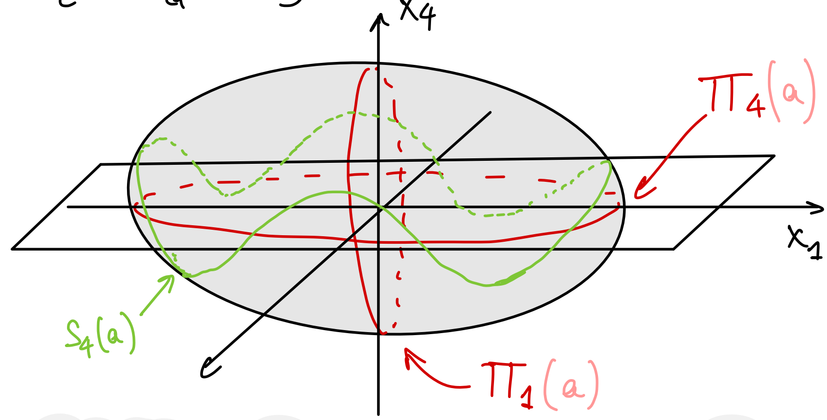
Toronto, 3/2022

BIFURCATING MINIMAL 2-SPHERES IN ELLIPSOIDS OF REVOLUTION

(joint work w/ P. Piccione)

Let $a \geq b \geq c \geq d > 0$ and

$$E(a,b,c,d) := \left\{ x \in \mathbb{R}^4 : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\} \cong (S^3, g)$$



Planar minimal 2-spheres:

$$\Pi_i = E(a,b,c,d) \cap \{x_i = 0\}$$

$i = 1, 2, 3, 4.$

$\mathrm{Emb}(S^2, S^3) \simeq \mathbb{RP}^3$ $\overset{\text{"Morse theory"}}{\rightsquigarrow}$ At least 4 embedded minimal 2-spheres in every (S^3, g)?

At least 3 closed geodesics in (S^2, g)
$E(a,b,c) \cong (S^2, g) \subset \mathbb{R}^3$ realizes that minimum.
$a \neq b \neq c$

$\underline{Q}$ (Yau, 1987). Are all minimal 2-spheres in $E(a,b,c,d)$ planar?

- Almgren '66: Yes, if $a = b = c = d$
- Haslhofer–Ketover '19: No, if $a \gg b$
  - $\hookrightarrow$ index$(\Pi_4) \nearrow +\infty$ as $a \nearrow +\infty$

Nonplanar $S_{HK} \subset E(a,b,c,d)$ realizes the 2-width:
$\mathrm{Area}(\Pi_1) < \mathrm{Area}(S_{HK}) < 2\,\mathrm{Area}(\Pi_1)$
As $a \nearrow +\infty$, $S_{HK} \xrightarrow{\text{varifold}} 2\Pi_1$

{Min-max + MCF}

Thm. (B.–Piccione, 21). If $b = c$ or $c = d$, then there are arbitrarily many geometrically distinct nonplanar embedded minimal 2-spheres in $E(a,b,c,d)$ as $a \nearrow +\infty$.

Henceforth: $b = c$ $\quad N(a) = \#\{\text{distinct solutions}\}$ satisfies $\displaystyle\liminf_{a \to +\infty} \frac{N(a)}{a} \geq \frac{1}{2d}$

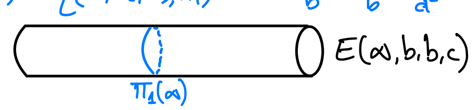
More precisely:

