EXTREMALITY AND RIGIDITY FOR SCALAR CURVATURE IN DIMENSION 4

Joint w/ M. Jackson Goodman NY, June 2022

Q: Suppose (S^2, g) has $K \geq 0$ (or $K > 0$)
Can you increase its Gauss curvature without shrinking any areas?

A: "Competitor" metric g' with $\begin{cases} K' \geq K \\ dA' \geq dA \end{cases}$

Gauss-Bonnet:

$$\int_{S^2} K\, dA = \int_{S^2} K'\, dA' = 4\pi$$

Uniformization:

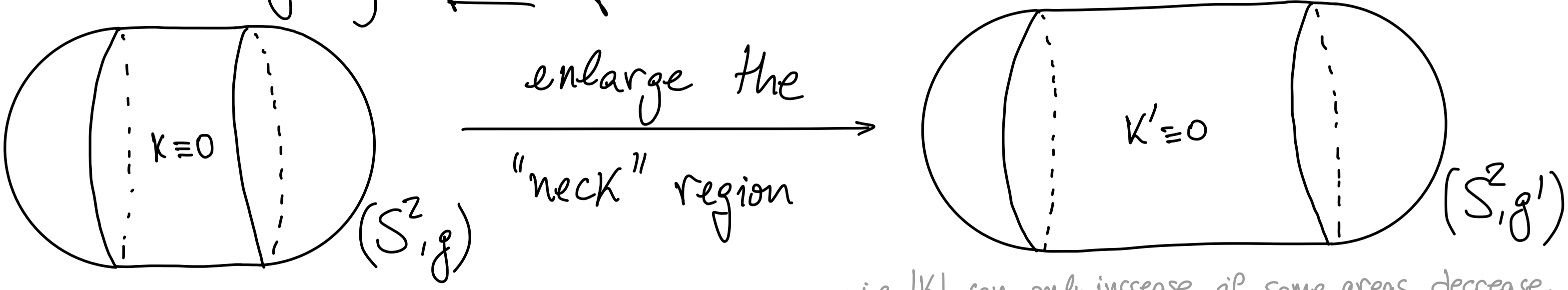
$g, g' \in [g_{\text{round}}]$ so WLOG, $g' = e^{2u} \cdot g$, $dA' = e^{2u} dA$, with $e^{2u} \geq 1$.

$$0 = \int_{S^2} K' dA' - \int_{S^2} K\, dA = \int_{S^2} \left(K' e^{2u} - K\right) dA \geq \int_{S^2} \underbrace{\left(K' - K\right)}_{\geq 0} dA$$

$\Longrightarrow \boxed{K' = K}$ "area-extremality"

If $K > 0$, then can divide both sides of $K' e^{2u} = K$ by $K = K' > 0$ and get $e^{2u} \equiv 1$ i.e. $\boxed{g' = g}$ "area-rigidity"

Remarks: Rigidity fails if $K \geq 0$ but not $K > 0$:



i.e. $|K|$ can only increase if some areas decrease

Same proof implies (Σ^2, g) is area-extremal if $K \leq 0$ and area-rigid in its conformal class if $K < 0$.

Def: A closed Riem. mfld (M,g) is area-extremal for scalar curvature if

$$g' \text{ with } \begin{cases} \mathrm{scal}' \geqslant \mathrm{scal} \\ \underbrace{\Lambda^2 g' \geqslant \Lambda^2 g} \end{cases} \Longrightarrow \mathrm{scal}' = \mathrm{scal} \qquad \left(\text{and area-rigid if } \cdots \Longrightarrow g' = g \right)$$

i.e. $dA' \geqslant dA$ on all surfaces $\Sigma^2 \subset M$

Llarull '98: $(S^n, g_{\mathrm{round}})$ is area-rigid.

Goette-Semmelmann '01-'02, building on Min-Oo '98:

$\chi(M) \neq 0$, $R_g \geqslant 0$, $R_g > 0$ or $\frac{\mathrm{scal}}{2} \cdot g > \mathrm{Ric}_g > 0$ $\Longrightarrow$ (M^n, g) is area-extremal
area-rigid

(M,g) Kähler, $\mathrm{Ric}_g \geqslant 0$, $\mathrm{Ric}_g > 0$

Note: Other than on S^n, only metrics w/ special holonomy!

In dimension 4:

Finsler-Thorpe trick: (M^4, g) has $\sec \geqslant 0 \iff \exists\, \tau : M \to \mathbb{R}$ s.t. $R_g + \tau * \geqslant 0$.

Thm A (B.–Goodman '22). (M^4, g) closed, simply-connected, with $\sec \geqslant 0$. If $\tau : M \to \mathbb{R}$ s.t. $R + \tau * \geqslant 0$ can be chosen $\tau \geqslant 0$ or $\tau \leqslant 0$, then g is area-extremal. If $\frac{\mathrm{scal}}{2} g > \mathrm{Ric} > 0$, then g is area-rigid.

