Geography of pinched 4-manifolds

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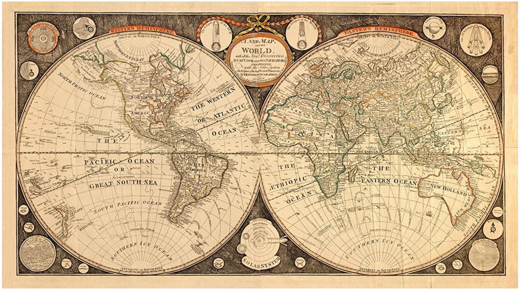

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“Classical” Geography Problem

Given a group G, which points $(\sigma(M), \chi(M)) \in \mathbb{Z}^2$ can be realized by a 4-manifold with $\pi_1(M) \cong G$?

Definition

(M^n, g) is δ-pinched if either

- $\delta \leq \mathrm{sec}_M \leq 1$,
- $-1 \leq \mathrm{sec}_M \leq -\delta$.

Map by Nicolaes Visscher II, ca. 1689

Pinched Geography Problem

Which $(\sigma, \chi) \in \mathbb{Z}^2$ are realized by δ-pinched 4-manifolds?

Topology of simply-connected 4-manifolds

- M^4 closed, $\pi_1(M) = \{1\} \implies$ $b_1(M) = b_3(M) = 0$; $\boxed{\begin{array}{l}\text{All information in intersection form} \\ Q_M \colon H_2(M) \times H_2(M) \longrightarrow \mathbb{Z}\end{array}}$

- Q_M has $b_+(M)$ positive eigenvalues
  $b_-(M)$ negative eigenvalues

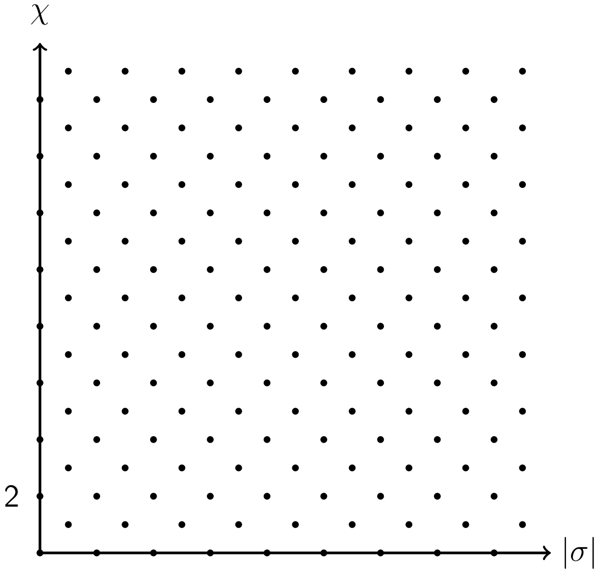
  $b_2(M) = \operatorname{rank} Q_M = b_+(M) + b_-(M)$

  $$\boxed{\chi(M) = \underline{2 + b_+(M) + b_-(M)}, \quad \sigma(M) = \underline{b_+(M) - b_-(M)}}$$

- Thus: $\underline{\chi(M) \equiv \sigma(M) \mod 2}$, and $\underline{\chi(M) \geq |\sigma(M)| + 2}$.

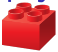
- If M is smooth, by Hodge Theory:

$$b_\pm(M) = \dim \left\{\alpha \in \Omega^2(M) : \Delta\alpha = 0, *\alpha = \pm\alpha\right\}$$



χ
2
$|\sigma|$

Simply-connected building blocks



$\mathbb{C}P^2$

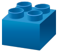
$b_+ = 1$

$b_- = 0$

$\overline{\mathbb{C}P^2}$

$b_+ = 0$

$b_- = 1$



$S^2 \times S^2$

$b_+ = 1$

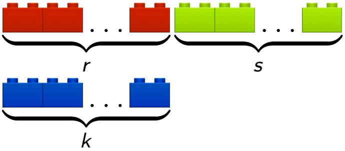
$b_- = 1$

By Freedman, Donaldson, Atiyah–Singer, Hirzebruch, ...

