How to find nontrivial solutions out of trivial ones?

Renato G. Bettiol

THE CITY UNIVERSITY OF NEW YORK

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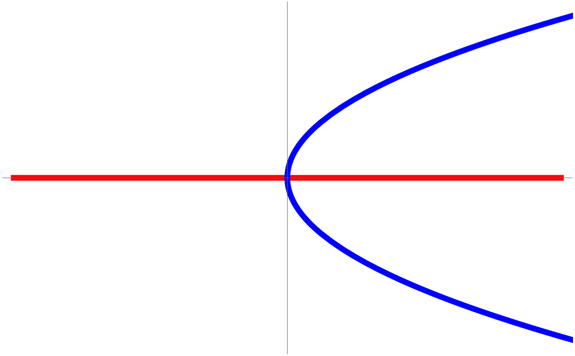

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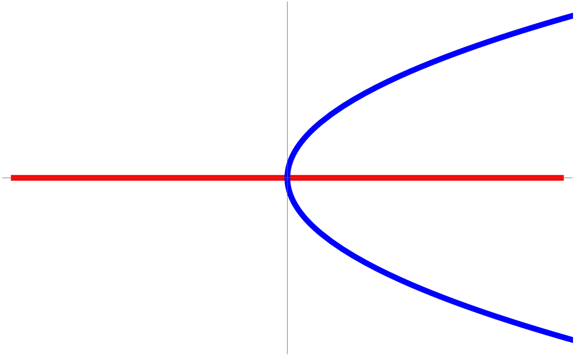

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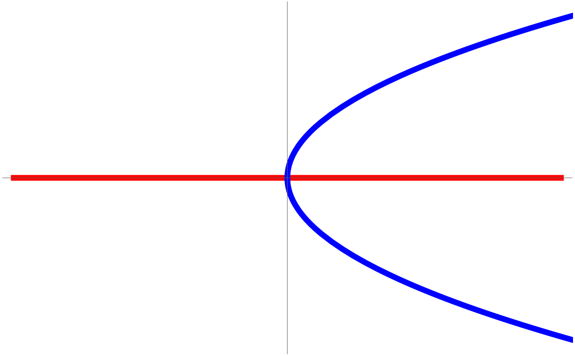

- What if we only saw $x = 0$? What happens at $a = 0$?

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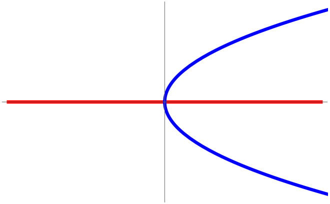
Theorem (Crandall–Rabinowitz)

Suppose $f(a, 0) = 0$ for all $a \in \mathbb{R}$, and

- $\dfrac{\partial f}{\partial x}(a_*, 0) = 0$
- $\dfrac{\partial^2 f}{\partial a \partial x}(a_*, 0) \neq 0$

then a bifurcation branch issues at $(a_*, 0)$.



Bifurcation

H. Poincaré. “L’Équilibre d’une masse fluide animée d’un mouvement de rotation”. Acta Math., vol. 7, pp. 259-380, 1885.

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Bifurcation value: $a = a_*$

Bifurcation

270 H. Poincaré.

Il pourra d'ailleurs arriver qu'une même forme d'équilibre appartienne à la fois à deux ou plusieurs séries linéaires. Nous dirons alors que c'est une *forme de bifurcation*. On peut en effet, pour une valeur de y infiniment voisine de celle qui correspond à cette forme, trouver *deux* formes d'équilibre qui diffèrent infiniment peu de la forme de bifurcation.

Il peut arriver également que deux séries linéaires de formes d'équi-

...

Avant de démontrer ce résultat général, donnons quelques exemples. Soit:

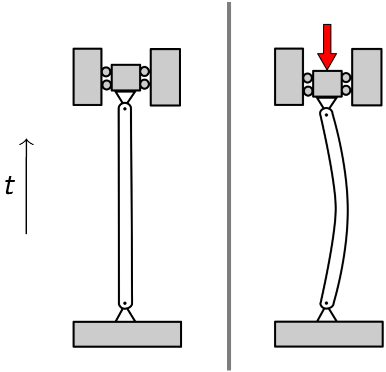
$$F = Ax_1^2 + \frac{1}{3}x_2^3 - y^2 x_2 - \alpha y x_2.$$

Il vient pour les équations d'équilibre:

$$x_1 = 0, \qquad x_2 = \pm\sqrt{y^2 + \alpha y}$$

d'où

Euler's Buckling Problem (1757)



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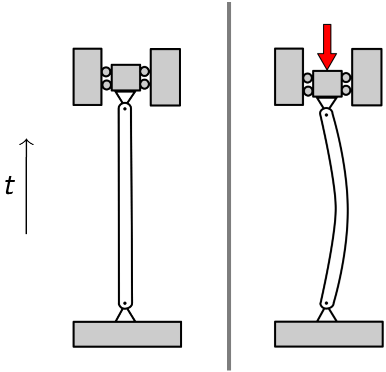
$t \in [0, L]$



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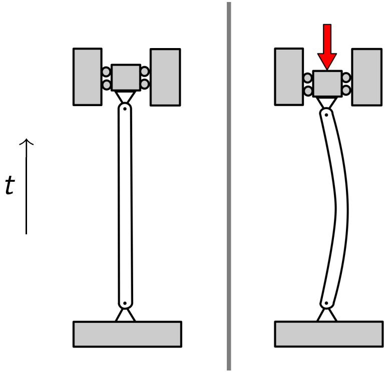

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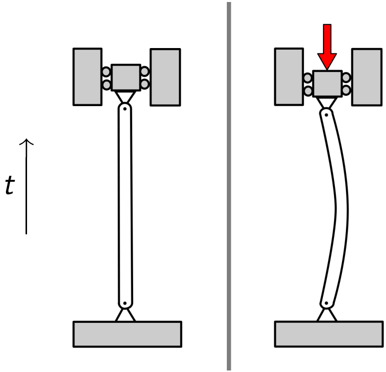

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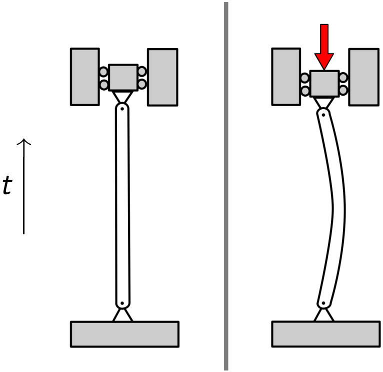

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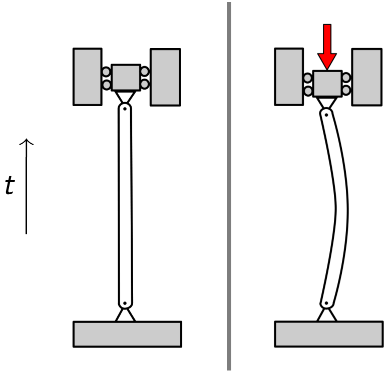

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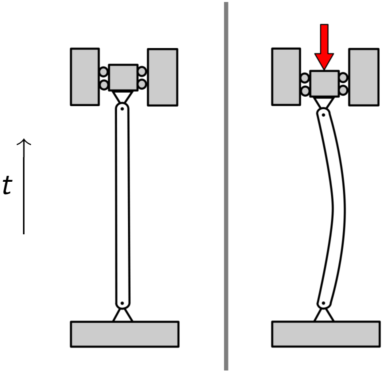

