Four-dimensional Ricci flow and sectional curvature NYC 2/2022

(joint work w/ A. Krishnan)

$(M^n, g(t))$, $t \geq 0$ $\quad \frac{\partial}{\partial t} g = -2\,\mathrm{Ric}_g$

$R: \Lambda^2 TM \to \Lambda^2 TM$ curvature operator of g

PDE $\qquad\qquad\qquad\qquad\qquad$ ODE

$$\frac{\partial}{\partial t} R = \Delta R + 2Q(R) \rightsquigarrow \frac{\partial}{\partial t} R = 2Q(R)$$

$Q(R) = R^2 + R^\#$ is quadratic on R

$R^\#(u,v) = -\frac{1}{2}\mathrm{tr}\left(\mathrm{ad}_u R\, \mathrm{ad}_v R\right)$

$\boxed{\begin{array}{l}\underline{\text{Hamilton's Max. Princ.:}}\ \text{Let } \mathcal{C} \subset \mathrm{Sym}_b^2(\Lambda^2\mathbb{R}^n) \text{ be } O(n)\text{-invariant.} \\ \quad \mathcal{C} \text{ preserved by ODE} \implies \mathcal{C} \text{ preserved by PDE}\end{array}}$

(i.e., curvature condition corresponding to $\mathcal{C}$ is preserved by R.F.)

<u>Ex:</u> $\mathcal{C}_{R\geq 0} = \{R \in \mathrm{Sym}_b^2(\Lambda^2\mathbb{R}^n) : R \succeq 0\}$

$\mathcal{C}_{scal\geq 0} = \{R \in \mathrm{Sym}_b^2(\Lambda^2\mathbb{R}^n) : \mathrm{tr}\, R \geq 0\}$

$\vdots$

$= \mathcal{C}_{sec\geq 0}$ if $n \leq 3$. What about $\mathcal{C}_{sec\geq 0}$ for $n \geq 4$?

<u>Lie-theoretic criterion [Wilking]:</u>

$S \subset \mathfrak{so}_{\mathbb{C}}$ $\mathrm{Ad}_{G_{\mathbb{C}}}$-invariant

$\Rightarrow C(S) = \{R : R(v, \bar{v}) \geq 0\ \forall v \in S\}$

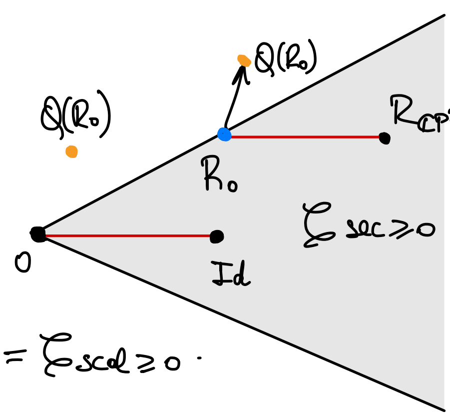
is preserved by ODE.

<u>Ex:</u> $R_0 = R_{\mathbb{CP}^2} - \mathrm{Id} \in \partial \mathcal{C}_{sec\geq 0} \subset \mathrm{Sym}_b^2(\Lambda^2\mathbb{R}^4)$

$Q(R_0) = 6R_{\mathbb{CP}^2} - 9\,\mathrm{Id} \notin \mathcal{C}_{sec\geq 0}$

$\Rightarrow \mathcal{C}_{sec\geq 0}$ <u>not</u> preserved by ODE.