Theorem

If (M^4, g) *is closed,* $\pi_1(M) = \{1\}$, *and* $\underline{\quad \text{scal} > 0 \quad}$ *then*

$$M^4 \cong_{homeo} \begin{cases} \underline{\#^r \mathbb{C}P^2 \#^s \overline{\mathbb{C}P^2}} \\ \underline{\#^k (S^2 \times S^2)} \end{cases}$$



Conversely, any connected sum of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$, $S^2 \times S^2$ has:

- Gromov–Lawson 1980; Schoen–Yau 1979: $\underline{\text{scal} > 0}$
- Sha–Yang 1993; Perelman 1997: $\underline{\text{Ric} > 0}$

Geography Problem for scal > 0 and Ric > 0

Any (σ, χ) *with* $\chi \equiv \sigma \mod 2$ *and* $\chi \geq |\sigma| + 2$ *is realized:*
$M = \#^r \mathbb{C}P^2 \#^s \overline{\mathbb{C}P^2}$, $r = \frac{\chi+\sigma-2}{2}$, $s = \frac{\chi-\sigma-2}{2}$.

Conjecture (Folklore)

(M^4, g) *closed,* $\pi_1(M) = \{1\}$, $\mathrm{sec}_M > 0$ $\implies M^4 \cong_{\textit{diffeo}} \underline{\;S^4 \text{ or } \mathbb{C}P^2\;}$.

Question

How small can we make $\delta > 0$ *and prove:*

(M^4, g) *closed,* $\pi_1(M) = \{1\}$, *positively* δ*-pinched* $\implies M^4 \cong_{\substack{\textit{diffeo?}\\ \textit{homeo?}}} \underline{\;S^4 \text{ or } \mathbb{C}P^2\;}$.

Homeomorphism Question has 2 parts

$\underbrace{\boxed{b_- = 0}}_{\textit{definite}}$ & $\underbrace{\boxed{2 - b_+ = \chi - 2\sigma > 0}}_{\textit{geography problem}} \Rightarrow \boxed{M^4 \cong_{\textit{homeo}} \underline{S^4 \text{ or } \mathbb{C}P^2}}$

$0 \quad \ldots \quad \delta$

Suppose (M^4, g) has $\pi_1(M) = \{1\}$ and is positively δ-pinched.

Theorem (Berger 1960, Klingenberg 1961)

$\delta > \frac{1}{4} \implies M \cong_{homeo} S^4$

Theorem (Brendle–Schoen, 2009)

$\delta > \frac{1}{4} \implies M \cong_{diffeo} S^4$

Theorem (Berger 1983; Petersen–Tao, 2009)

$\exists \varepsilon > 0, \quad \delta > \frac{1}{4} - \varepsilon \implies M \cong_{diffeo} S^4$ *or* $\mathbb{C}P^2$

Theorem (Ville, 1989)

$\delta \geq \frac{4}{19} \cong \underline{\ 0.2105\ } \implies M \cong_{homeo} S^4$ *or* $\mathbb{C}P^2$.

Theorem (Seaman, 1989)

$\delta \geq \frac{1}{3\sqrt{1+(2^{5/4}/5^{1/2})+1}} \cong \underline{\ 0.1883\ } \implies M \cong_{homeo} S^4$ *or* $\mathbb{C}P^2$.