B.–Mendes, '17: If $\tau \geqslant 0$ or $\tau \leqslant 0$, then either $M^4 \underset{\text{homeo}}{\cong} \#^k \mathbb{CP}^2$ (definite)
or $M^4 \underset{\text{isom}}{\cong} (S^2 \times S^2, g_1 \oplus g_2)$. area-rigid by [GS]!

Using Thm A, can produce examples w/ generic holonomy: $R_{g_{FS}} + * > 0$

Cor: (i) $\mathbb{CP}^2$ has an open set of area-rigid metrics containing g_{FS};
(ii) $\mathbb{CP}^2 \# \mathbb{CP}^2$ has area-rigid metrics (Cheeger metrics).

w/ $\sec \geqslant 0$ but not $\sec > 0$, cf. pictures of S^2

Note: $\mathbb{CP}^2 \# \mathbb{CP}^2$ does not admit metrics w/ $R_g \geqslant 0$ nor Kähler metrics!

Only known M^4 with $\sec \geq 0$ to which Thm A does not apply is $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, which has Kähler metrics with $\mathrm{Ric} > 0$, hence area-rigid by [G-S]

(widely conjectured to be all...)

Upshot:

Among known simply-connected examples: $\overbrace{\underbrace{S^4,\ \mathbb{CP}^2}_{\sec>0},\ \underbrace{S^2 \times S^2}_{R\geq 0},\ \mathbb{CP}^2 \# \mathbb{CP}^2,\ \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}}^{\sec \geq 0}$

- M^4 admits $\sec \geq 0 \implies M^4$ admits area-extremal metrics (w/ $\sec \geq 0$)
- M^4 admits $\sec > 0 \implies$ Open set of area-rigid metrics (w/ $\sec > 0$) on M^4

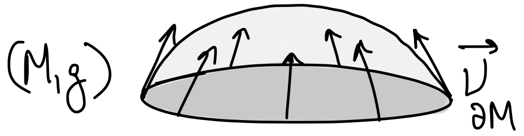
"Local version" cf. works of Cecchini, Lott, Räde, Zeidler, ...

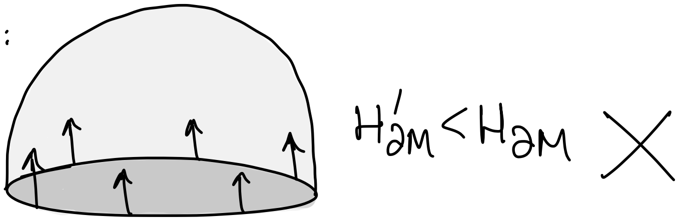
Def: A Riem. mfld w/ boundary (M,g) is area-extremal (for scalar curvature) if

$$g' \text{ with } \begin{cases} \mathrm{scal}' \geq \mathrm{scal} \\ \Lambda^2 g' \geq \Lambda^2 g \\ g'|_{\partial M} = g|_{\partial M} \\ H'_{\partial M} \geq H_{\partial M} \end{cases} \implies \begin{cases} \mathrm{scal}' = \mathrm{scal} \\ H'_{\partial M} = H_{\partial M} \end{cases}$$

(and area-rigid if $\cdots \implies g' = g$)

Need condition on $H_{\partial M}$ for interesting results:





Convention: $\mathrm{II}_{\partial M}(X,Y) = \langle \nabla_X Y, \vec{\nu}_{\partial M} \rangle$, $H_{\partial M} = \mathrm{tr}\, \mathrm{II}_{\partial M}$, $\vec{\nu}_{\partial M}$ inward unit normal.

Thm B (B.–Goodman '22). If (X^4, g) has $\sec > 0$ at $p \in X$, then sufficiently small convex neighborhoods of $p \in X$ are area-extremal.

"Cannot increase scal nor $H_{\partial M}$ without decreasing areas (leaving ∂M unchanged)".

Extension of Thm A to mfld w/ boundary:

Thm C (B.–Goodman '22). (M^4, g) mfld w/ boundary, with $\sec \geq 0$ and $\mathrm{II}_{\partial M} \geq 0$. If $\tau : M \to \mathbb{R}$ s.t. $R + \tau * \geq 0$ can be chosen $\tau \geq 0$ or $\tau \leq 0$, then g is area-extremal. If $\frac{\mathrm{scal}}{2} g > \mathrm{Ric} > 0$, then g is area-rigid.

Outline of proof of Thm A:

- Let S_g, $S_{g'}$ be the spinor bundles over M^4 w.r.t. g, g'; which are locally defined but $S_{g'} \otimes S_g$ is globally defined.

- Twisted Dirac operator $D_{g,g'} : \Gamma(S_{g'} \otimes S_g^+) \to \Gamma(S_{g'} \otimes S_g^+)$

$$D_{g,g'}(\phi \otimes \psi) = \sum_{i=1}^{4} e_i \nabla^{g'}_{e_i} \phi \otimes \psi + (e_i \phi) \otimes \nabla^{g}_{e_i} \psi$$

splits as $D_{g,g'} = \begin{pmatrix} 0 & D^-_{g,g'} \\ D^+_{g,g'} & 0 \end{pmatrix}$ w.r.t. $S_{g'} \otimes S_g^+ = \underbrace{(S^+_{g'} \otimes S^+_g)}_{\cong \Lambda^{+,\text{even}}_{\mathbb{C}} TM} \oplus \underbrace{(S^-_{g'} \otimes S^+_g)}_{\cong \Lambda^{+,\text{odd}}_{\mathbb{C}} TM}$.