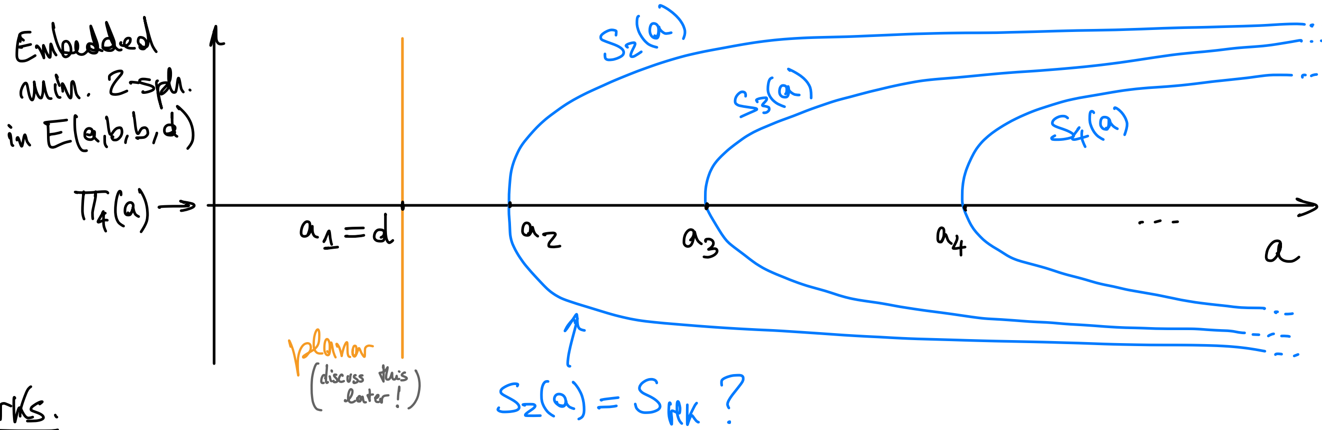
$\forall m \geq 2$, $\exists\, S_m(a) \subset E(a,b,b,d)$, $a > a_m$, nonplanar embedded minimal 2-sphere

$S_m(a)$ "scarring" at $\Pi_1(\infty)$, cf. Song–Zhu '21

- $S_m(a) \cap \Pi_4(a) = m$ disjoint circles
- $S_m(a) \longrightarrow m \cdot \Pi_1(\infty)$ smoothly away from $(0,0,0,\pm d)$ $\leftarrow$ so $\mathrm{Area}(S_m(a)) \to m \cdot \mathrm{Area}(\Pi_1(\infty))$
- $\displaystyle\liminf_{m \to \infty} \frac{\mathrm{index}(S_m(a_m+\varepsilon))}{\mathrm{Area}(S_m(a_m+\varepsilon))} \geq \frac{1}{\mathrm{Area}(\Pi_1(\infty))} = \frac{3}{4\pi b^2 d}$

$$\Pi_1(\infty) = \left\{ (0, x_2, x_3, x_4) \in \mathbb{R}^4 : \frac{x_2^2}{b^2} + \frac{x_3^2}{b^2} + \frac{x_4^2}{d^2} = 1 \right\}$$





Remarks.

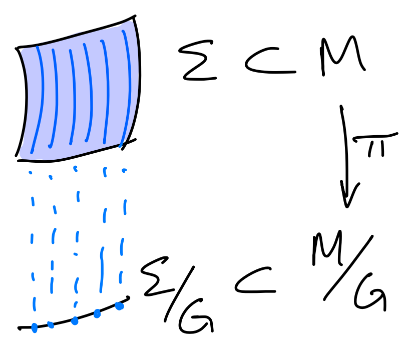
- m is even: $S_m^+(a) = S_m(a) \cap \{x_1 \geq 0\}$ is a free boundary minimal disk in $E^+(a,b,b,d) = E(a,b,b,d) \cap \{x_1 \geq 0\}$.
- $S_2(a) = S_{HK}$? Do any $S_m(a)$ realize some p-width?
- $(a_m)_{m \geq 2}$ solve a continued-fraction (arithmetic) equation.