0 .... δ

0 .... δ

0 .... δ

0 .. δ

Theorem A (B., Kummer, Mendes)

If (M^4, g) *has* $\pi_1(M) = \{1\}$ *and is positively* δ*-pinched,*

$$\delta \geq \frac{1}{1+3\sqrt{3}} \cong \underline{\ 0.16139\ },$$

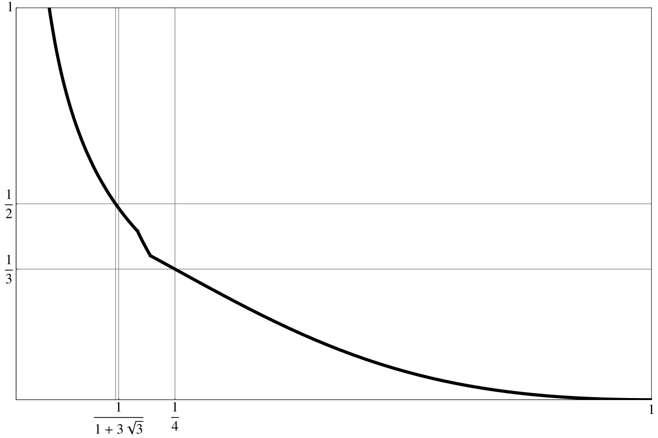
then $M \cong_{homeo} S^4$ *or* $\mathbb{C}P^2$.


MATHEMATICIANS ARE WEIRD
YOU KNOW THAT THING THAT WAS 2.3728642?
YES?
I GOT IT DOWN TO 2.3728639.
Thunderous applause
smbc-comics.com

Actually, there's much more behind it...

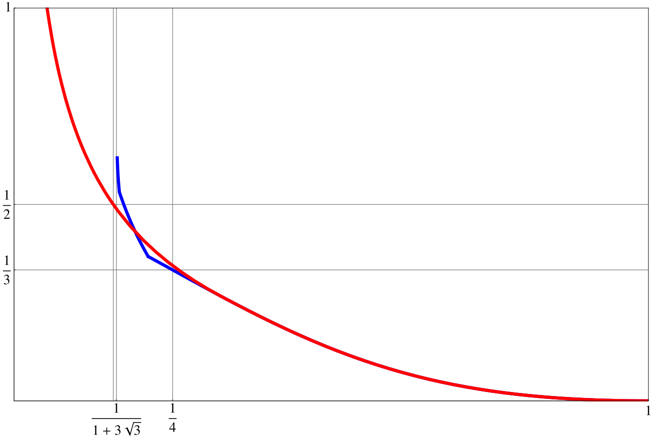
Theorem B (B., Kummer, Mendes)

For all $\delta > 0$, if (M^4, g) is closed, oriented, and δ-pinched, then $|\sigma(M)| \leq \lambda(\delta)\,\chi(M)$, where $\lambda\colon (0,1] \to \mathbb{R}$ is an **explicit function**.



- $\lim_{\delta \searrow 0} \lambda(\delta) = +\infty$
- $\lambda(1) = 0$
  cf. $\sigma(M^4_\kappa) = 0$
- $\lambda\left(\frac{1}{1+3\sqrt{3}}\right) < \frac{1}{2}$
  cf. Thm A
- $\lambda\left(\frac{1}{4}\right) = \frac{1}{3}$
  [Ville, 1985]

$$\lambda(\delta) = \begin{cases} \dfrac{\sqrt{\frac{24}{\delta}+8-8\delta+\delta^2}+\delta-4}{6(3-\delta)}, & \text{if } 0 < \delta < 0.069, \\ \dfrac{4}{3\sqrt{15}}\dfrac{1-\delta}{\sqrt{\delta(\delta+2)}}, & \text{if } 0.069 \le \delta < 0.191, \\ \dfrac{26\delta^2+8\delta+2-2\sqrt{3}\sqrt{55\delta^4+40\delta^3+6\delta^2+8\delta-1}}{3(1-\delta)^2}, & \text{if } 0.191 \le \delta \le 0.211, \\ \dfrac{8(1-\delta)^2}{24\delta^2-12\delta+15}, & \text{if } 0.211 \le \delta \le 1. \end{cases}$$



Different methods:

[BKM, 2021]

Any $0 < \delta \le 1$

[Ville, 1985, 1989]

$\delta = \frac{1}{4}, \quad \delta = \frac{4}{19}$

$\rightsquigarrow 0.163 \le \delta \le 1$

Positive δ-pinching

$|\sigma(M)| \leq \lambda(\delta)\,\chi(M)$

gives new information only if $\lambda^{-1}(1) \cong 0.052 < \delta < \frac{1}{4} - \varepsilon$.