- $\operatorname{ind}(D^+_{g,g'})$ does not depend on g, g', so, assuming $g = g'$

Via $S \otimes S \cong \Lambda^* TM$, $D_{g,g'}$ is conjugate to $d + d^*$.

$$\begin{aligned} \operatorname{ind}(D^+_{g,g'}) &= \operatorname{ind}(d+d^*)\big|_{\Lambda^+_{\mathbb{C}} TM} \\ &= \dim \operatorname{Ker}(d+d^*)\big|_{\Lambda^{+,\text{even}}_{\mathbb{C}} TM} - \dim \operatorname{Ker}(d+d^*)\big|_{\Lambda^{-,\text{odd}}_{\mathbb{C}} TM} \\ &= 1 + b_2^+(M) - \cancelto{0}{b_1}(M) > 0 \end{aligned}$$

thus $\exists\, \xi \in \Gamma(S^+_{g'} \otimes S^+_g)$, $\xi \neq 0$, with $D_{g,g'} \xi = 0$.

- Bochner–Lichnerowicz–Weitzenböck formula, $\Lambda^2 g' \geq \Lambda^2 g$ and $\sec_g \geq 0$ imply:

$$D^2_{g,g'} \geq \nabla^* \nabla + \frac{1}{4}(\operatorname{scal}_{g'} - \operatorname{scal}_g) + \mathcal{T}(R),$$

curvature term for Hodge Laplacian on $\Lambda^* TM$

where $\mathcal{T}(R) \geq 0$ if $R \geq 0$, and $\mathcal{T}(*)\big|_{S^+_{g'} \otimes S^+_g} \geq 0$.

- Using that $\mathcal{T}(R) = \mathcal{T}(R + \tau *) - \tau \mathcal{T}(*)$ and, up to reversing orientation, $\tau \leq 0$, get $D^2_{g,g'} \geq \nabla^* \nabla + \frac{1}{4}(\operatorname{scal}_{g'} - \operatorname{scal}_g)$.

- $$0 = \int_M \langle D^2_{g,g'} \xi, \xi \rangle \geq \int_M |\nabla \xi|^2 + \frac{1}{4} \underbrace{(\operatorname{scal}_{g'} - \operatorname{scal}_g)}_{\geq 0} |\xi|^2 \Rightarrow \boxed{\operatorname{scal}_{g'} = \operatorname{scal}_g}$$

□

Adaptations to prove Thm C:

- Atiyah–Patodi–Singer Index Theorem:

$$\operatorname{ind}(D^+_{g,g'}) = \frac{1}{2}\Big(\chi(M) + \sigma(M) + b_0(\partial M) + b_2(\partial M)\Big) > 0$$

Soul theorem and $\mathrm{II}_{\partial M} \geq 0$

signature of bilinear form induced by cup product on image of $H^2(M,\partial M)$ in $H^2(M)$.

thus $\exists\, \xi \in \Gamma(S^+_{g'} \otimes S^+_g)$, $\xi \neq 0$, with $D_{g,g'}\xi = 0$.

As before, and $\mathrm{II}_{\partial M} \geq 0$

- $$0 = \int_M \langle D^2_{g,g'}\xi, \xi\rangle \overset{\downarrow}{\geq} \int_M |\nabla\xi|^2 + \frac{1}{4}\underbrace{(\mathrm{scal}_{g'} - \mathrm{scal}_g)}_{\geq 0}|\xi|^2 + \frac{1}{2}\int_{\partial M}\underbrace{(H'_{\partial M} - H_{\partial M})}_{\geq 0}|\xi|^2.$$

$\square$

Remarks:

* Rigidity statements from $\frac{\mathrm{scal}}{2} g > \mathrm{Ric}_g > 0$ as in Goette–Semmelmann.
* Round hemisphere S^4_+ is area-rigid by Thm C, so counter-examples to Min-Oo conjecture (by Brendle–Marques–Neves) shrink some area

  cf. Miao-Tam '12
* Theorem C applies to normal bundle of $\mathbb{CP}^1 \subset \mathbb{CP}^2$ with Cheeger metric; this mfld does not admit metrics w/ $R \geq 0$ and $\mathrm{II}_{\partial M} \geq 0$.

  Soul Thm $\Rightarrow$ trivial bundle over soul
* Theorem B follows from Theorem C taking M to be small convex neighborhood of $p \in X$ where $\mathcal{Z}$ does not change sign.
* Corvino '00: For generic g, any deformation of scal near $p \in X$ is realized!
* Can prove more general results w/ "topologically modified" competitors $f: (N^4, g') \to (M^4, g)$, with conditions on $\deg f$.