Conjecture: If $b=c=d=1$, then $a_m = m$; i.e., m^{th} soliton appears when $\text{Area}(\Pi_4(a)) = m \cdot \text{Area}(\Pi_i(a))$, $i = 2,3$

Steps of proof:

1. Symmetry reduction
2. Existence of Geodesics
3. Local bifurcation
4. Global bifurcation

1. Symmetry reduction (Hsiang–Lawson)



$G \curvearrowright (M,g)$ isometric action

$M_{pr} \subset M$ principal part (open, dense, connected)

$\pi : (M_{pr}, g) \to (M_{pr}/G, \check{g})$ Riem. submersion

$V : M_{pr}/G \to \mathbb{R}$ Volume function is smooth, exends to $V: M/G \to \mathbb{R}$
$x \mapsto \text{Vol}(\pi^{-1}(x))$ with $V|_{\partial(M/G)} \equiv 0$.

Palais Symmetric Criticality Principle + Fubini Thm.

$\boxed{\text{G-invariant hypersurface } \Sigma \subset M \text{ is minimal}} \iff \boxed{\Sigma_{pr}/G \subset (M_{pr}/G, V^{2/k}\check{g}) \text{ is minimal}}$

"cohomogeneity" of $\Sigma \subset M$, i.e. codim of principal orbits in Σ.

$k = \dim M_{pr}/G - 1$

$G = O(2) \curvearrowright M = E(a, \underline{b, b}, d)$

$$G(x) = \{ (x_1, x_2\cos\theta, x_3 \sin\theta, x_4) \in E(a,b,b,d) : \theta \in \mathbb{R} \}$$

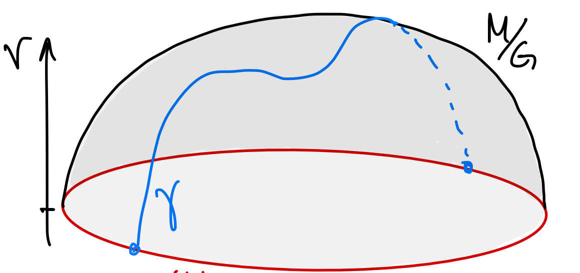
$G(x)$ is a circle of radius $r = b\sqrt{1 - \frac{x_1^2}{a^2} - \frac{x_4^2}{d^2}}$

$M^G = E(a,b,b,d) \cap \{x_2 = x_3 = 0\}$ fixed points

$\pi(x) = (x_1, \underbrace{\sqrt{x_2^2 + x_3^2}}_{r}, x_4) \in M/G$

$M_{pr} = M \setminus M^G$

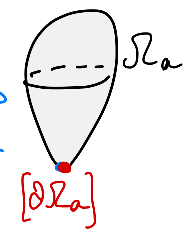
$$(M/G, \check{g}) = \{ (x_1, r, x_4) \in \mathbb{R}^3 : \frac{x_1^2}{a^2} + \frac{r^2}{b^2} + \frac{x_4^2}{d^2} = 1,\ r \geq 0 \}$$



$$V = 2\pi r = 2\pi b \sqrt{1 - \frac{x_1^2}{a^2} - \frac{x_4^2}{d^2}}$$

$$\Omega_a := (M_{pr}/G,\ V^2 \check{g})$$

degenerates along $\partial\Omega_a$

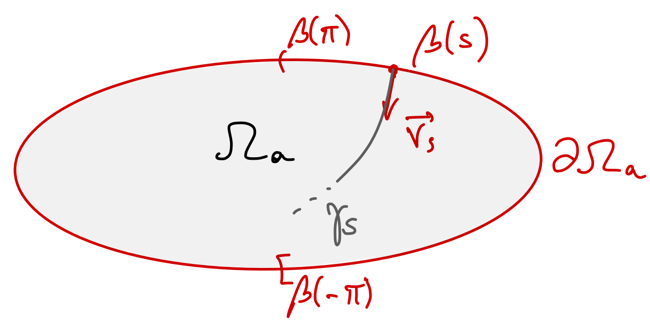


G-invariant minimal 2-spheres in $E(a,b,b,d)$	$\leftrightsquigarrow$	Free boundary geodesics in Ω_a