Theorem C (BKM)

If (M^4, g) is positively δ-pinched, oriented, then $M \cong_{diffeo} S^4$ or

- $\chi(M) \leq \underline{\frac{8}{9}\left(\frac{1}{\delta} - 1\right)^2}$,
- $|\sigma(M)| \leq \underline{\frac{8}{27}\left(\frac{1}{\delta} - 1\right)^2}$.

Gromov: $\chi(M) \leq \underline{10^{1440}}$.

Corollary

Explicit list *of homeotypes for positively δ-pinched 4-manifolds.*

Negative δ-pinching

$|\sigma(M)| \leq \lambda(\delta)\,\chi(M)$

gives new information for all $0 < \delta \leq 1$.

$\rightsquigarrow b_1 \leq 1 + \frac{\lambda-1}{2\lambda} b_+ + \frac{\lambda+1}{2\lambda} b_-$.

Theorem D (BKM)

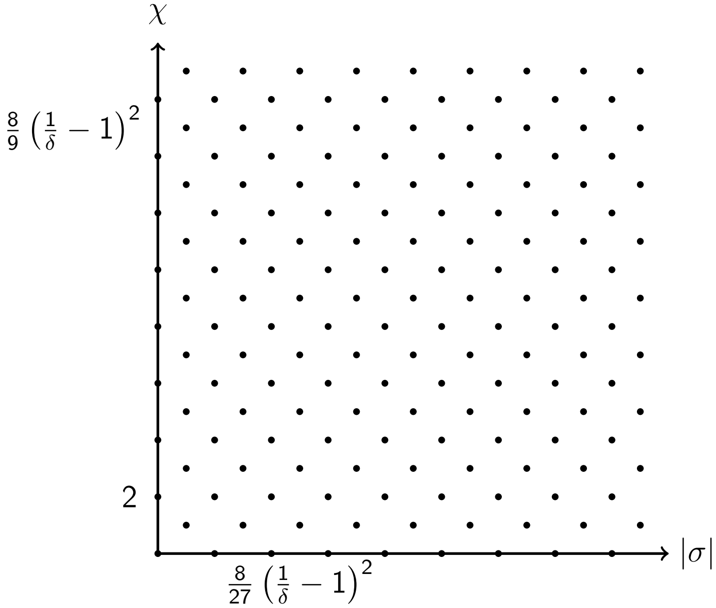
If (M^4, g) is negatively δ-pinched, closed, oriented, then:

- $\chi(M) \leq \underline{\frac{3}{4\pi^2}\mathrm{Vol}(M, \mathrm{g})}$,
- $|\sigma(M)| \leq \underline{\frac{2(1-\delta)^2}{9\pi^2}\mathrm{Vol}(M, \mathrm{g})}$.

Corollary

Quantify *Gromov's volume and diameter bounds for negatively δ-pinched M with $\sigma(M) \neq 0$.*

Bounding the geography of δ-pinched 4-manifolds



Curvature operator

Eigenspaces of Hodge star $*$

$$\wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM$$

Curvature operator canonical form, using $\mathrm{O}(4) \curvearrowright \wedge^2 TM$

$$R\colon \ \wedge^2 \ TM \longrightarrow \wedge^2 TM$$

$$R = \begin{pmatrix} u\mathrm{Id} + W_+ & C \\ C^t & u\mathrm{Id} + W_- \end{pmatrix} \in \mathrm{Sym}^2_b(\wedge^2 TM)$$