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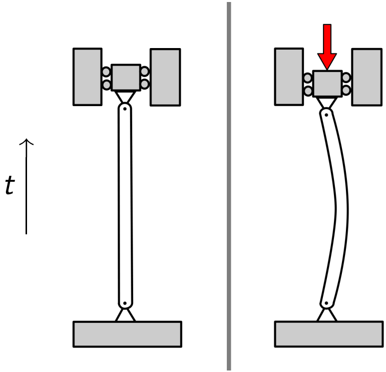

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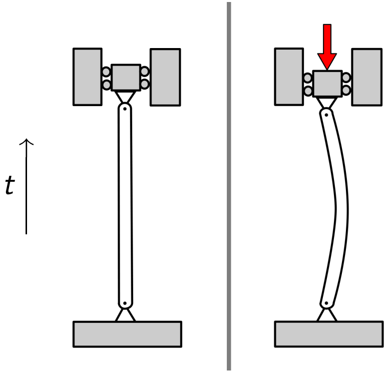

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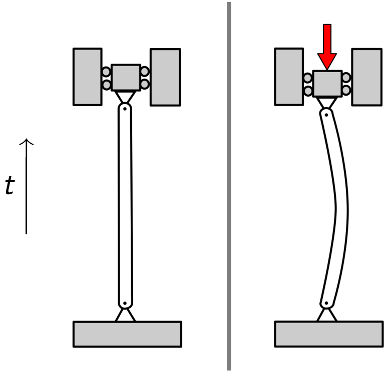

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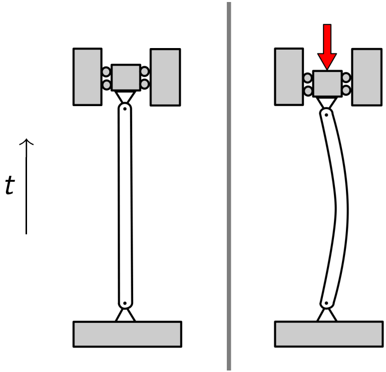

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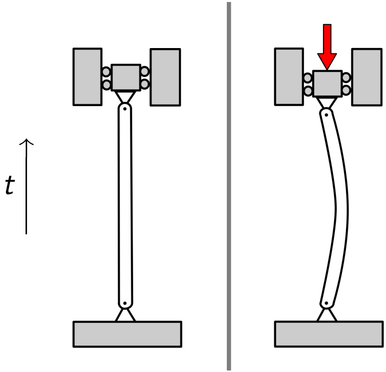

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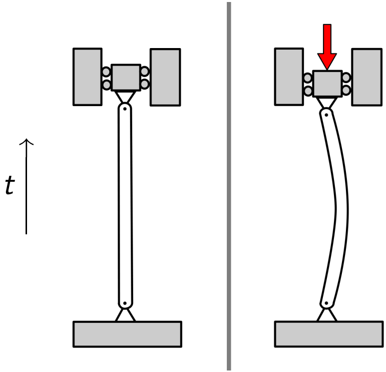

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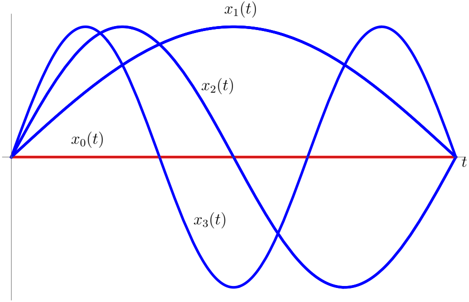

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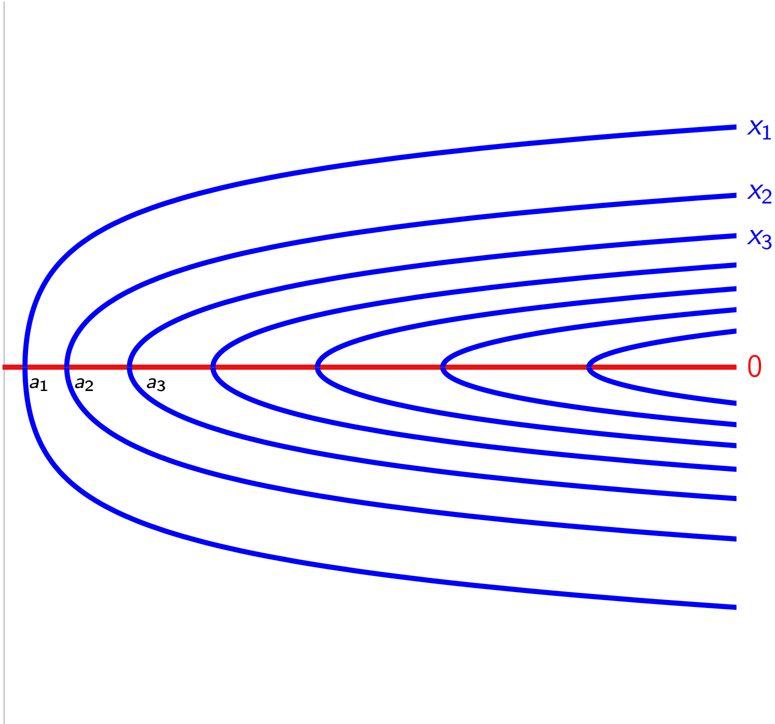

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x_2
x_3
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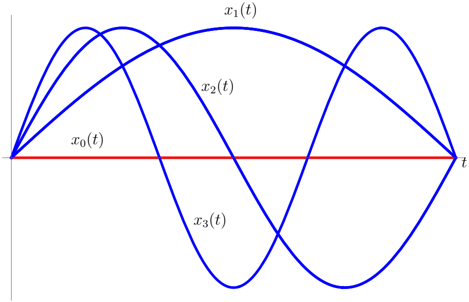

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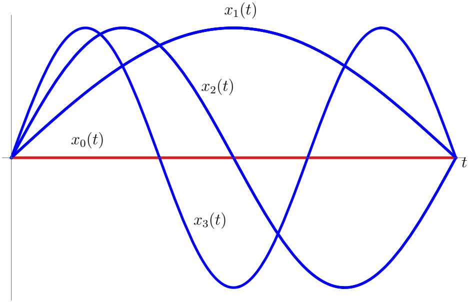

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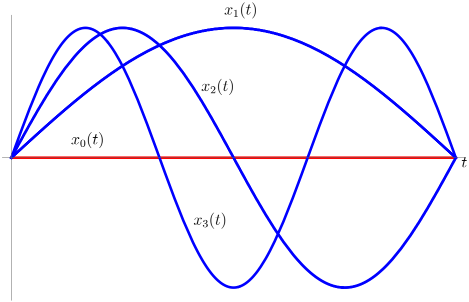

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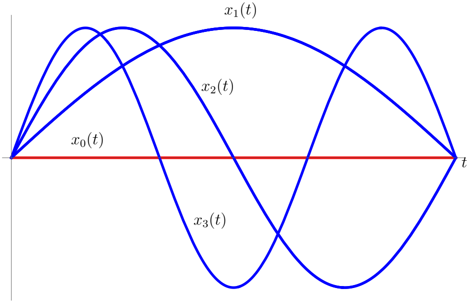

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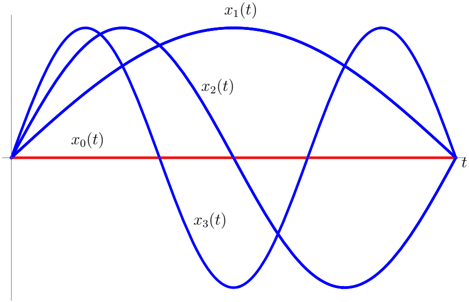