2. Existence of Geodesics

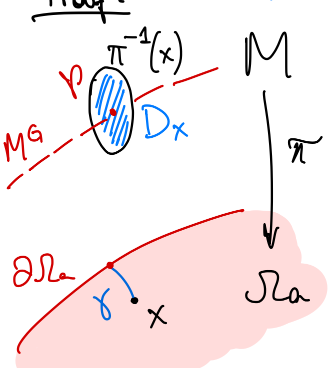
$$\beta(s) = (a\cos s, 0, d\sin s)$$

$\vec{V}_s$ = unit normal at $\beta(s)$



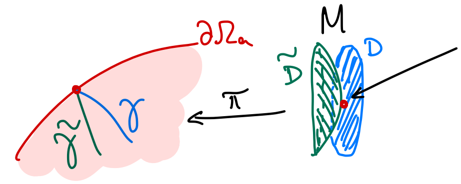
Thm. $\forall s \in [-\pi, \pi)$, $\exists!\ \gamma_s : (0, \ell_s) \to \Omega_a$ maximal geodesic starting transversal to $\partial\Omega_a$ at $\beta(s)$. Moreover, $\lim_{t \to 0} \frac{\gamma_s'(t)}{\|\gamma_s'(t)\|_g} = \vec{V}_s$

Proof: ← Inspired by Hass–Norbury–Rubinstein '03



- $x \in \Omega_a$ close to $\partial\Omega_a \rightsquigarrow \pi^{-1}(x) \subset E(a,b,b,d)$ real-analytic, extremal
- Plateau problem: $\exists D_x \subset E(a,b,d,d)$ least-area disk w/ $\partial D_x = \pi^{-1}(x)$; is unique, smooth, embedded, G-invariant.
- $D_x \cap M^G = \{p\}$, $\pi(p) = \beta(s)$ for some s
- Symm. reduction: $\pi(D_x \setminus \{p\}) \subset \Omega_a$ is a geodesic γ

- $\lim_{t\downarrow 0} \gamma(t) = \pi(p) = \beta(s) \in \partial\Omega_a$
- D_x smooth $\Longrightarrow \lim_{t\downarrow 0} \dfrac{\gamma'(t)}{\|\gamma'(t)\|} = \vec{v}_s$
- Removable Singularity Thm + Max. Princ.: Only such geodesic is γ. $\square$

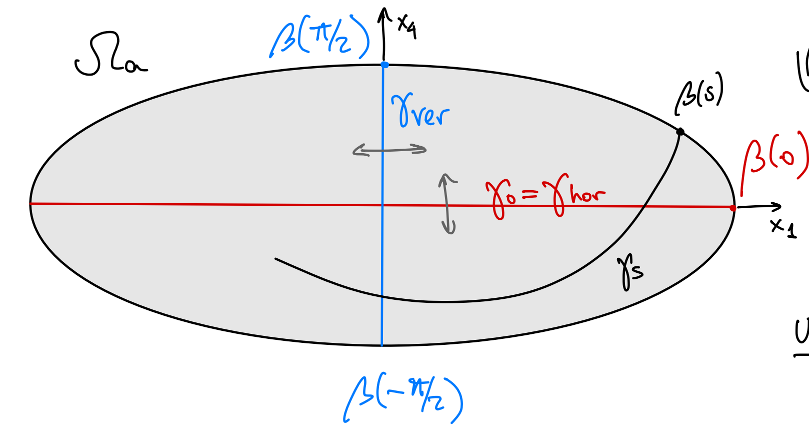


Remove singularity (punctured min. surface) [Choi-Schoen]

$\Downarrow$

Tangent min. surf.

Reflections: $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \curvearrowright E(\underline{a},b,b,\underline{d}) \rightsquigarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \curvearrowright \Omega_a$

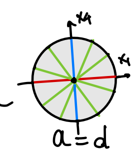


Only G-inv. planar min. 2-spheres:

$\pi^{-1}(\gamma_{ver}) = \Pi_1(a)$

$\pi^{-1}(\gamma_{hor}) = \Pi_4(a)$

unless $a = d$, then get $S^1 \underset{\text{isom.}}{\curvearrowright} \Omega_a$



Prop: All γ_s, $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$, do not self-intersect and intersect γ_{ver} transversely.