"pointwise statement"



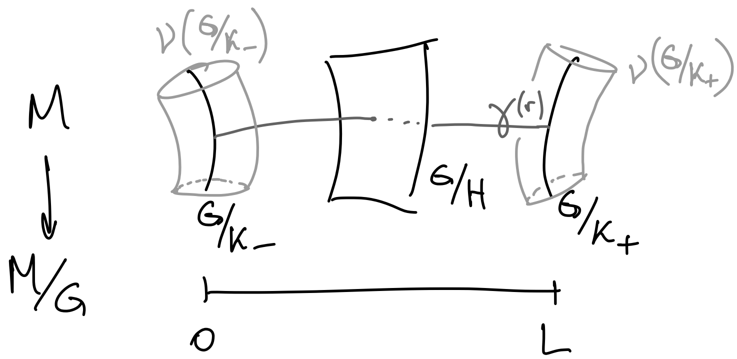
<u>Richard-Seshadri '15:</u> $\mathcal{C} \supseteq \mathcal{C}_{sec\geq 0}$ preserved by ODE $\Rightarrow \mathcal{C} = \mathcal{C}_{scal\geq 0}$.

"Global statement":

Theorem A (B. – Krishnan, 21). There are smooth Riem. metrics on S^4 and $\mathbb{CP}^2$ with $\sec > 0$ that lose that property when evolved under Ricci flow.

- <u>Starting point:</u> Grove-Ziller gluing construction of $\sec \geq 0$ ← Used to endow all exotic 7-spheres with $\sec \geq 0$!

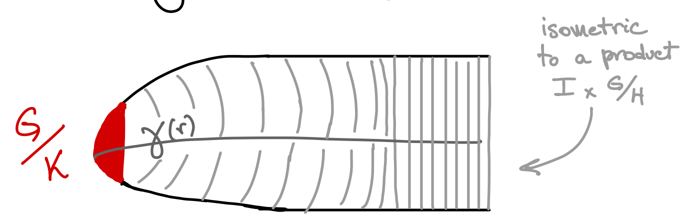
$G \curvearrowright M$ cohom 1 action



$$M = \nu(G/K_-) \cup \nu(G/K_+)$$

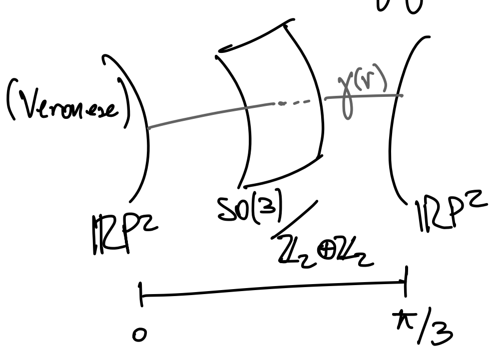
If $\operatorname{codim}(G/K) = 2$, then can produce a metric w/ $\sec \geq 0$:

$$g = dr^2 + g_r$$



$\nu(G/K) \cong G \times_K K/H$ disk bundle

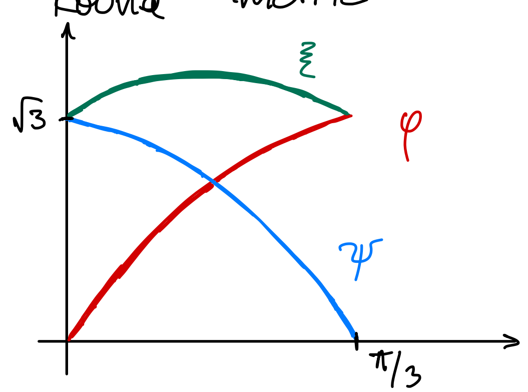
<u>Example:</u> $SO(3) \curvearrowright \mathbb{R}^5 \cong \{A \in \mathbb{R}^{3\times 3} : A = A^T, \operatorname{tr} A = 0\}$
conjugation



$$g_r = \begin{pmatrix} \varphi(r)^2 & & \\ & \psi(r)^2 & \\ & & \xi(r)^2 \end{pmatrix}$$

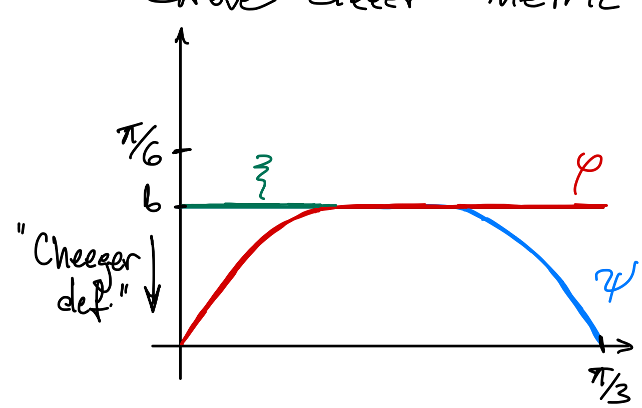
1-par. family of left-inv. metrics on $SO(3)$

