Introduction to Convex Algebraic Geometry

§1. Hierarchy of convex sets (in $\mathbb{R}^N$)

$$\left\{\begin{matrix}\text{convex}\\ \text{semialgebraic sets}\end{matrix}\right\} \supseteq \left\{\begin{matrix}\text{spectrahedral}\\ \text{shadows}\end{matrix}\right\} \supset \{\text{spectrahedra}\} \supset \{\text{polyhedra}\}$$

$\overset{?}{=}$ ← Helton–Nie Conjecture

- convex semialgebraic sets: Finitely many polynomial equalities and inequalities
- spectrahedral shadows: Linear projections of spectrahedra
- spectrahedra: "Semidefinite representation" "Linear matrix inequality"
- polyhedra: Intersection of finitely many half-spaces

Preserved under linear projections: ✓ (Tarski – Seidenberg Thm) ✓ ✗ ✓

Def: A (basic) <u>semialgebraic set</u> in $\mathbb{R}^N$ is a set of the form

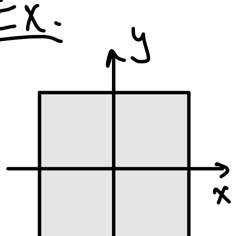
$$W(p_1, p_2, \dots, p_r) = \{x \in \mathbb{R}^N : p_1(x) \geq 0,\ p_2(x) \geq 0, \dots, p_r(x) \geq 0\}.$$

Def. A <u>spectrahedron</u> is a set $S \subset \mathbb{R}^N$ of the form

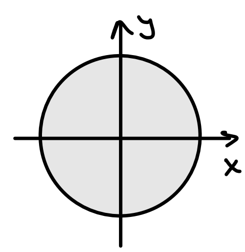
$$S = \left\{x \in \mathbb{R}^N : A_0 + \sum_{i=1}^{N} x_i A_i \succeq 0\right\} \text{ where } A_0, A_1, \dots, A_N \in \mathrm{Sym}^2(\mathbb{R}^d)$$

<u>Note</u>: If A_i are all <u>diagonal</u>, then S is a <u>polyhedron</u>. (Intersection of d half-spaces)

<u>Ex.</u>



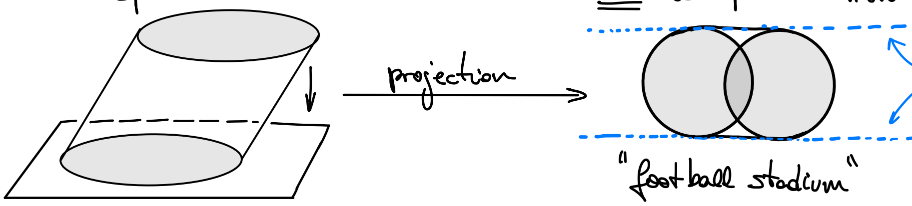
$$\begin{pmatrix} 1+x & & & \\ & 1-x & & \\ & & 1+y & \\ & & & 1-y \end{pmatrix} \succeq 0$$



$$\begin{pmatrix} 1+x & y \\ y & 1-x \end{pmatrix} \succeq 0$$

$$B^n: \begin{pmatrix} 1+x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & 1-x_1 & 0 & \cdots & 0 \\ x_3 & 0 & 1-x_1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ x_n & 0 & 0 & \cdots & 1-x_1 \end{pmatrix} \succeq 0$$

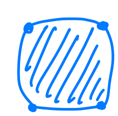
spectrahedron — <u>not</u> a spectrahedron



"football stadium"

Defining polynomial (as an algebraic interior) would vanish on the whole lines

<u>Another example:</u> "TV screen" $x^4 + y^4 \leq 1$



Def: A *spectrahedral shadow* is a set $S \subset \mathbb{R}^N$ of the form

$$S = \left\{ x \in \mathbb{R}^N : \exists y \in \mathbb{R}^M,\ A_0 + \sum_{i=1}^{N} x_i A_i + \sum_{j=1}^{M} y_j B_j \succeq 0 \right\}$$

where $A_i, B_j \in \mathrm{Sym}^2(\mathbb{R}^d)$.

