–Singer Index Theorem,

$$\begin{aligned} \operatorname{ind} D_{\mathrm{g}',\mathrm{g}}^+ &= \operatorname{ind}(\mathrm{d} + \mathrm{d}^*)|_{\wedge_{\mathbb{C}}^{+,\mathrm{even}} TM} \qquad (D_{\mathrm{g},\mathrm{g}} \text{ conjugate to } d+d^* \text{ via } S \otimes S \cong \wedge^* TM) \\ &= \dim\ker(\mathrm{d} + \mathrm{d}^*)|_{\wedge_{\mathbb{C}}^{+,\mathrm{even}} TM} - \dim\ker(\mathrm{d} + \mathrm{d}^*)|_{\wedge_{\mathbb{C}}^{-,\mathrm{odd}} TM} \\ &= 1 + b_2^+(M) - \cancel{b_1(M)} > 0. \end{aligned}$$

$$\underbrace{S \otimes S}_{\wedge_{\mathbb{C}}^*} \cong \underbrace{(S^+ \otimes S^+)}_{\wedge_{\mathbb{C}}^{+,\mathrm{even}}} \oplus \underbrace{(S^- \otimes S^-)}_{\wedge_{\mathbb{C}}^{-,\mathrm{even}}} \oplus \underbrace{(S^+ \otimes S^-)}_{\wedge_{\mathbb{C}}^{-,\mathrm{odd}}} \oplus \underbrace{(S^- \otimes S^+)}_{\wedge_{\mathbb{C}}^{+,\mathrm{odd}}}$$

- Thus $\exists\, \xi \in \Gamma(S_{\mathrm{g}'}^+ \otimes S_{\mathrm{g}}^+)$, $\xi \not\equiv 0$, with $D_{\mathrm{g}',\mathrm{g}}^+ \xi = 0$.

Outline of Proofs

Fix orientation of (M^4, g) so that $R_{\mathrm{g}} + \tau * \succeq 0$ with $\tau \leq 0$.

Part II: Bochner–Lichnerowicz–Weitzenböck formula

- $D^2_{\mathrm{g}',\mathrm{g}} = \nabla^*\nabla + \frac{1}{4}\mathrm{Sc}(\mathrm{g}') - \frac{1}{8}\mathrm{Sc}(\mathrm{g}) - \frac{1}{4}\mathrm{tr}\big(T^* R_{\mathrm{g}} T\big) + \mathcal{L}(R_{\mathrm{g}})$, where $\wedge^2\mathrm{g}' \xrightarrow{T} \wedge^2\mathrm{g}$, $\mathcal{L}(R) \succeq 0$ if $R \succeq 0$, and $\mathcal{L}(*)|_{S^+_{\mathrm{g}'}\otimes S^+_{\mathrm{g}}} \succeq 0$
- $\mathcal{L}(R_{\mathrm{g}}) = \underline{\mathcal{L}(R_{\mathrm{g}} + \tau *) - \tau\,\mathcal{L}(*) \succeq 0}$ on $S^+_{\mathrm{g}'} \otimes S^+_{\mathrm{g}}$.
- Using $\sec_{\mathrm{g}} \geq 0$ and $\wedge^2\mathrm{g}' \succeq \wedge^2\mathrm{g}$, we have $\mathrm{tr}\big(T^* R_{\mathrm{g}} T\big) \leq \frac{1}{2}\mathrm{Sc}(\mathrm{g})$.
- Thus $D^2_{\mathrm{g}',\mathrm{g}} \succeq \underline{\nabla^*\nabla + \frac{1}{4}\big(\mathrm{Sc}(\mathrm{g}') - \mathrm{Sc}(\mathrm{g})\big)}$, so g is extremal:

$$0 = \int_M \langle D^2_{\mathrm{g}',\mathrm{g}}\xi, \xi\rangle \geq \int_M \|\nabla\xi\|^2 + \frac{1}{4}\underbrace{\big(\mathrm{Sc}(\mathrm{g}') - \mathrm{Sc}(\mathrm{g})\big)}_{\geq 0}\|\xi\|^2.$$

- Rigidity by same argument from Goette–Semmelmann. □

Adaptations to case with boundary

Part I: Index Theory

- By the Atiyah–Patodi–Singer Index Theorem,

$$\operatorname{ind} D^+_{g',g} = \underline{\ \tfrac{1}{2}\big(\chi(M) + \sigma(M) + b_0(\partial M) + b_2(\partial M)\big)\qquad\qquad\qquad}$$

- Using $\mathrm{I\!I}_{\partial M} \succeq 0$ and $\underline{\quad\text{Soul Theorem}\quad}$, obtain $\operatorname{ind} D^+_{g',g} > 0$, so $\exists\, \xi \in \Gamma(S^+_{g'} \otimes S^+_{g})$, $\xi \not\equiv 0$, with $D^+_{g',g}\xi = 0$.