Scalar curvature: $u = \frac{1}{12}\,\mathrm{scal}$

Weyl tensor: $W_\pm = \mathrm{diag}(w_1^\pm, w_2^\pm, w_3^\pm)$

Traceless Ricci: $C = \mathring{\mathrm{Ric}}$

R <u>diagonal</u> $\iff$ $C =$ <u>0</u> $\iff$ <u>M is Einstein</u>

Chern–Gauss–Bonnet, Hirzebruch

Integral formulas for topological invariants:

$$\chi(M) = \frac{1}{\pi^2} \int_M \underline{\chi}(R), \qquad \sigma(M) = \frac{1}{\pi^2} \int_M \underline{\sigma}(R).$$

Integrands are *indefinite* and SO(4)-*invariant* quadratic forms:

$$\underline{\chi}, \underline{\sigma} \colon \operatorname{Sym}^2_b(\wedge^2 \mathbb{R}^4) \longrightarrow \mathbb{R}$$

$$\underline{\chi}(R) = \tfrac{1}{8}\big(6u^2 + |W_+|^2 + |W_-|^2 - 2|C|^2\big)$$

$$\underline{\sigma}(R) = \tfrac{1}{12}\big(|W_+|^2 - |W_-|^2\big)$$

Since there are *no linear terms*:

$$\underline{\chi}(-R) = \underline{\chi}(R), \qquad \underline{\sigma}(-R) = \underline{\sigma}(R)$$

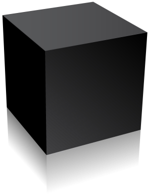
Blackbox optimization lemmas (more on this later)

Definition

$\Omega_\delta := \underline{\left\{ R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4) : \delta \leq \sec_R \leq 1 \right\}}$

Lemma A

If $\delta \geq \frac{1}{1+3\sqrt{3}}$, *then* $\min\limits_{R\in\Omega_\delta} \underline{\chi}(R) - 2\,\underline{\sigma}(R) > 0.$



Lemma B

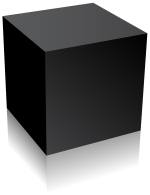
For all $\delta > 0$, *and* $t \in [-1,1]$,

$$\max_{R\in\Omega_\delta} |W_+|^2 + t\,|W_-|^2 = \tfrac{8}{3}(1-\delta)^2.$$

In particular,

$(t=+1)$ $\max\limits_{R\in\Omega_\delta} |W|^2 = \tfrac{8}{3}(1-\delta)^2,$

$(t=-1)$ $\max\limits_{R\in\Omega_\delta} \underline{\sigma}(R) = \tfrac{2}{9}(1-\delta)^2.$



Proof of Theorem A

Let (M^4, g) be simply-connected and positively δ-pinched,

$$\delta \geq \frac{1}{1+3\sqrt{3}} \cong 0.16139.$$

Theorem (Diógenes–Ribeiro, 2019)

M^4 is $\underline{\text{ definite }}$, i.e., $\underline{\ b_-(M) = 0\ }$.

Lemma A

$\chi(M) - 2\,\sigma(M) > 0.$

$$\underbrace{\boxed{b_- = 0}}_{\textit{definite}} \;\&\; \underbrace{\boxed{2 - b_+ = \chi - 2\sigma > 0}}_{\textit{geography problem}} \Rightarrow \boxed{M^4 \cong_{homeo} \underline{S^4 \text{ or } \mathbb{C}P^2}}$$

$\square$

Proof of Theorem C (Euler Characteristic)

Let (M^4, g) be positively δ-pinched and oriented.

Theorem (Chang–Gurksy–Yang, 2003)

If $\int_M |W|^2 < 4\pi^2\chi(M)$, *then* $\underline{M \cong_{diffeo} S^4}$.