Motivated by Nemirovski 2006 ICM plenary:

Conjecture (Helton – Nie, 2009). Every convex semialgebraic set $S \subset \mathbb{R}^N$ is a spectrahedral shadow

Thm (Scheiderer, 2018). The Helton - Nie Conjecture is

- TRUE if $N \leq 2$
- FALSE in general (counter-examples known for $N \geq 14$)

↳ Necessary and sufficient criterion for a convex semialgebraic set to be a spectrahedral shadow.

B. - Kummer - Mendes 2021: Refinement of Scheiderer's criterion / New examples.

§ 2. NONNEGATIVE v. SUM OF SQUARES

Q: $p(x) \in \mathbb{R}[x_1, \dots, x_N]_{2d}$ homogeneous polynomial of degree $2d$.

$$\boxed{p(x) \geq 0,\ \forall x \in \mathbb{R}^N} \quad \overset{?}{\Longleftrightarrow} \quad \boxed{p(x) = \sum q_i(x)^2, \text{ for some } q_i(x) \in \mathbb{R}[x_1, \dots, x_N]_d}$$

Example (Motzkin, 1967). $p(x,y,z) = z^6 + x^2y^4 + x^4y^2 - 3x^2y^2z^2$ $\begin{pmatrix} N=3 \\ 2d=6 \end{pmatrix}$

$p(x,y,z) \geq 0,\ \forall (x,y,z) \in \mathbb{R}^3$, but p is not a S.O.S.

↖ Arithmetic Mean ≥ Geometric Mean

↖ Newton polytope: $p(x,y,z) = (ax^2y + bxy^2 + cxyz + dz^3)^2$
Cannot have negative $x^2y^2z^2$ term.

Thm (Hilbert, 1893). Yes if and only if:

$$\underbrace{2d = 2,\ \forall N}_{\downarrow} \quad \text{or} \quad \underbrace{N \leq 2,\ \forall d}_{\downarrow} \quad \text{or} \quad N = 3,\ 2d = 4$$

"ternary quadrics"

quadratic polynomials ($\mathbb{R}[x_1, \ldots, x_N]_{2d} \cong \mathrm{Sym}^2(\mathbb{R}^N)$ spectral thm: symm $\Rightarrow$ diag.)

univariate (inhomogeneous) polynomials (2 squares suffice!)

$\rightsquigarrow$ Hilbert's 17^{th} problem:

sum of squares of rational functions (instead of polynomials)

$$\boxed{p(x) \geq 0,\ \forall x \in \mathbb{R}^N} \quad \overset{?}{\Longleftrightarrow} \quad \boxed{q(x)^2 p(x) = \sum q_i(x)^2, \text{ for some } q(x), q_i(x) \in \mathbb{R}[x_1, \ldots, x_N]}$$

~~**Def:** Given $p_1, \ldots, p_r \in \mathbb{R}[x_1, \ldots, x_N]$, let $\mathcal{P}(p_1, \ldots, p_r)$ be the smallest subset closed under $+$ and $\cdot$ containing $p_1, \ldots, p_r$ and all S.O.S.,~~

$$\mathcal{P}(p_1, \ldots, p_r) = \left\{ \sum_{\varepsilon \in \{0,1\}^r} \sigma_\varepsilon\, p_1^{\varepsilon_1} \cdots p_r^{\varepsilon_r} : \sigma_\varepsilon \in \mathbb{R}[x_1, \ldots, x_N] \text{ S.O.S.} \right\}.$$

"preordering"

SKIP!

~~**Thm** (Nichtnegativstellensatz). The following are equivalent for $p, p_1, \ldots, p_r \in \mathbb{R}[x_1, \ldots, x_N]$~~

(i) $p \geq 0$ on $W(p_1, \ldots, p_r) = \{x \in \mathbb{R}^N : p_1(x) \geq 0, \ldots, p_r(x) \geq 0\}$

(ii) $f \cdot p = p^{2\varepsilon} + g$ for some $f, g \in \mathcal{P}(p_1, \ldots, p_r)$, $f \neq 0$, $\varepsilon \in \mathbb{N}$.