Part II: Bochner–Lichnerowicz–Weitzenböck formula

- $$0 = \int_M \langle D^2_{g',g}\xi, \xi\rangle \geq \int_M \|\nabla\xi\|^2 + \tfrac{1}{4}\underbrace{\big(\mathrm{Sc}(g') - \mathrm{Sc}(g)\big)}_{\geq 0}\|\xi\|^2$$
$$+\tfrac{1}{2}\int_{\partial M}\underbrace{\big(H_{\partial M}(g') - H_{\partial M}(g)\big)}_{\geq 0}\|\xi\|^2,$$

  thus g is extremal. □

Topologically modified competitors

Definition

Let (M, g) be an oriented Riemannian manifold,

$$\mathcal{C} = \{ f \colon (N, \mathrm{g}') \longrightarrow (M, \mathrm{g}) \}$$

be a *class of competitors*, where $\underline{\dim N = \dim M}$, $f \colon N \to M$ are smooth spin maps with $\underline{\deg f \neq 0}$,

$$\mathrm{Sc}(\mathrm{g}') \geq \mathrm{Sc}(\mathrm{g}) \circ f \quad \text{and} \quad \wedge^2 \mathrm{g}' \succeq f^* \wedge^2 \mathrm{g}.$$

Then:

- g is $\mathcal{C}$-***extremal*** if $\underline{\mathrm{Sc}(\mathrm{g}') = \mathrm{Sc}(\mathrm{g}) \circ f, \forall f \in \mathcal{C}}$
- g is $\mathcal{C}$-***rigid*** if $\underline{\mathrm{g}' = f^* \mathrm{g}, \ \forall f \in \mathcal{C}}$

Similarly for the case of manifolds with boundary.

Kandinsky, "Stars" (1938)

Theorem (B. – Goodman, 2022)

Let (M^4, g) *be an oriented Riemannian manifold,*

$$\mathcal{C} = \left\{ f : (N, \mathrm{g}') \to (M, \mathrm{g}) : 2\chi(M) + 3\sigma(M) > \frac{\sigma(N)}{\deg f} \right\}$$

- *If* $R_{\mathrm{g}} + \tau * \succeq 0$ *with* $\tau \leq 0$*, then* g *is* $\underline{\quad \mathcal{C}\text{-extremal} \quad}$
- *If, in addition,* $\frac{\mathrm{Sc(g)}}{2}\mathrm{g} \succ \mathrm{Ric}_{\mathrm{g}} \succ 0$*, then* g *is* $\underline{\quad \mathcal{C}\text{-rigid} \quad}$

Corollary

Assuming $N = M$ *is simply-connected and definite,* $\mathcal{C}$ *simplifies to*

$$\mathcal{C}^{\mathrm{self}} = \left\{ f : (M, \mathrm{g}') \to (M, \mathrm{g}) : 4 + \left(\frac{1}{\deg f} - 1 \right) b_2(M) > 0 \right\},$$

hence includes self-maps of any degree if $\underline{\quad b_2(M) \leq 4 \quad}$

In particular, this applies to $\mathbb{C}P^2$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$.

Theorem (B. – Goodman, 2022)

Let (M^4, g) *be an oriented Riemannian manifold with boundary,*

$$\mathcal{C} = \left\{ f\colon (N, \mathrm{g}') \to (M, \mathrm{g}) : \begin{array}{l} f|_{\partial N} \textit{ is an oriented isometry onto } \partial M \textit{ and} \\ 2\chi(M) + 3\sigma(M) + 2b_0(\partial M) + 2b_2(\partial M) > \sigma(N) \end{array} \right\}$$

- *If* $\mathrm{II}_{\partial M} \succeq 0$ *and* $R_{\mathrm{g}} + \tau * \succeq 0$ *with* $\tau \leq 0$, *then* g *is* <u>$\mathcal{C}$*-extremal*</u>
- *If, in addition,* $\frac{\mathrm{Sc(g)}}{2}\mathrm{g} \succ \mathrm{Ric}_{\mathrm{g}} \succ 0$, *then* g *is* <u>$\mathcal{C}$*-rigid*</u>

Remark

By the Soul Theorem, if (M^4, g) has $\sec_{\mathrm{g}} \geq 0$ and $\mathrm{II}_{\partial M} \succeq 0$, then the class $\mathcal{C}$ with $N = M$ simplifies to

$$\mathcal{C} = \left\{ f\colon (M, \mathrm{g}') \to (M, \mathrm{g}) : \ f|_{\partial M} \text{ is an oriented isometry onto } \partial M \right\}$$

Thank you for your attention!

Finsler–Thorpe trick

(M^4, g) has $\sec_{\mathrm{g}} \geq 0 \iff \exists\, \tau \colon M \to \mathbb{R}$ with $R_{\mathrm{g}} + \tau * \succeq 0$.

Recall that $\sec_{\mathrm{g}} \colon \mathrm{Gr}_2(\mathbb{R}^4) \to \mathbb{R}$ is given by:

$$\sec_{\mathrm{g}}(\sigma) = \langle R_{\mathrm{g}}\, \sigma, \sigma \rangle, \quad \mathrm{Gr}_2(\mathbb{R}^4) = \{\sigma \in \wedge^2 \mathbb{R}^4 : \langle *\, \sigma, \sigma \rangle = 0\}$$

Lemma (Finsler, 1936)

Let $A, B \in \mathsf{Sym}^2(\mathbb{R}^d)$. The following are equivalent:

(i) $\langle Ax, x \rangle \geq 0$ *for all* $x \in \mathbb{R}^d$ *such that* $\langle Bx, x \rangle = 0$;

(ii) $\exists\, \tau \in \mathbb{R}$ *such that* $A + \tau B \succeq 0$.