Reparametrize γ_s so that $\gamma_s(1) \in \gamma_{ver}$, $\forall s$.

Strategy: $f : (0, +\infty) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ real-analytic functions

$$f_{even}(a,s) := [\gamma_s'(1)]_{x_4}$$

$$f_{odd}(a,s) := [\gamma_s(1)]_{x_4}$$

$\boxed{f(a,s) = 0} \rightsquigarrow \boxed{\pi^{-1}(\gamma_s) \text{ is a minimal 2-sphere}}$

Note: $f_{even}(a,0) = f_{odd}(a,0) = 0$, $\forall a > 0$.

Look for bifurcations from $(a,0)$!

3. Local Bifurcation

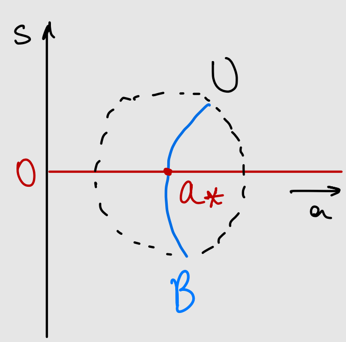
Clever use of Implicit Function Theorem

Thm (Crandall-Rabinowitz). Suppose $f(a,0) = 0$ for all $a > 0$, and

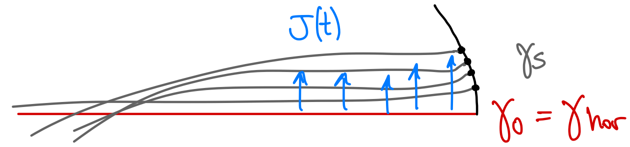
(i) $\dfrac{\partial f}{\partial s}(a_*, 0) = 0$

(ii) $\dfrac{\partial^2 f}{\partial a \partial s}(a_*, 0) \neq 0$

Then $\exists U \ni (a_*, 0)$ s.t. $f^{-1}(0) \cap U = \{(a,0) \in U\} \cup \mathcal{B}$, where $\mathcal{B}$ is a bifurcation branch.



$J(t) = \frac{d}{ds}\gamma_s\Big|_{s=0}$ Jacobi field, $\quad v_a(t) = [J(t)]_{x_4}$



$$\left.\begin{aligned} \frac{\partial f_{even}}{\partial s}(a,0) &= v_a'(1), & \frac{\partial f_{odd}}{\partial s}(a,0) &= v_a(1), \\ \frac{\partial^2 f_{even}}{\partial a\,\partial s}(a,0) &= \frac{d}{da}v_a'(1), & \frac{\partial^2 f_{odd}}{\partial a\,\partial s}(a,0) &= \frac{d}{da}v_a(1). \end{aligned}\right\}$$

Determined by values of $J(t)$ where γ_{hor} and γ_{ver} meet.

Jacobi equation for $J(t)$

$\rightsquigarrow$

Sturm-Liouville equation for $v_a(t)$ satisfied with $\lambda = 0$:

$$(SL)_a \begin{cases} -(p_a v_a')' + q_a v_a = \overset{=0}{\lambda p_a v_a} \\ a^2 v_a(0) + d^2 v_a'(0) = 0 \quad (\text{IC: orthog. to } \partial\Omega_a) \\ \underbrace{v_a'(1) = 0}_{\text{even}} \text{ or } \underbrace{v_a(1) = 0}_{\text{odd}} \quad (\text{BC: even or odd}) \end{cases}$$