In particular, $W(p_1, \ldots, p_r) = \emptyset \iff -1 \in \mathcal{P}(p_1, \ldots, p_r)$.

numeric computation w/ semidefinite program

symbolic computation w/ Gröbner basis

cf. Hilbert's Nullenstellensatz:

$$\{x \in \mathbb{R}^N : p_1(x) = \cdots = p_r(x) = 0\} = \emptyset \iff 1 \notin \langle p_1, \ldots, p_r \rangle \subset \mathbb{R}[x_1, \ldots, x_N]$$

~~Case $r=0$ (Artin, 1927): $\mathcal{P}(p_1,\dots,p_r)=\{\sigma_e\in\mathbb{R}[x_1,\dots,x_n]\ \text{s.o.s.}\}$~~

~~(i) $p\geq 0$ on $W(p_1,\dots,p_n)=\mathbb{R}^n$~~ SKIP

~~(ii) $q^2\cdot p = q_1^2+\dots+q_s^2$ for some $q,q_1,\dots,q_s\in\mathbb{R}[x_1,\dots,x_n]$, $q\neq 0$.~~

Affirmative answer to Hilbert's 17^{th} problem!

Local version (for quadratic forms) and connection to spectrahedral shadows:

X real projective variety ← $X\subset\mathbb{CP}^n$ real, irreducible, full, $X(\mathbb{R})$ Zariski-dense

$$\Sigma_X := \left\{p\in\mathbb{R}[X]_2 : p=\sum q_i^2,\ q_i\in\mathbb{R}[X]_1\right\}$$

← Always a spectrahedral shadow!

Image of PSD cone under PSD $\ni A\longmapsto \tilde{x}^{\top}A\tilde{x}\in\mathbb{R}[X]_2$

$\cap$

$$P_X := \left\{p\in\mathbb{R}[X]_2 : p(x)\geq 0\ \ \forall x\in X\right\}$$

← Value $p(x)$ not defined, but sign is b/c $f(\lambda\tilde{x})=\lambda^2 f(\tilde{x})$, $\forall\tilde{x}\in\mathbb{R}^{n+1}$ $\lambda\in\mathbb{R}\setminus\{0\}$ (representative)

$\boxed{P_X=\Sigma_X\ ?}$

$\Sigma_X\subsetneq P_X$ is first step to show P_X is not a spectrahedral shadow!

Finsler's Lemma (1936). If X is a quadric, then $P_X=\Sigma_X$.

Thm (Blekherman-Smith-Velasco, 2016). $P_X=\Sigma_X \iff X$ has minimal degree ($\deg X=\operatorname{codim} X+1$)

Example: $X=Gr_2(n)=\{2\text{-dim subspaces in } \mathbb{R}^n\}$

has minimal degree $\iff n\leq 4$ ← $\begin{cases}\deg Gr_2(n)=\dfrac{(2(n-2))!}{(n-2)!\,(n-1)!}\\ \operatorname{codim} Gr_2(n)=(n-2)(n-3)/2\end{cases}$

By [BSV'16]: $\Sigma_{Gr_2(n)}=P_{Gr_2(n)}\iff n\leq 4$.

Key input to show $P_{Gr_2(n)}$ is not a spectrahedral shadow is $P\in P_{Gr_2(n)}\setminus\Sigma_{Gr_2(n)}$:

based on Scheiderer 2018

Thm (B.–Kummer–Mendes, 2021). Let $L \subset \mathbb{R}[x_1,\dots,x_n]$ be a finite-dim vector space with $1 \in L$, and $f \in \mathbb{R}[x_1,\dots,x_n]$ s.t. $f \geq 0$ but f is not a S.O.S. Suppose that $\forall y \in \mathbb{R}^n$, the coeff. of $f^h(t, x_1-y_1,\dots,x_n-y_n)$, considered as polynomials in t, belong to L. Then

homogenization of $f \in \mathbb{R}[x_1,\dots,x_n]$
Unique homog. poly $f^h \in \mathbb{R}[t,x_1,\dots,x_n]$ w/ $\deg f = \deg f^h$
and $f^h(1,x_1,\dots,x_n) = f(x_1,\dots,x_n)$

$$K = \{ g \in L : g(x) \geq 0 \text{ for all } x \in \mathbb{R}^n \}$$

is <u>not</u> a spectrahedral shadow.

Cor. The following are equivalent:

(i) $P_{Gr_2(n)}$ is <u>not</u> a spectrahedral shadow

(ii) $P_{Gr_2(n)} \not\supseteq \Sigma_{Gr_2(n)}$

(iii) $n \geq 5$.

Remark: Same for $X = Gr_k(n)$ and $2 \leq k \leq n-2$.