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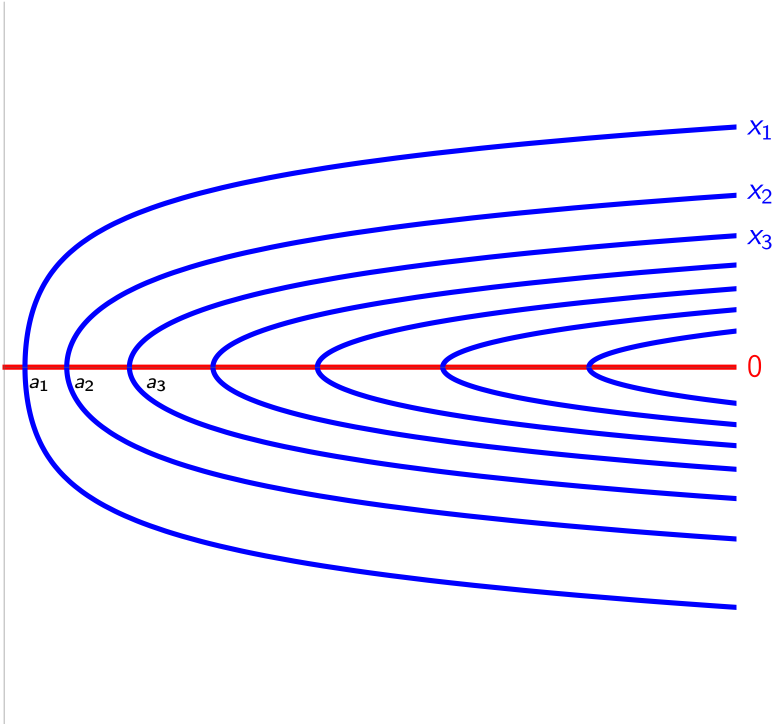
- $\frac{\partial^2 f}{\partial a \partial x}(a_n, 0) = \frac{\partial}{\partial a}\lambda(a)\Big|_{a=a_n} = \frac{1}{2\sqrt{a_n E}} = \frac{L}{2n\pi E} \neq 0$



x_1
x_2
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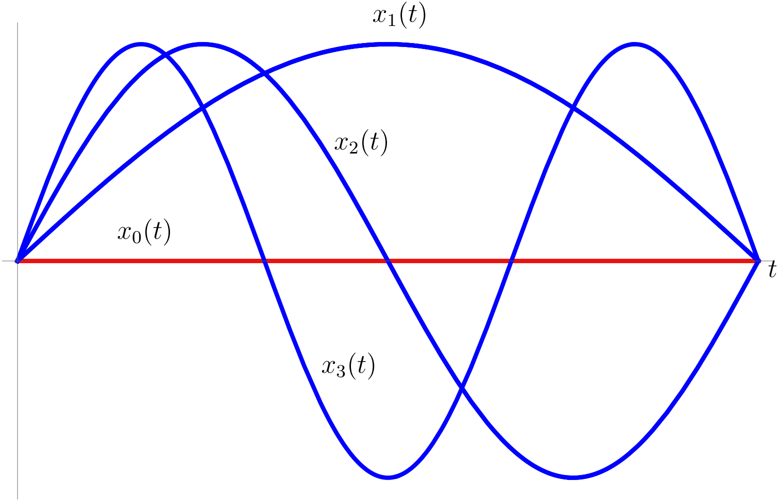
How realistic are these?



Sunkink on train tracks

Applications to Geometric Analysis

Can we *buckle/bifurcate* interesting geometric objects into new ones?

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4. Solutions to the constant Q-curvature problem (with Sammy Sbiti)
5. ...

Applications to Geometric Analysis

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Shameless advertising:

Instability and Bifurcation

Renato G. Bettiol and Paolo Piccione

By the Principle of Least Action, physical systems governed by conservative forces typically assume energy-minimizing states. As such, these ground states are *stable* stationary points of the corresponding energy functional. On the other hand, *unstable* stationary points may also have very interesting features from a mathematical viewpoint, although (or, perhaps, because) they are harder to find in nature.

If the way energy is measured depends on a parameter, a family of stationary points may lose stability when that parameter crosses a certain threshold. Remarkably, this loss of stability creates a new branch of stationary points that splits from the family. This phenomenon was first explored by Poincaré [Poi85], who called it a *bifurcation*, marking the dawn of a multifaceted theory with applications to Dynamical Systems, Analysis, PDEs, and, more recently, to Differential Geometry and Geometric Analysis.

In this article, we give an overview of classical results in variational Bifurcation Theory and some geometric applications, including multiplicity results for Geodesics, Constant Mean Curvature Surfaces, and the Yamabe problem. These are obtained by exploiting the growing instability of families of trivial (often highly symmetric) solutions as they degenerate. The resulting bifurcating solutions are often less symmetric, and give rise to interesting examples where ground states need not be the most symmetric ones.

Renato G. Bettiol is an assistant professor of mathematics at CUNY, Lehman College. His email address is r.bettiol@lehman.cuny.edu.

Paolo Piccione is a professor of mathematics at the University of São Paulo, Brazil, president of the Brazilian Mathematical Society, and member of the IMU Executive Committee. His email address is piccione@ime.usp.br.

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DOI: https://doi.org/10.1090/noti2185

1. Constant Mean Curvature surfaces
2. Minimal surfaces
3. Solutions to the Yamabe problem
4. Solutions to the constant Q-curvature problem (with Sammy Sbiti)
5. ...

`https://www.ams.org/journals/notices/202011/rnoti-p1679.pdf`

Applications to Geometric Analysis

Can we *buckle/bifurcate* interesting geometric objects into new ones?

Shameless advertising:

1. Constant Mean Curvature surfaces
2. Minimal surfaces
3. Solutions to the Yamabe problem
4. Solutions to the constant Q-curvature problem (with Sammy Sbiti)
5. ...

Instability and Bifurcation

Renato G. Bettiol and Paolo Piccione

By the Principle of Least Action, physical systems governed by conservative forces typically assume energy-minimizing states. As such, these ground states are *stable* stationary points of the corresponding energy functional. On the other hand, *unstable* stationary points may also have very interesting features from a mathematical viewpoint, although (or, perhaps, because) they are harder to find in nature.

If the way energy is measured depends on a parameter, a family of stationary points may lose stability when that parameter crosses a certain threshold. Remarkably, this loss of stability creates a new branch of stationary points that splits from the family. This phenomenon was first explored by Poincaré [Poi85], who called it a *bifurcation*, marking the dawn of a multifaceted theory with applications to Dynamical Systems, Analysis, PDEs, and, more recently, to Differential Geometry and Geometric Analysis.

In this article, we give an overview of classical results in variational Bifurcation Theory and some geometric applications, including multiplicity results for Geodesics, Constant Mean Curvature Surfaces, and the Yamabe problem. These are obtained by exploiting the growing instability of families of trivial (often highly symmetric) solutions as they degenerate. The resulting bifurcating solutions are often less symmetric, and give rise to interesting examples where ground states need not be the most symmetric ones.