⚠ Singular: $p_a(0) = 0$

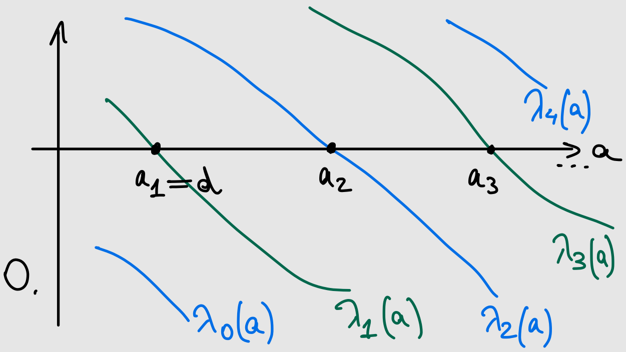
$\frac{\partial}{\partial a} p_a < 0, \quad \frac{\partial}{\partial a} q_a < 0$

$p_a \longrightarrow 0$ and $q_a \longrightarrow -\infty$ uniformly

$\rightsquigarrow$ Oscillation theory for singular SL eqn

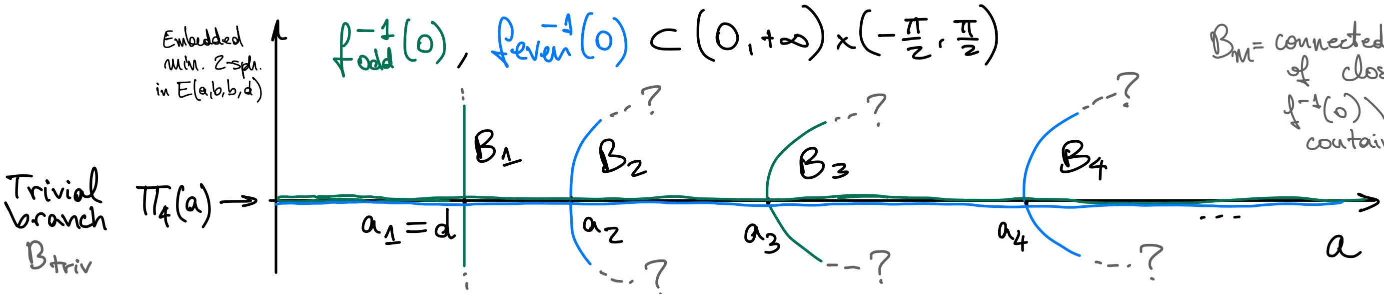
Prop. $\exists (a_m)_{m\geq 1}$ s.t. $\lambda = 0$ is an even/odd eigenvalue of $(SL)_a$ iff $a = a_m$ with m even/odd.

$\exists\, \lambda_m : (a_m - \varepsilon, +\infty) \longrightarrow \mathbb{R}$ s.t. $\lambda_m(a)$ is an eigenvalue of $(SL)_a$ with $\lambda_m(a_m) = 0$ and $\lambda_m'(a) < 0$.



By Crandall-Rabinowitz: Local bif. branches issuing from $(a_m, 0)$, $m \in \mathbb{N}$.

$f_{odd}^{-1}(0)$, $f_{even}^{-1}(0) \subset (0,+\infty) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



B_m = connected component of closure of $f^{-1}(0) \setminus B_{triv}$ that contains $(a_m, 0)$.

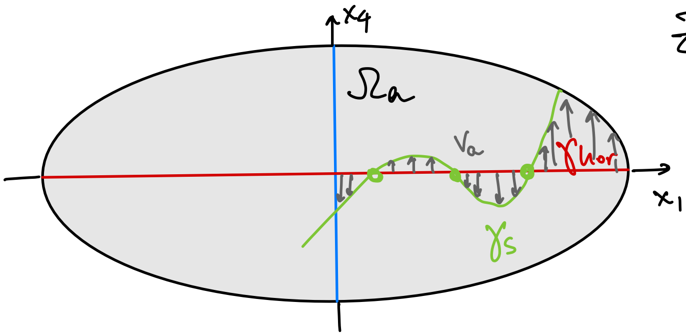
4. Global Bifurcation

Thm (Rabinowitz). If the restriction of $(a,s) \longmapsto a$ to $f^{-1}(0)$ is proper, then every bifurcation branch $\mathcal{B}$ either is noncompact or reattaches to trivial branch.