Note: $P_{Gr_2(n)}$ is the dual cone to an <u>orbitope</u>, i.e., dual cone to the convex hull of a highest weight orbit of an $SO(n)$-representation:

$$SO(n) \curvearrowright Sym^2(\Lambda^k \mathbb{R}^n), \quad 2 \leq k \leq n-2.$$

On the other hand, for $SO(n) \curvearrowright Sym^2(\mathbb{R}^n)$, $Sym^{2d}(\mathbb{R}^2)$, $\Lambda^2\mathbb{R}^n$ these cones are even <u>spectrahedra</u> [Sanyal, Sottile, Sturmfels, 2011].

Bonus: Let $p \in \mathbb{R}[x_1,\dots,x_n]$, $e \in \mathbb{R}^n$.

Geom: any real line through e intersects cplx surface $p=0$ in only real points.

- p is <u>hyperbolic</u> w.r.t. e if $p(e) \neq 0$ and $\forall v \in \mathbb{R}^n$, the polynomial $p_v(t) = p(e+tv) \in \mathbb{R}[t]$ has only real roots.
- In this case, $\mathcal{H}_e(p) = \{ a \in \mathbb{R}^n : \forall \lambda \in [0,1],\ p(\lambda e + (1-\lambda)a) \neq 0 \}$ is the <u>hyperbolicity region</u> of h w.r.t. e. (Always convex semialgebraic!)

Fact: Every spectrahedron is the hyperbolicity region of a hyp. poly.

<u>Geometric Lax Conjecture</u>: Every hyperbolicity region is a spectrahedron.

Applications of Convex Algebraic Geometry to Geometric Analysis

(M^n,g) Riem. mfld., $p\in M$, $R:\Lambda^2T_pM\longrightarrow\Lambda^2T_pM$ symmetric endom.

$$\langle R(X\wedge Y),Z\wedge W\rangle=\langle R(X,Y)Z,W\rangle$$

<u>Bianchi map:</u> $b:\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n)\longrightarrow\Lambda^4\mathbb{R}^n\subset\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n)$

$$R\longmapsto b(R)(X,Y,Z,W)=\tfrac{1}{3}\Big(\langle R(X,Y)Z,W\rangle+\langle R(Y,Z)X,W\rangle+\langle R(Z,X)Y,W\rangle\Big)$$

$\Lambda^4\mathbb{R}^n\hookrightarrow\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n)$, $\langle\omega(\alpha),\beta\rangle=\langle\omega,\alpha\wedge\beta\rangle$, $\forall\alpha,\beta\in\Lambda^2\mathbb{R}^n$.

$$\begin{tikzcd}
0 \arrow[r, hook] & \Lambda^4\mathbb{R}^n \arrow[r, shift left] \arrow[d, phantom, "\shortparallel"] & \mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \arrow[l, shift left, "b"] \arrow[r, "\pi"] & \boxed{\mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^n)} \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\
& \mathrm{Im}\,b & & \ker b &
\end{tikzcd}$$

"Algebraic curvature operators"

<u>Sectional curvature:</u>

$$\mathrm{Gr}_2(\mathbb{R}^n)=\{\sigma\in\Lambda^2\mathbb{R}^n:\ \underbrace{\sigma\wedge\sigma=0}_{\Leftrightarrow\ \langle\omega(\sigma),\sigma\rangle=0\ \forall\omega\in\Lambda^4\mathbb{R}^n},\ |\sigma|=1\}\quad\text{Grassmannian}$$

$$\sec_R:\mathrm{Gr}_2(\mathbb{R}^n)\longrightarrow\mathbb{R}$$
$$\sigma\longmapsto\langle R(\sigma),\sigma\rangle$$

$\forall\sigma\in\mathrm{Gr}_2\mathbb{R}^n$ $\sec_R(\sigma)=\langle R\sigma,\sigma\rangle\geq0$

<u>Def:</u> $\mathcal{R}_{\sec\geq0}(n):=\{R\in\mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^n):\ \sec_R\geq0\}$
Analogously for $\mathcal{R}_{\sec\geq k}(n)$ and $\mathcal{R}_{\sec\leq k}(n)$.

As an application of Tarski's Quantifier Elimination:

<u>Thm</u> (A. Weinstein, 1971). $\mathcal{R}_{\sec\geq0}(n)$ is a convex semialgebraic set.