Round metric



$\varphi(r) = 2\sin r$, $\psi(r) = \sqrt{3}\cos r + \sin r$,
$\xi(r) = \sqrt{3}\cos r - \sin r$.

Grove-Ziller metric



φ, ψ, ξ are C^∞ but not C^ω.

Theorem B (B–Krishnan, '19). The Ricci flow evolution of Grove–Ziller metrics on S^4, $\mathbb{C}P^2$, $S^2\times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ immediately develops $\sec<0$.

Pf: $\sec_{g_0}(\dot\gamma(r)\wedge e_3)=0$ for all $r\approx 0$.

$$\frac{d}{dt}\sec_{g_t}(\dot\gamma(r)\wedge e_3)\Big|_{t=0} = -\frac{4(\varphi')^2+4\varphi''\varphi}{\xi^4}<0 \quad \text{for } r\approx 0.$$

Subtle issue: Ansatz $g=dr^2+g_r$ "is preserved" under R.F. $\square$

Theorem C (B.–Krishnan '21). Every Grove–Ziller metric on S^4 and $\mathbb{C}P^2$ is the limit (in C^∞-topology) of cohom. 1 metrics with $\sec>0$.

$$\boxed{\text{Thm B}}+\boxed{\text{Thm C}} \xrightarrow[\text{on initial data}]{\text{R.F. continuous dependence}} \boxed{\text{Thm A}}$$

Hamilton's Compactness Thm, or Bahuaud–Guenther–Isenberg '20 convergence stability results.

Pf: Grove-Ziller; Round/Fubini-Study

$$\begin{aligned}
\varphi_s(r) &= (1-s)\,\varphi_0(r)+s\,\varphi_1(r)\\
\psi_s(r) &= (1-s)\,\psi_0(r)+s\,\psi_1(r)\\
\xi_s(r) &= (1-s)\,\xi_0(r)+s\,\xi_1(r)
\end{aligned}
\qquad
g_{r,s}=\begin{pmatrix}\varphi_s(r)^2 & & \\ & \psi_s(r)^2 & \\ & & \xi_s(r)^2\end{pmatrix}$$

Of course $g\longmapsto \sec_g$ is highly nonlinear, but still:

Claim: $g_s = dr^2+g_{r,s}$ has $\sec>0$ if $s>0$ is sufficiently small.

Might be true for $s\in(0,1]$...

Finsler–Thorpe Trick: The following are equivalent for $R\in \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^4)$:

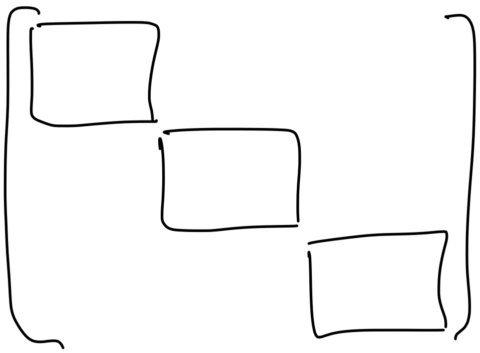
30's 70's

(i) $\sec_R>0$ (≥ 0) ← computationally hard

(ii) $\exists\, \tau\in\mathbb{R}$ s.t. $R+\tau *>0$ (≥ 0) ← computationally easier "semidefinite programming"

$R_{g_s},\ *_{g_s} : \Lambda^2 T_{\gamma(r)} M \longrightarrow \Lambda^2 T_{\gamma(r)} M$ are block-diagonal w/ 2×2 blocks:

On an orthonormal basis w.r.t. $\Lambda^2 g_s$, so that $*_{g_s} = *_{g_0} = *$. Can choose since being >0 (≥ 0) does not depend on basis!