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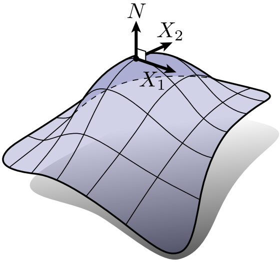
Constant Mean Curvature surfaces

$\Sigma^n \subset \mathbb{R}^{n+1}$
hypersurface

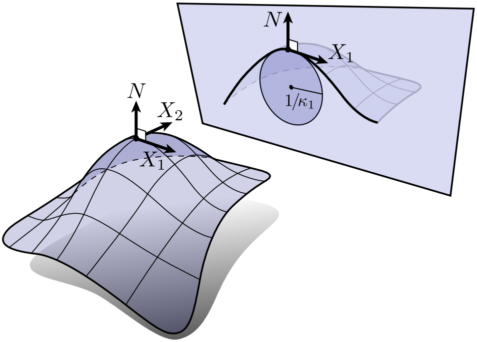


Constant Mean Curvature surfaces

$\Sigma^n \subset \mathbb{R}^{n+1}$
hypersurface

Principal curvatures:

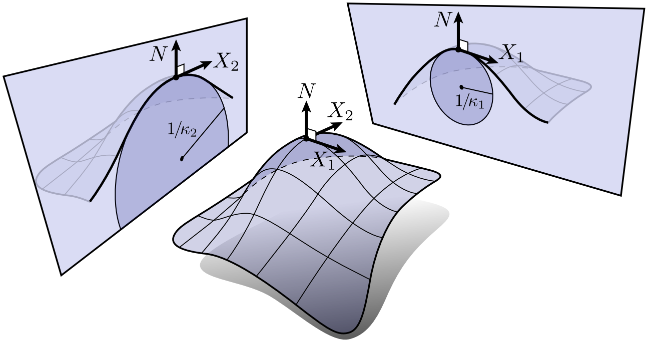
κ_1



Constant Mean Curvature surfaces

$\Sigma^n \subset \mathbb{R}^{n+1}$
hypersurface

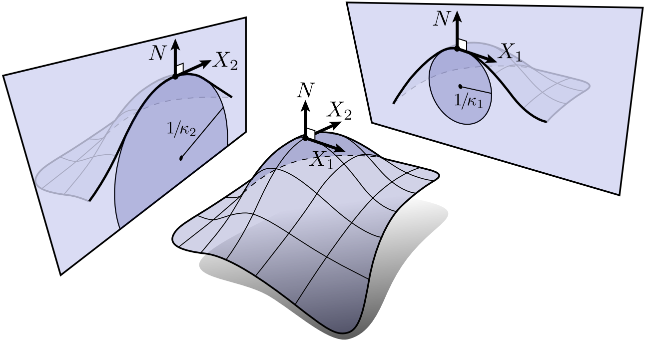
Principal curvatures:
κ_1, κ_2



Constant Mean Curvature surfaces

$\Sigma^n \subset \mathbb{R}^{n+1}$
hypersurface

Principal curvatures:
$\kappa_1, \kappa_2, \ldots, \kappa_n$.



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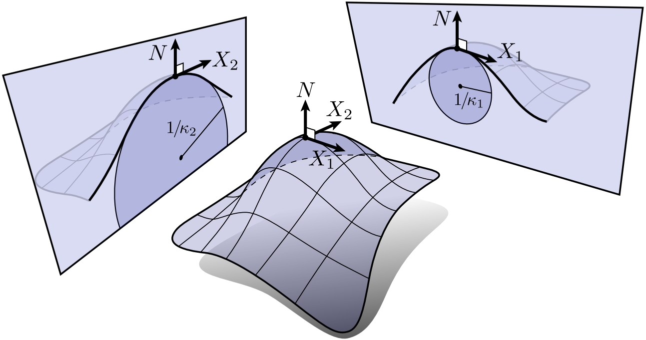
Definition

$\Sigma^n \subset \mathbb{R}^{n+1}$ has Constant Mean Curvature (CMC) if



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$\Sigma^n \subset \mathbb{R}^{n+1}$
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Principal curvatures:
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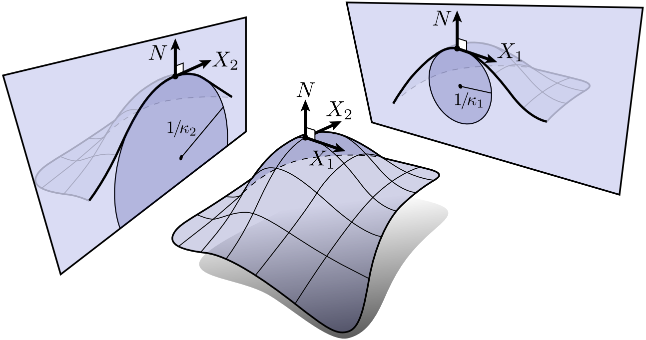


Definition

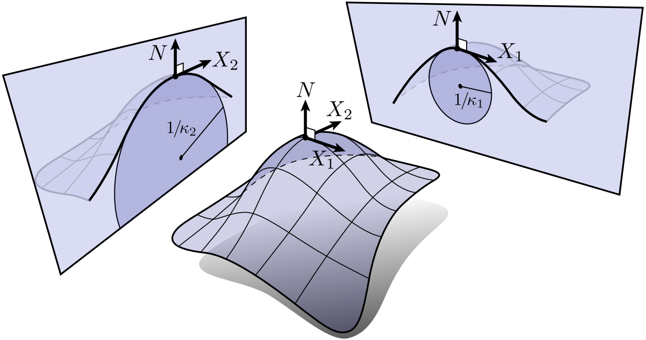
$\Sigma^n \subset \mathbb{R}^{n+1}$ has Constant Mean Curvature (CMC) if

$$\underbrace{\kappa_1 + \cdots + \kappa_n}_{H(\Sigma)} = c$$

Constant Mean Curvature surfaces

$\Sigma^n \subset \mathbb{R}^{n+1}$ hypersurface

Principal curvatures: $\kappa_1, \kappa_2, \ldots, \kappa_n$.



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- Soap bubbles in $\mathbb{R}^3$ are CMC surfaces: round spheres

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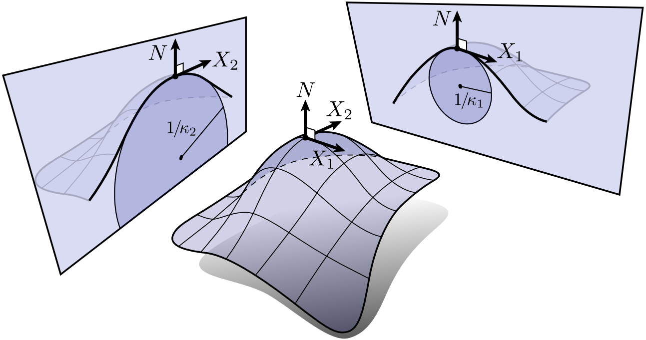

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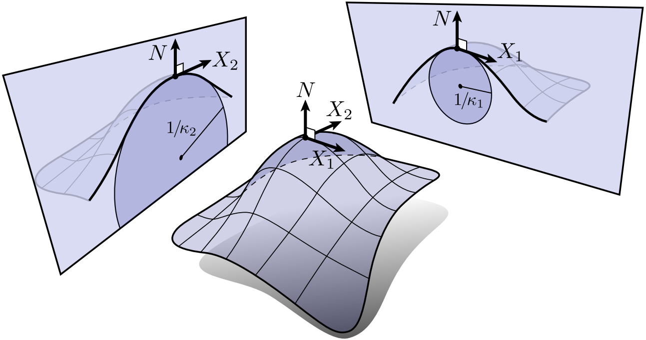

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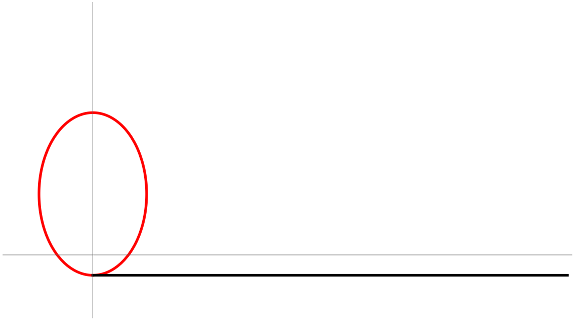
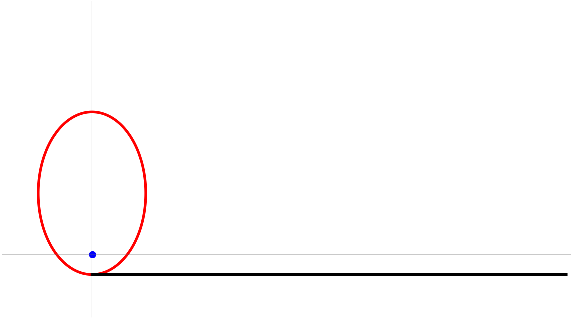
- Soap bubbles in $\mathbb{R}^3$ are CMC surfaces: round spheres
- General *isoperimetric regions* have CMC boundary
- Center of Mass in General Relativity: talk to Dan Lee!