Letting $X=\mathrm{Gr}_2\mathbb{R}^n\subset\Lambda^2\mathbb{R}^n$, we have $\mathcal{R}_{\sec\geq0}(n)=P_X=\{p\in\mathbb{R}[X]_2: p(x)\geq0,\forall x\in X\}$.

$$\begin{tikzcd}
\Lambda^4\mathbb{R}^n \arrow[r, hook] \arrow[d, phantom, "\shortparallel"] & \mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \arrow[r, "\pi"] \arrow[d, phantom, "\shortparallel"] & \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^n) \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\shortparallel"] & \mathcal{R}_{\sec\geq0}(n) \arrow[d, phantom, "\shortparallel"] & & \\
I_2 \arrow[r] & \mathbb{R}[x_{ij}]_2 \arrow[r] & \mathbb{R}[X]_2 \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\shortparallel"] & P_X \arrow[r, phantom, "\supset"] & \Sigma_X & \text{S.O.S.} \\
& & \mathbb{R}[X]_2/I_2 & & &
\end{tikzcd}$$

b/c: $\mathrm{Gr}_2\mathbb{R}^n=\{\sigma\in\Lambda^2\mathbb{R}^n:\ \sigma\wedge\sigma=0,\ |\sigma|=1\}$
$\sigma\wedge\sigma=0\iff\langle\omega(\sigma),\sigma\rangle=0\quad\forall\omega\in\Lambda^4\mathbb{R}^n$

"strongly positive curvature"

Thm. (B.–Kummer–Mendes, 2021). The set $\mathcal{R}_{\sec\geq 0}(n)$ is
(i) not a spectrahedral shadow, if $n \geq 5$
(ii) a spectrahedral shadow, but not a spectrahedron, if $n=4$
(iii) a spectrahedron, if $n \leq 3$.

Sketch: (i) Refinement of Scheiderer's criterion for $X = \mathrm{Gr}_2\mathbb{R}^n$:

$$\boxed{P_{\mathrm{Gr}_2(n)} = \Sigma_{\mathrm{Gr}_2(n)} \iff P_{\mathrm{Gr}_2(n)} \text{ is a spectrahedral shadow.}}$$

and $P_{\mathrm{Gr}_2(n)} \supsetneq \Sigma_{\mathrm{Gr}_2(n)}$ if $n \geq 5$, from Part 1 (yesterday).

(ii) Finsler's Lemma: $X = \mathrm{Gr}_2(4) \implies \mathcal{R}_{\sec\geq 0}(4) = P_{\mathrm{Gr}_2(4)} = \Sigma_{\mathrm{Gr}_2(4)}$
X is a quadric/has minimal degree
is a spectrahedral shadow.

(iii) $\sec_R \geq 0 \iff R \succeq 0$ if $n \leq 3$.

$\Pi : \mathrm{Sym}^2(\Lambda^2\mathbb{R}^4) \to \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^4)$
orthogonal projection ($\ker \Pi = \mathrm{span}\, * \cong \Lambda^4\mathbb{R}^4$).

DIMENSION $n=4$.

Finsler–Thorpe Trick. $\underbrace{\mathcal{R}_{\sec\geq 0}(4)}_{\text{spectrahedral shadow}} = \Pi\Big(\underbrace{\{R \in \mathrm{Sym}^2(\Lambda^2\mathbb{R}^4) : R \succeq 0\}}_{\text{spectrahedron}}\Big)$

$$= \{R \in \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^4) : \exists\, a \in \mathbb{R},\ R + a\underset{\text{Hodge star}}{*} \succeq 0\}$$