Thus, if $\underline{M \not\cong_{\text{diffeo}} S^4}$, then:

$$\chi(M) \leq \frac{1}{4\pi^2}\int_M |W|^2$$

Lemma B

$$\leq \underline{\frac{1}{4\pi^2}\,\frac{8}{3}(1-\delta)^2\ \mathrm{Vol}(M)}$$

Bishop Volume Comparison
Diameter Sphere Theorem

$$\leq \underline{\frac{1}{4\pi^2}\,\frac{8}{3}(1-\delta)^2\ \mathrm{Vol}\left(S^4_+\left(\tfrac{1}{\sqrt{\delta}}\right)\right)}$$

$$\boxed{\mathrm{Vol}\left(S^4_+\left(\tfrac{1}{\sqrt{\delta}}\right)\right) = \frac{4\pi^2}{3\delta^2}}$$

$$= \frac{8}{9}\left(\frac{1}{\delta} - 1\right)^2.$$

Proof of Theorems C & D (Signature)

Let (M^4, g) be positively or negatively δ-pinched and oriented.

$$\sigma(M) = \frac{1}{\pi^2}\int_M \underline{\sigma}(R)$$

Lemma B

$$\leq \underline{\frac{1}{\pi^2}\,\frac{2}{9}(1-\delta)^2\ \mathrm{Vol}(M)}$$

– stop here if negatively pinched –

Bishop Volume Comparison
Diameter Sphere Theorem

$$\leq \underline{\frac{1}{\pi^2}\,\frac{2}{9}(1-\delta)^2\ \mathrm{Vol}\left(S^4_+\left(\tfrac{1}{\sqrt{\delta}}\right)\right)}$$

$$\boxed{\mathrm{Vol}\left(S^4_+\left(\tfrac{1}{\sqrt{\delta}}\right)\right) = \frac{4\pi^2}{3\delta^2}}$$

$$= \frac{8}{27}\left(\frac{1}{\delta}-1\right)^2. \qquad \square$$

Proof of Theorem D (Euler Characteristic)

Analogous: $\max\limits_{R\in\Omega_\delta} \underline{\chi}(R) \leq \max\limits_{R\in\Omega_\delta} \frac{1}{8}\left(6u^2 + |W_+|^2 + |W_-|^2\right) = \frac{3}{4}$.

Inside the blackbox

- $q\colon \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4) \longrightarrow \mathbb{R}$ quadratic form

$$q(R) = R^t \cdot A_q \cdot R + b_q \cdot R + c_q$$

- $\Omega_\delta \subset \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ is a compact *spectrahedral shadow* (Thorpe's trick / Finsler's Lemma)

$$R \in \Omega_\delta \iff \exists\, \alpha, \beta \in \mathbb{R}, \quad \begin{array}{l} R - \delta \mathrm{Id} + \alpha * \succeq 0 \\ \mathrm{Id} - R + \beta * \succeq 0 \end{array}$$

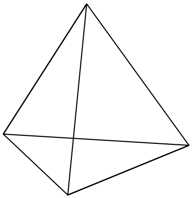
- If A_q is *indefinite*, "brute force" semidefinite programming on Ω_δ does not work, nor do other *convex* methods

$$\min_{R \in \Omega_\delta} q(R) = ? \qquad \max_{R \in \Omega_\delta} q(R) = ?$$

The Einstein simplex

Proposition

The set of δ-pinched Einstein curvature operators $E_\delta := \mathrm{Diag} \cap \Omega_\delta$ is a $\underline{\text{ 5-simplex }}$, and $\underline{\ \mathrm{proj}(\Omega_\delta) = E_\delta\ }$, where $\mathrm{proj}\colon \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4) \longrightarrow \mathrm{Diag}$.



E_δ

Proposition

The set of modified δ-pinched Einstein operators $\widetilde{E}_\delta := \big\{(R,\alpha) \in E_\delta \times \mathbb{R} : R - \delta\mathrm{Id} + \alpha * \succeq 0\big\}$ is a $\underline{\text{ 6-simplex }}$.