Roulette of a conic section

Roulette of a conic section



Roulette of a conic section



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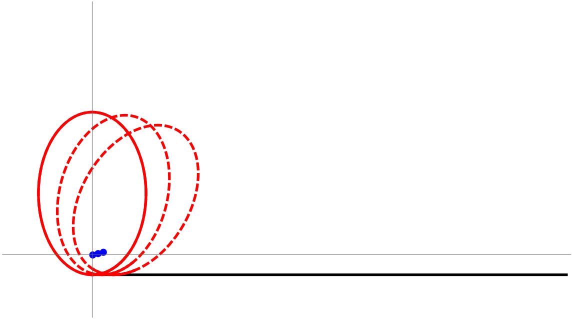

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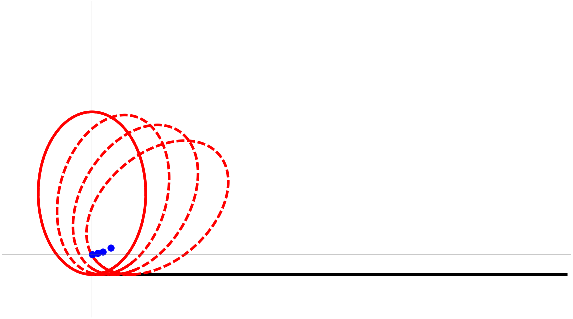

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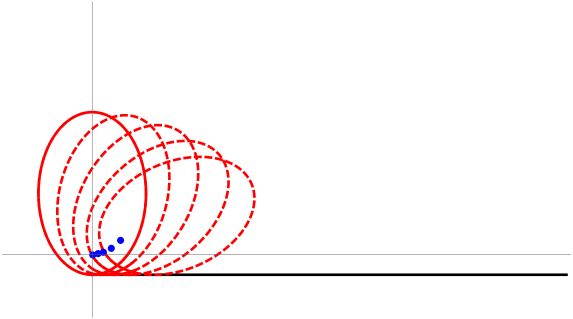

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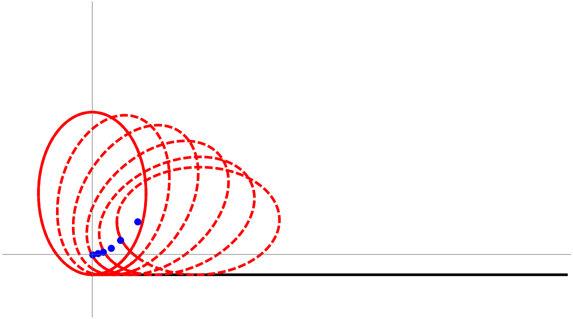

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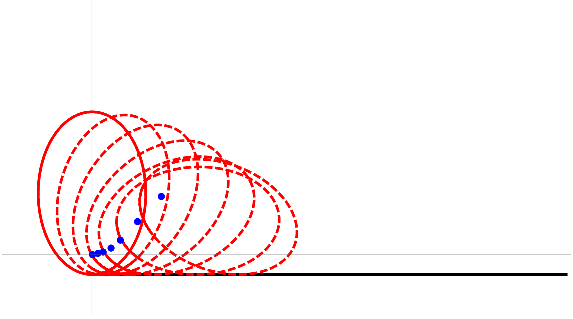

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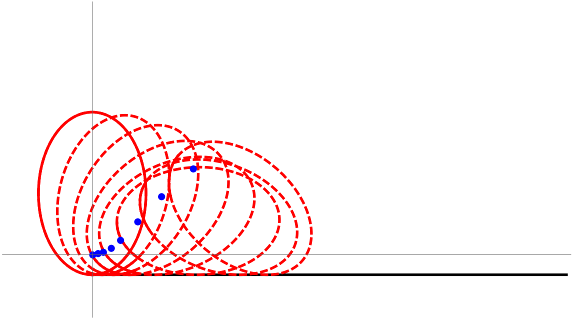

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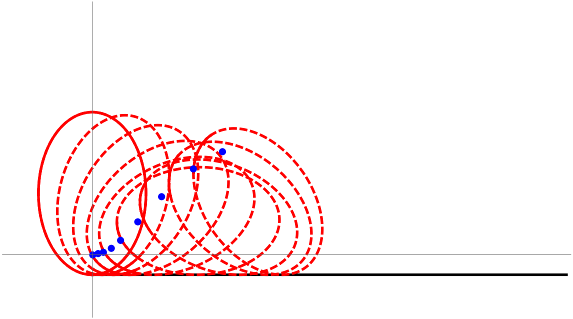

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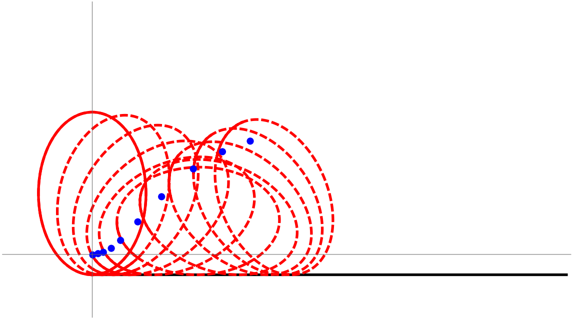

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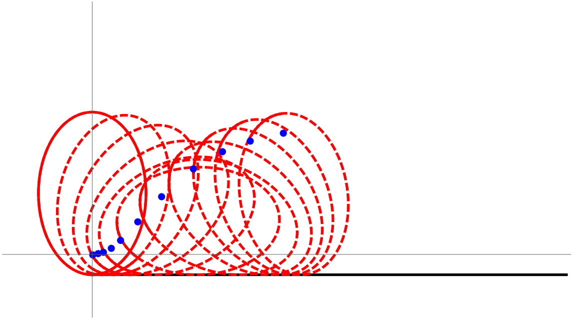

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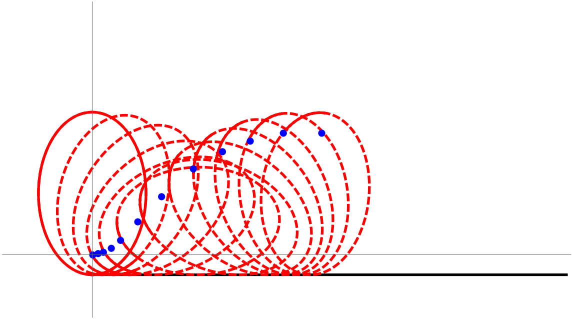

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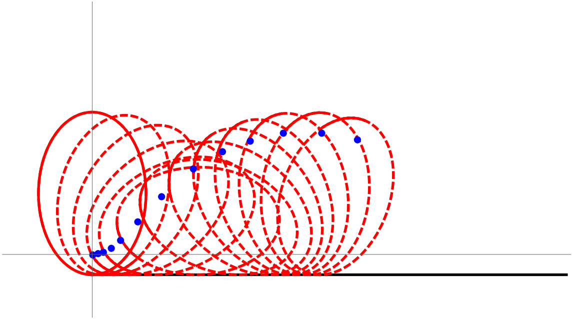