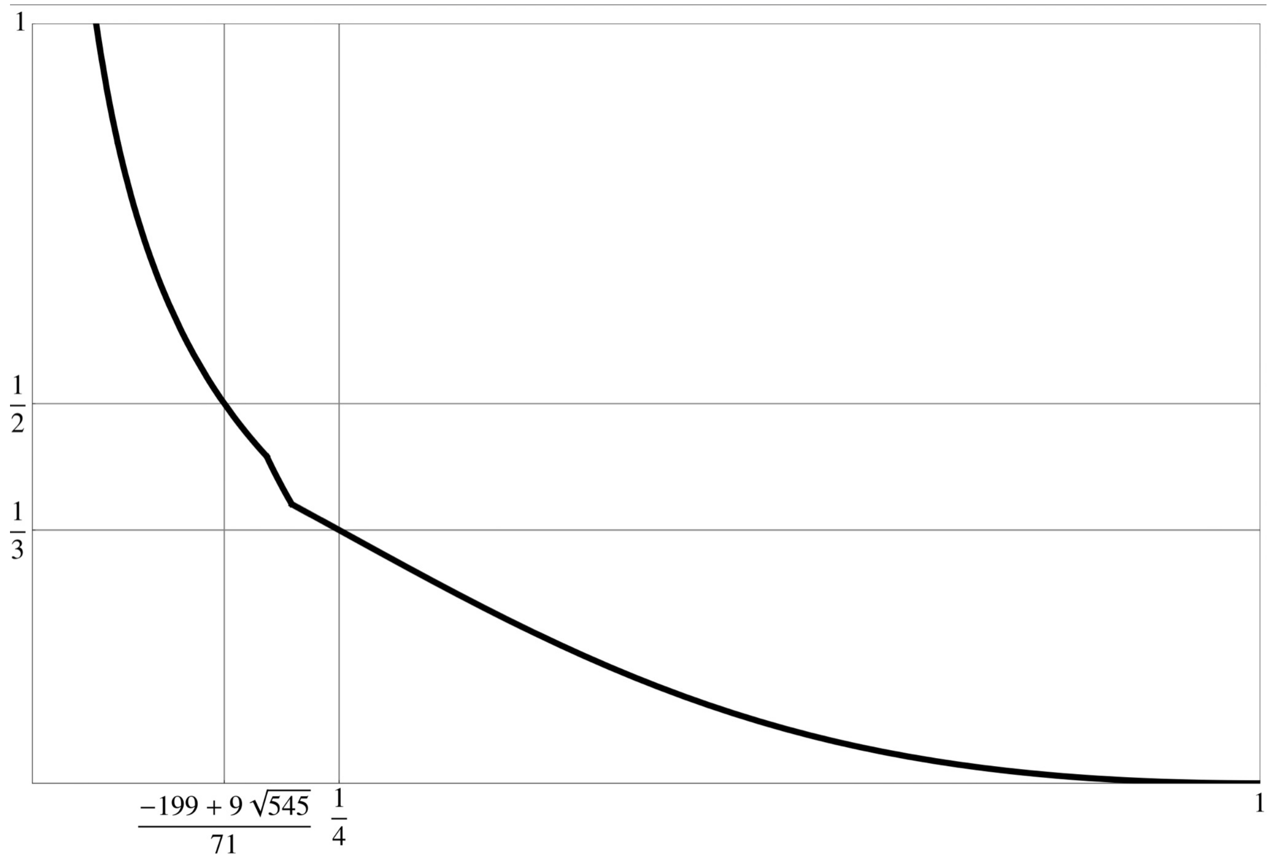
Cor. (M^4, g) has $\sec \geq 0 \iff \exists f : M \to \mathbb{R},\ R + f* \succeq 0$.

Applications: Area-extremality/rigidity à la Gromov, evolution of $\sec \geq 0$ under Ricci flow...

Thm (B.–Kummer–Mendes, 2021). If (M^4, g) is oriented and has $\delta \leq \sec \leq 1$ or $-1 \leq \sec \leq -\delta$ and finite volume, then

$$|\sigma(M^4)| \leq \lambda(\delta) \cdot \chi(M^4)$$

here $\lambda : (0,1] \to [0, +\infty)$ is an *explicit* function of δ.



Notable values: $\lambda\left(\frac{1}{4}\right)=\frac{1}{3}$ [Ville, 80's], $\lambda\left(\frac{1}{1+3\sqrt{3}}\right)<\frac{1}{2}$, $\lambda(1)=0$.

Sharp: $\mathbb{CP}^2$, $\mathbb{CH}^2/\Gamma$

$\sigma(M)\neq 0$ and $\delta\nearrow 1$ (sec ≈ -1) $\Rightarrow \chi(M)\nearrow +\infty$

Sketch: Let $\varphi_\lambda(R)=\lambda\cdot\underbrace{\chi(R)}_{\text{Chern-Gauss-Bonnet integrand}}-\underbrace{\sigma(R)}_{\text{Signature integrand}}$, so that $\int_M \varphi_\lambda(R)=\lambda\chi(M)-\sigma(M)$.