• Properness: $f^{-1}(0) \cap \{(a_*, s)\}$ is finite! (or [Choi–Schoen])

$\boxed{\text{Prop: Branches } B_m \text{ issuing from } (a_m, 0) \text{ are pairwise disjoint.}}$

Pf: $Z(a,s) = \#\{\gamma_s \cap \gamma_{hor} \subset \Omega_a\}$ is locally constant



$Z|_{B_m} = m$, because $Z(a,s) = \# V_a^{-1}(0)$

and k^{th} eigenfunction $V_a : [0,1] \to \mathbb{R}$

has k zeros by Sturm Oscillation Thm.

$\Downarrow$ $\left(\begin{array}{l} m = 2k \text{ for } k^{th} \text{ \underline{even} eigenfunction} \\ m = 2k+1 \text{ for } k^{th} \text{ \underline{odd} eigenfunction} \end{array} \right)$

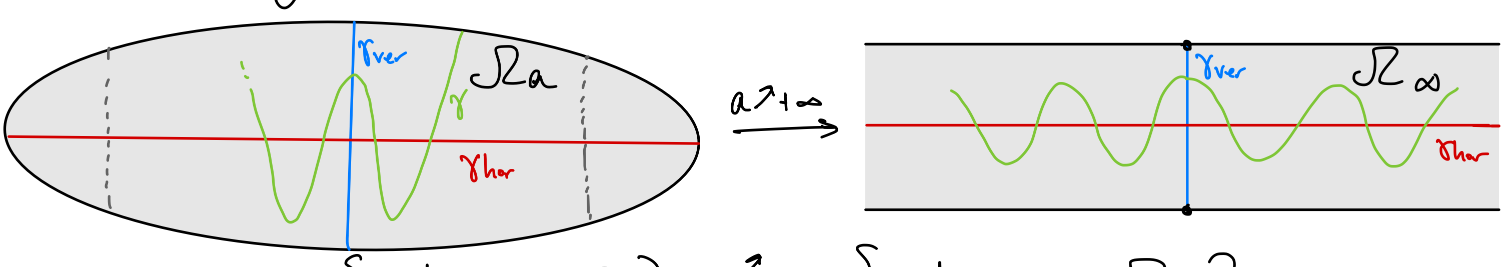
$B_m \cap B_n = \emptyset$ if $m \neq n$. $\square$

Thus each $(B_m)_{m \geq 2}$ is <u>noncompact</u>, hence has points (a,s) for all $a > a_m$.

$\begin{array}{c}(a,s) \in B_m \\ a > a_m\end{array} \rightsquigarrow S_m(a) \subset E(a,b,b,d)$ nonplanar minimal 2-sphere.

$\#\big(S_m(a) \cap \Pi_4(a)\big) = m \implies$ geometrically distinct. $\square$

Bonus: Convergence $S_m(a) \longrightarrow m \cdot \Pi_1(\infty)$ as $a \nearrow +\infty$



$\{\text{geodesics in } \Omega_a\} \xrightarrow{a \nearrow +\infty} \{\text{geodesics in } \Omega_\infty\}$

$\# \gamma \cap \gamma_{hor} = m$

geodesic γ in Ω_∞ is either:

→ periodic $(\# \gamma \cap \gamma_{hor} = +\infty)$ ✗

→ vertical segment ✓

$$\implies \begin{array}{c}\Omega_a \\ \cup \\ \gamma\end{array} \longrightarrow m \cdot \begin{array}{c}\Omega_\infty \\ \cup \\ \gamma_{ver}\end{array} \text{ as } a \nearrow +\infty \implies S_m(a) \longrightarrow m \cdot \Pi_1(\infty) \text{ as } a \nearrow +\infty. \quad \square$$