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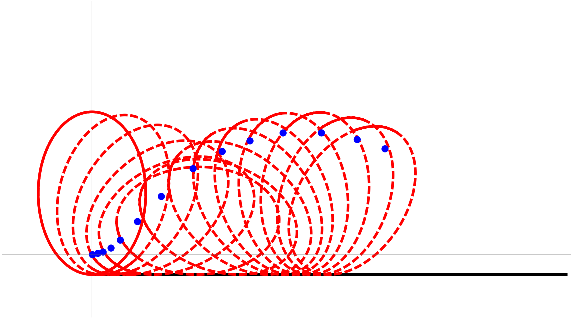

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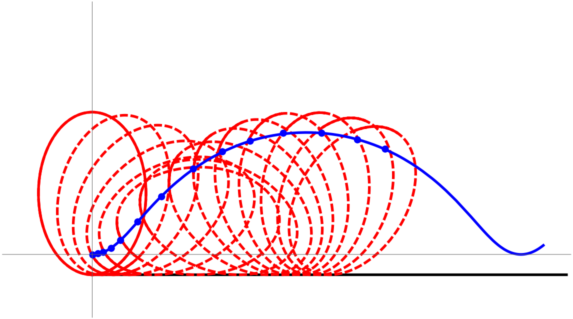

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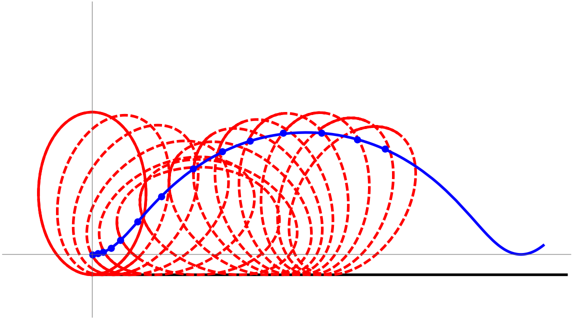

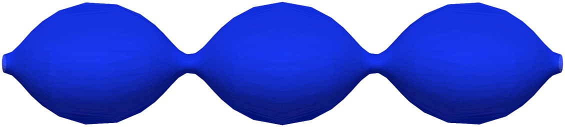

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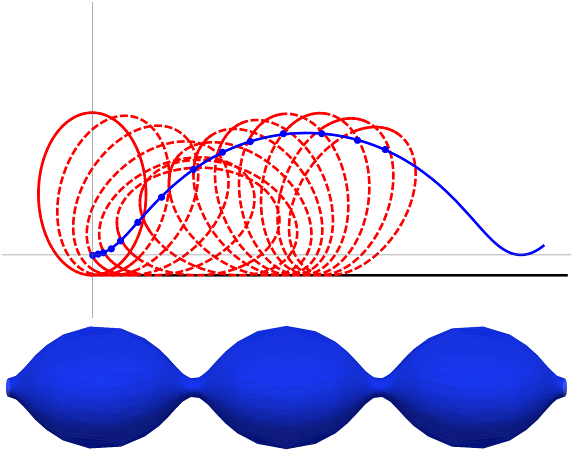
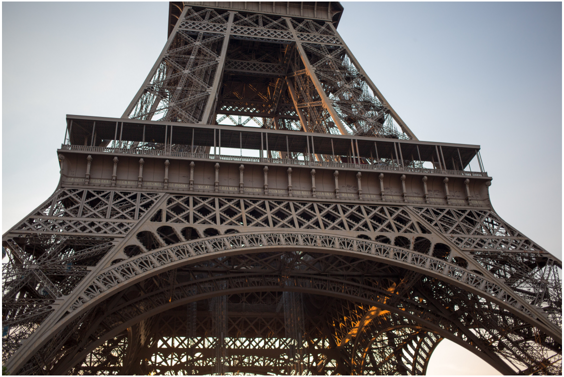


Theorem (Delaunay, 1841)

Surface of revolution $\Sigma \subset \mathbb{R}^3$ has CMC	$\Longleftrightarrow$	Profile curve of Σ is the roulette of a conic section.

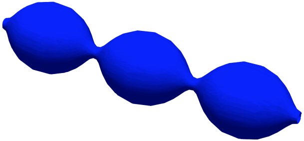
Delaunay



C.-E. Delaunay

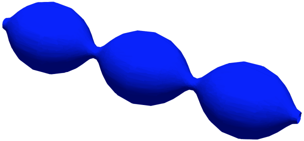
Southeast side of the Eiffel tower:

Delaunay surfaces



Unduloid
(ellipse)

Delaunay surfaces



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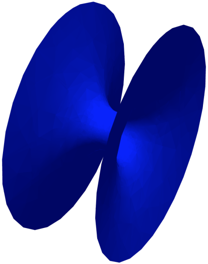
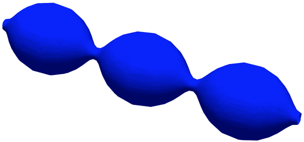
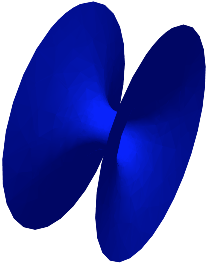


Catenoid
(parabola)

Delaunay surfaces



Unduloid
(ellipse)



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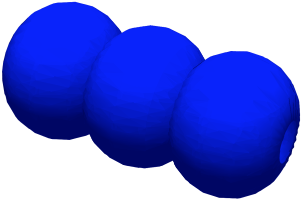
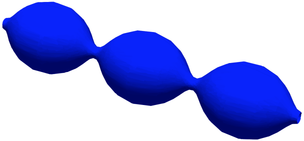
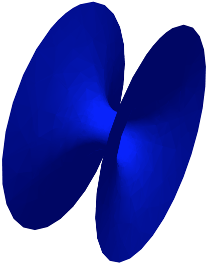


Nodoid
(hyperbola)

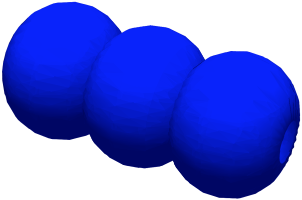
Delaunay surfaces



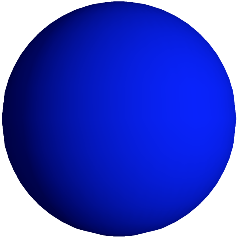
Unduloid
(ellipse)



Catenoid
(parabola)

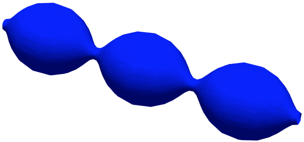


Nodoid
(hyperbola)

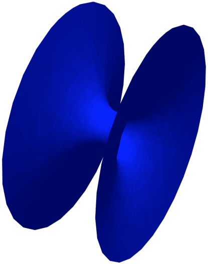


Sphere

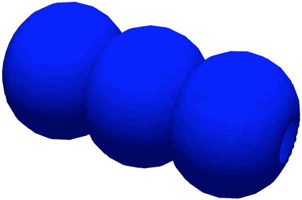
Delaunay surfaces



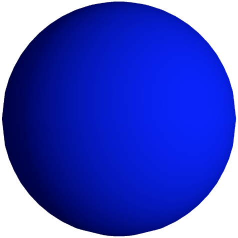
Unduloid
(ellipse)



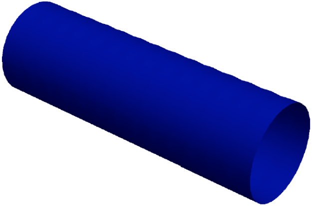
Catenoid
(parabola)