Optimize $\varphi_\lambda:\underbrace{\mathcal{R}_{\delta\le\sec\le 1}(4)}_{\text{Spectrahedral shadow}}\longrightarrow\mathbb{R}$ to get $\Omega=\left\{(\delta,\lambda):\min_{R\in\mathcal{R}_{\delta\le\sec\le1}(4)}\varphi_\lambda\ge 0\right\}$

Use cylindrical algebraic decomposition to write $\Omega=\{(\delta,\lambda):\lambda\geq\underline{\lambda(\delta)}\}$. $\square$

Q: Which simply-connected (M^4,g) have $\sec>0$?

Widely conjectured answer: $M^4\cong S^4$, or $\mathbb{CP}^2$.

Hopf Question (1932): Does $S^2\times S^2$ have $\sec>0$?

previously known for slightly stronger pinching...

Cor: If (M^4,g) is simply-connected and $\underbrace{\frac{1}{1+3\sqrt{3}}}_{0.161\ldots}\le\sec\le 1$,

then $M^4\underset{\text{homeo}}{\cong}S^4$ or $\mathbb{CP}^2$.

Pf: By [Diógenes-Ribeiro, 2019], M^4 has definite intersection form:

$$b_2(M) = b_+(M) + \underbrace{b_-(M)}_{=0}.$$

so $\sigma(M) = b_+(M)$, and $\chi(M) = 2 + b_+(M)$.

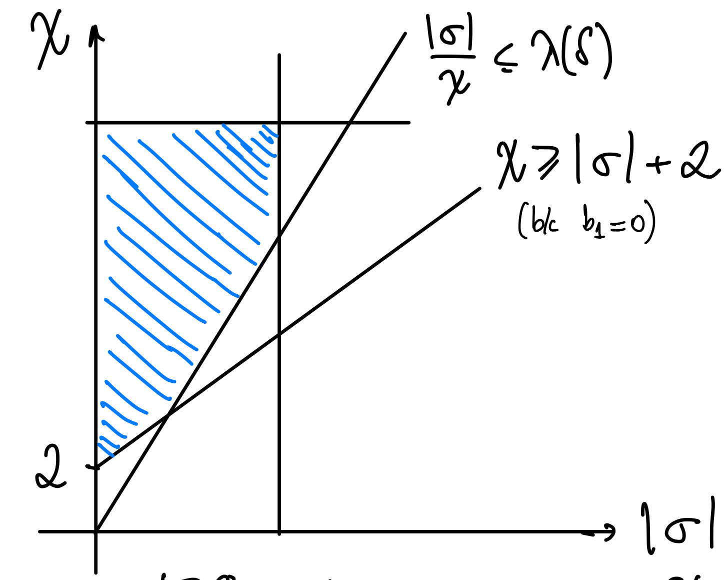
By Thm, $|\sigma| \leq \lambda\left(\frac{1}{1+3\sqrt{3}}\right)\chi < \frac{1}{2}\chi$ hence $|b_+| < 1 + \frac{1}{2}b_+$ so $b_+ \leq 1$.

Donaldson-Freedman: $b_+ = 0 \Rightarrow M \cong S^4$, $b_+ = 1 \Rightarrow M \cong \mathbb{CP}^2$. $\square$

"Geography of 4-manifolds": Which $(\sigma, \chi) \in \mathbb{Z}^2$ are realized?

Thm (BKM). If (M^4, g) is oriented, $\delta \leq \sec \leq 1$, then either $M \underset{\text{diff}}{\cong} S^4$ or $\chi(M) \leq \frac{8}{9}\left(\frac{1}{\delta} - 1\right)^2$ and $|\sigma(M)| \leq \frac{8}{27}\left(\frac{1}{\delta} - 1\right)^2$.

improves bounds of [Berger, 1962]



Cor: For each $\delta > 0$, there is an explicit finite list of homeom. types for M^4 with $\delta \leq \sec \leq 1$.