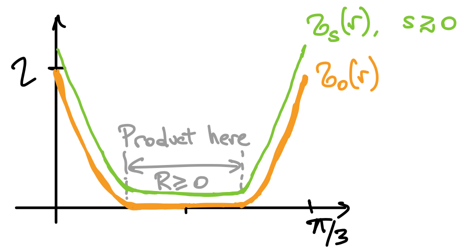


$$R_{g_s} = R_{g_0} + s \cdot \Delta_s + O(s^2)$$

R_{g_0} is Grove–Ziller curv. op., has sec ≥ 0:

$\exists\, \tau_0 : [0, \pi/3] \to \mathbb{R}$ s.t. $R_{g_0} + \tau_0 \cdot * \succeq 0$.

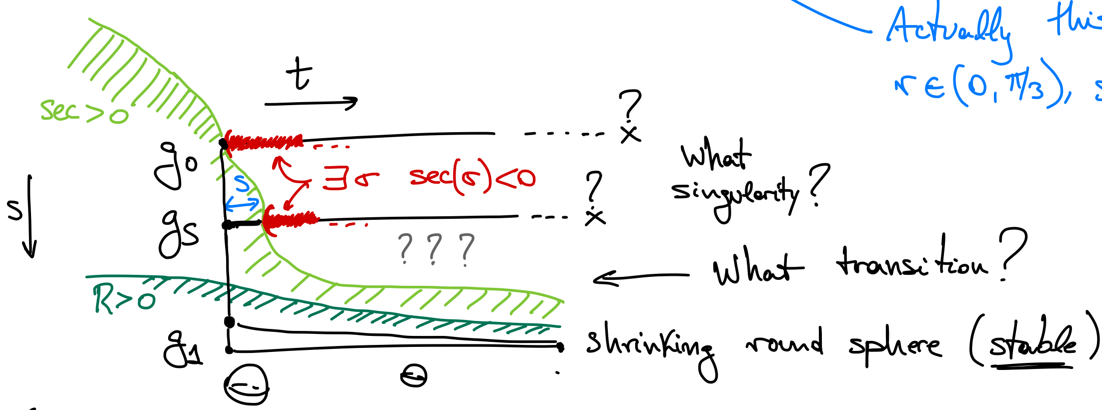
$$\tau_0(r) = \begin{cases} -\dfrac{\varphi'(r)}{2b^2} & \text{if } 0 < r \leq \pi/6, \\ -\dfrac{\psi'(r)}{2b^2} & \text{if } \pi/6 \leq r < \pi/3. \end{cases}$$



<u>Claim:</u> $\exists\, \tau_s(r) = \tau_0(r) + s \cdot \dot{\tau}(r)$ such that $R_{g_s} + \tau_s \cdot * \succ 0$ if $s \gtrsim 0$.

$$\frac{d}{ds}\left(R_{g_s} + \tau_s \cdot *\right)\Big|_{s=0} = \Delta_0 + \dot{\tau} * \succ 0 \quad \text{w/ Sylvester's criterion.}$$

Actually this holds only if $r \in (0, \pi/3)$, so also need $\frac{d^2}{ds^2}\big|_{s=0} \ldots$ $\square$



<u>Remarks:</u>

1) Ricci flow decouples on product metrics on $M^n = S^4 \times S^{n-4}$, so obtain examples w/ sec ≥ 0 that lose that property $\forall n \geq 4$. (But not sec > 0...)

Homogeneous examples known in dim 6, 12, 24.

2) <u>Open questions:</u>
- Is the moduli space of metrics w/ sec > 0 on S^4 connected?
- Does the space of cohom 1 metrics w/ sec ≥ 0 coincide with the closure of the space of cohom 1 metrics w/ sec > 0?
- On S^4, $\mathbb{CP}^2$, is the latter "star shaped" around round/Fubini–Study?