Nodoid
(hyperbola)

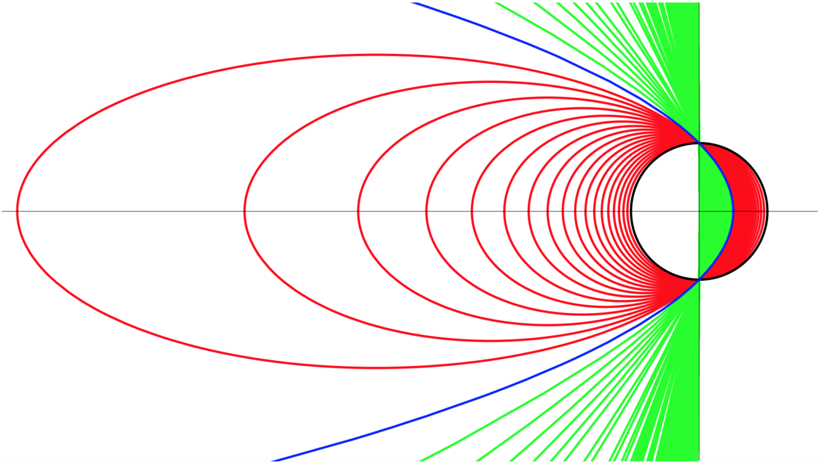


Sphere



Cylinder

Conics of varying eccentricity

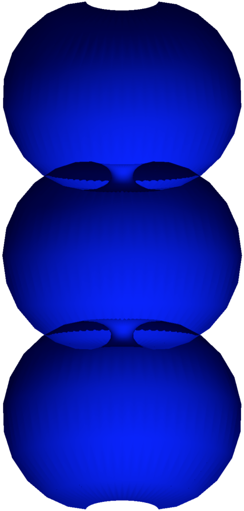


ellipses
$0 < e < 1$

parabola
$e = 1$

hyperbolae
$1 < e < +\infty$

Bifurcating Nodoids



Theorem (Mazzeo–Pacard, 2002)

There are infinitely many families of CMC surfaces in $\mathbb{R}^3$ that bifurcate from nodoids as their eccentricity goes to $+\infty$.

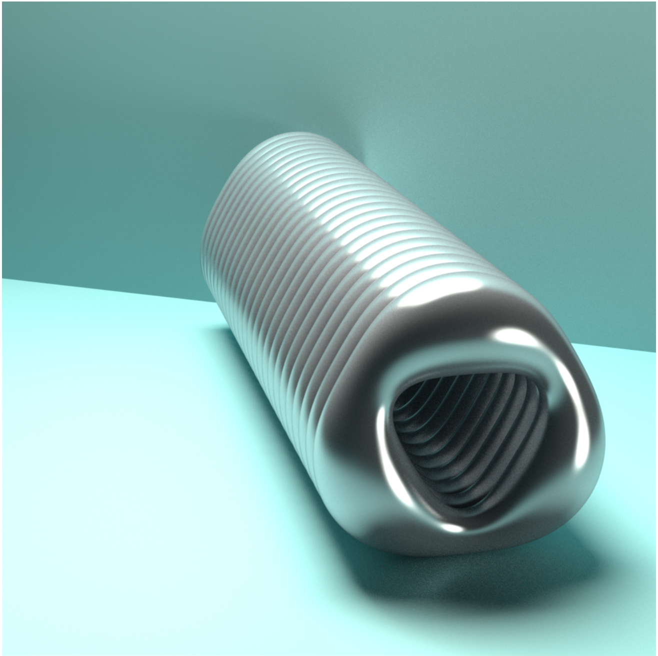


Image credit: GeometrieWerkstatt Gallery

http://service.ifam.uni-hannover.de/˜geometriewerkstatt/gallery/0003.html

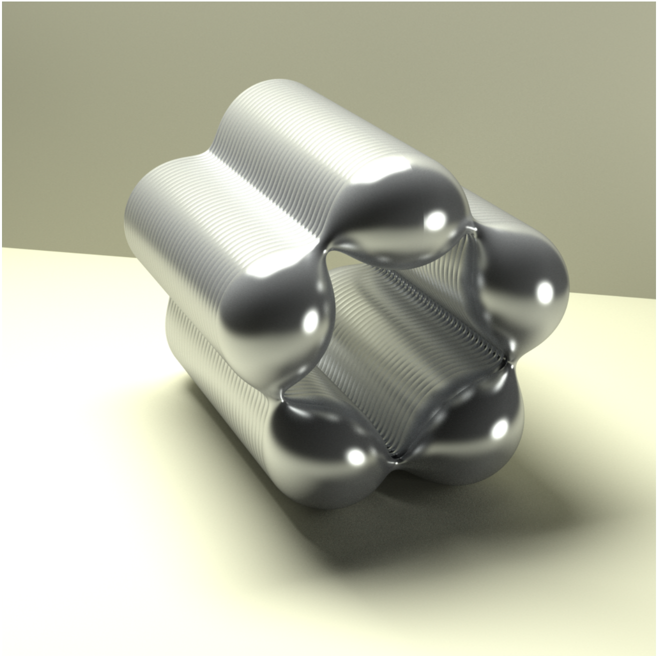


Image credit: GeometrieWerkstatt Gallery
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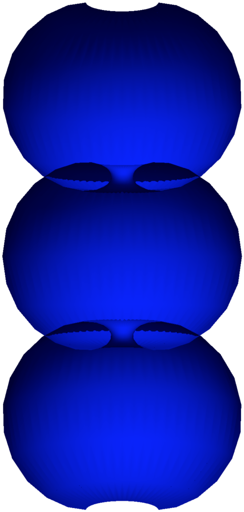
Bifurcating Nodoids



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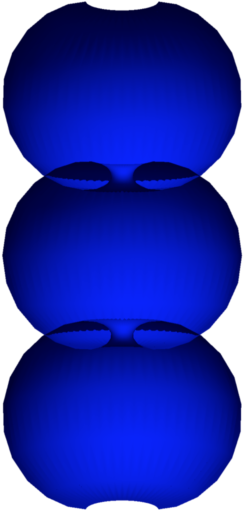
Bifurcating Nodoids



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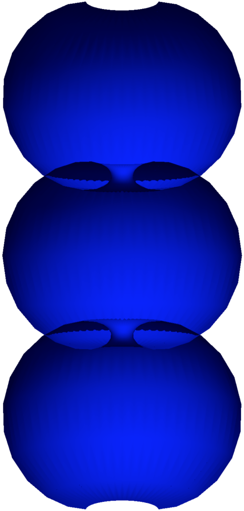
Symmetry-breaking: Bifurcating surfaces are not of revolution!

Bifurcating Nodoids



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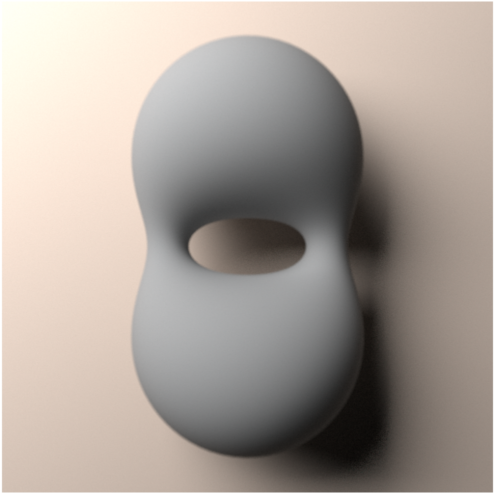
There are infinitely many families of CMC surfaces in $\mathbb{R}^3$ that bifurcate from nodoids as their eccentricity goes to $+\infty$.