Gromov '78: $\forall D > 0$, $V > 0$, $\exists\, \delta(D, V) \in (0,1)$ s.t. $-1 \leq \sec \leq -\delta$, $\sigma(M) \neq 0$ $\Rightarrow \begin{cases} \mathrm{Vol}(M) \geq V \\ \mathrm{diam}(M) \geq D. \end{cases}$

Quantified version:

Thm (BKM). If (M^4, g) is oriented, $-1 \leq \sec \leq -\delta$, and has finite volume, then

$$\chi(M) \leq \frac{3}{4\pi^2}\mathrm{Vol}(M, g) \quad \text{and} \quad |\sigma(M)| \leq \frac{2}{9\pi^2}(1-\delta)^2\,\mathrm{Vol}(M, g)$$

$= \Leftrightarrow (M, g)$ is hyperbolic

"L^2-signature" if M noncompact.

Using Bishop Volume Comparison, can replace $\mathrm{Vol}(M, g)$ with $\mathrm{diam}(M, g)$.

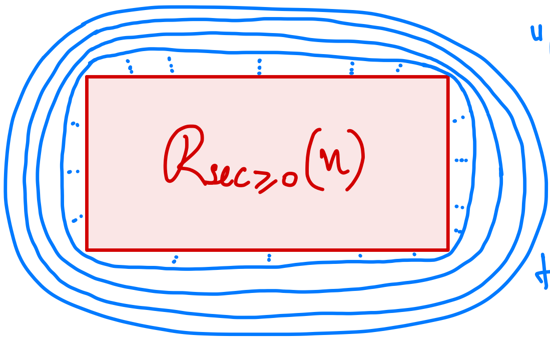
DIMENSIONS $n \geq 5$

"No Finsler–Thorpe trick, but..." ...convex algebro-geometric point of view is still fruitful.

Thm (B.–Mendes, 2017)

$\Delta_L = \nabla^*\nabla + \mathcal{K}(R, \mathrm{Sym}^p_0(\mathbb{R}^n))$

$$\mathcal{R}_{\sec\geq 0}(n) = \bigcap_{p\geq 2} \underbrace{\left\{R\in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^n) : \mathcal{K}(R,\mathrm{Sym}^p_0(\mathbb{R}^n))\succeq 0\right\}}_{\text{these are spectrahedra, for each } p\geq 2}$$



"Relaxation" by spectrahedra

Prove thm's by taking limits as $p\nearrow+\infty$.

(M^n,g), $\sec\geq 0$, compact

$\Rightarrow \underbrace{\exists N\in\mathbb{N}}_{\text{likely impossible to compute!}}$ s.t. $\forall x\in M$, $R_x\in\bigcap_{p=2}^{N}\dots$

Or switch to other curvature conditions... e.g.:

Def: $R\in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^n)$ is <u>k-positive</u> if $\lambda_1+\dots+\lambda_k>0$

$$\begin{array}{ccccccccc}
k=1 & \Rightarrow & \dots & \Rightarrow & k=n-1 & \Rightarrow & \dots & \Rightarrow & k=\binom{n}{2} \\
\Updownarrow & & \xleftarrow{\text{stronger}} & & \Downarrow & & \xrightarrow{\text{weaker}} & & \Updownarrow \\
R>0 & \Rightarrow & \dots & \Rightarrow & \mathrm{Ric}>0 & \Rightarrow & \dots & \Rightarrow & \mathrm{scal}>0
\end{array}$$

Note: For all $1\leq k\leq\binom{n}{2}$, this defines a spectrahedron!

Thm (Petersen–Wink, 2021). (M^n,g) closed, with $(n-p)$-positive R, then $b_1(M)=\dots=b_p(M)=0$ and $b_{n-p}(M)=\dots=b_n(M)=0$.
In particular, if $\lceil\frac{n}{2}\rceil$-positive, then M^n is a rational homology sphere.

Thm (B.–Goodman, 2021). If (M^{2m},g) is closed and spin, with k-positive R where $k\leq\frac{m(2m+7)}{m+8}$, and $\frac{\mathrm{scal}}{8}-\mathrm{Ric}\succeq 0$, then: $\underbrace{\langle\hat{A}(TM)\cdot\mathrm{ch}(TM_{\mathbb{C}}),[M]\rangle}_{\text{Elliptic genus associated to } \not{D}_{TM}:\, \mathcal{S}\otimes TM\to\mathcal{S}\otimes TM}=0$.

Cor: If (M^8,g) is spin, Einstein, and has 5-positive R, then M^8 is null-cobordant: $\hat{A}(M^8)=0$ and $\sigma(M^8)=0$.

In particular, $\mathbb{HP}^2$ does not have an Einstein metric w/ 5-positive R.