Symmetry-breaking:
Bifurcating surfaces are not of revolution!

Theorem (B.–Piccione, 2016)

There are infinitely many families of CMC surfaces in cohomogeneity one manifolds that bifurcate from homogeneous surfaces.



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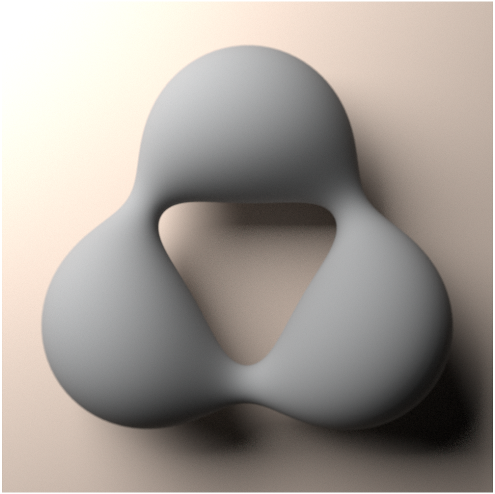

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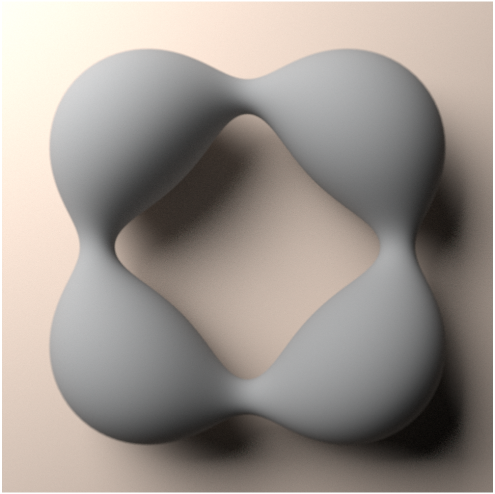

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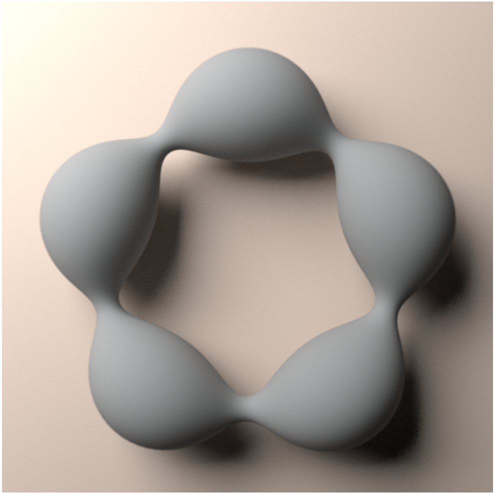

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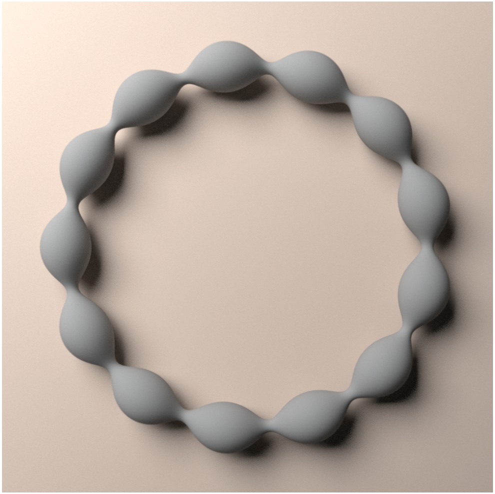

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Applications to Geometric Analysis

Can we *buckle/bifurcate* interesting geometric objects into new ones?

Shameless advertising:

1. ~~Constant Mean Curvature surfaces~~
2. Minimal surfaces
3. Solutions to the Yamabe problem
4. Solutions to the constant Q-curvature problem (with Sammy Sbiti)
5. ...

Instability and Bifurcation

Renato G. Bettiol and Paolo Piccione

By the Principle of Least Action, physical systems governed by conservative forces typically assume energy-minimizing states. As such, these ground states are *stable* stationary points of the corresponding energy functional. On the other hand, *unstable* stationary points may also have very interesting features from a mathematical viewpoint, although (or, perhaps, because) they are harder to find in nature.

Renato G. Bettiol is an assistant professor of mathematics at CUNY, Lehman College. His email address is r.bettiol@lehman.cuny.edu.
Paolo Piccione is a professor of mathematics at the University of São Paulo, Brazil, president of the Brazilian Mathematical Society, and member of the IMU Executive Committee. His email address is piccione@ime.usp.br.

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If the way energy is measured depends on a parameter, a family of stationary points may lose stability when that parameter crosses a certain threshold. Remarkably, this loss of stability creates a new branch of stationary points that splits from the family. This phenomenon was first explored by Poincaré [Poi85], who called it a *bifurcation*, marking the dawn of a multifaceted theory with applications to Dynamical Systems, Analysis, PDEs, and, more recently, to Differential Geometry and Geometric Analysis.

In this article, we give an overview of classical results in variational Bifurcation Theory and some geometric applications, including multiplicity results for Geodesics, Constant Mean Curvature Surfaces, and the Yamabe problem. These are obtained by exploiting the growing instability of families of trivial (often highly symmetric) solutions as they degenerate. The resulting bifurcating solutions are often less symmetric, and give rise to interesting examples where ground states need not be the most symmetric ones.

https://www.ams.org/journals/notices/202011/rnoti-p1679.pdf

Minimal surfaces

Definition

A surface with constant mean curvature $H = 0$ is *minimal*.

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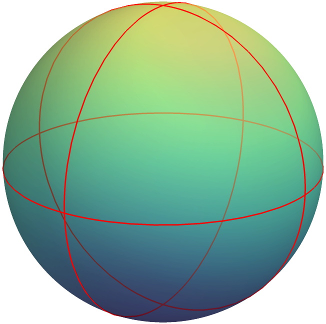
Trivial example on round spheres: equators



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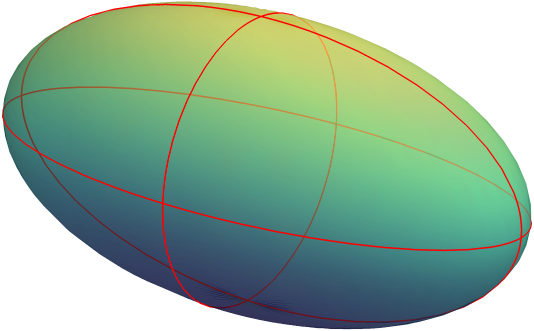

Question (Yau, 1987)

Are all minimal spheres in ellipsoids planar?

$$E(a,b,c,d) := \left\{ \vec{x} \in \mathbb{R}^4 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\}$$

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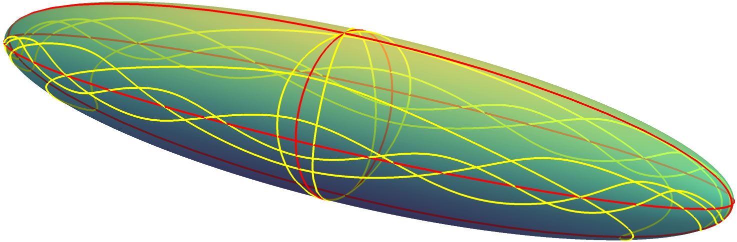
Theorem (B.–Piccione, 2022)

As $a \nearrow +\infty$ in the 3-dimensional ellipsoid $E(a,b,c,d)$, nonplanar minimal spheres bifurcate from (planar) equators.



Thank you for your attention

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