Vertices are *affine functions* of δ:

$$\begin{gathered} \delta R_{S^4} + \varepsilon(R_{\pm\mathbb{C}P^2} - R_{S^4}) \\ R_{S^4} + \varepsilon(R_{\pm\mathbb{C}H^2} + R_{S^4}), \quad \varepsilon \in \left\{0, \tfrac{1-\delta}{3}\right\} \end{gathered}$$

Quadratic optimization on simplices

Lemma

$\Delta^n = \mathrm{conv}(V) \subset \mathbb{R}^n$ *simplex,* $q\colon \mathbb{R}^n \to \mathbb{R}$ *quadratic form,* $A_q = \mathrm{Hess}\, q$

A_q <u>*indefinite*</u> $\Longrightarrow \max\limits_{\Delta^n} q = \max\limits_{\partial\Delta^n} q$ *(induction on n...)*

A_q <u>*positive-semidefinite*</u> $\Longrightarrow \max\limits_{\Delta^n} q = \max\limits_{V} q$

A_q <u>*negative-definite*</u> $\Longrightarrow$ *Use Calculus to find* $\max\limits_{\mathbb{R}^n} q$

Proof of Lemma B.

For all $\delta > 0$, and $t \in [-1, 1]$, since $\mathrm{proj}(\Omega_\delta) = E_\delta$,

$$\max_{R \in \Omega_\delta} |W_+|^2 + t\,|W_-|^2 \overset{\text{Prop}}{=} \max_{R \in E_\delta} \underbrace{|W_+|^2 + t\,|W_-|^2}_{q_t(R)}$$

$$\overset{\text{Lemma}}{=} \frac{8}{3}(1-\delta)^2.$$

Need to inspect faces of dimension $\leq \mathrm{ind}(A_{q_t}) = 2$ if $t < 0$. □

Integrand in Lemma A depends on $|C|^2$...

Lemma

Given $\lambda_i \geq 0$, $\mu_i \geq 0$, $C = (c_{ij})$,

$$\begin{pmatrix} \operatorname{diag}(\lambda_i) & C \\ C^t & \operatorname{diag}(\mu_i) \end{pmatrix} \succeq 0 \implies |C|^2 \leq \sum_{i=1}^{3} \lambda_i \mu_i .$$

Proof.

- Schur complements: $D = (d_{ij}) = (c_{ij}/\sqrt{\lambda_i \mu_j}) \in \underline{B_1^{\text{spec}}}$
- $|C|^2 = \sum_{ij} \lambda_i \mu_j d_{ij}^2$ is maximal if $D \in \underline{O(3) \subset B_1^{\text{spec}}}$
- Birkhoff–von Neumann Theorem: $D_2 = (d_{ij}^2) \in \mathfrak{S}_3$ $\square$

Corollary

If $R \in \Omega_\delta$ *is such that* $R - \delta \mathrm{Id} + \alpha * \succeq 0$, *then*

$$|C|^2 \leq \sum_{i=1}^{3} (u - \delta + w_i^+ + \alpha)(u - \delta + w_i^- - \alpha)$$

$$= 3(u - \delta)^2 - 3\alpha^2 + \langle W_+, W_- \rangle .$$

Proof of Lemma A

- Let $\delta \geq \frac{1}{1+3\sqrt{3}}$, $R \in \Omega_\delta$, and $\alpha \in \mathbb{R}$ s.t. $R - \delta \mathrm{Id} + \alpha * \succeq 0$.
- Recall that $\mathrm{proj}(\Omega_\delta) = E_\delta$, hence $(\mathrm{proj}(R), \alpha) \in \widetilde{E}_\delta$.
- Use Corollary to eliminate $|C|^2$:

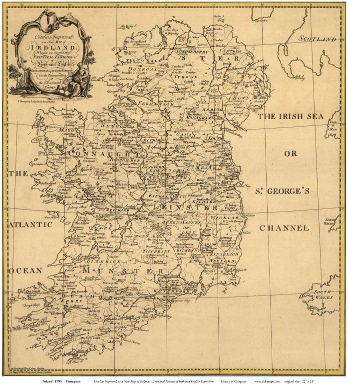
$$\begin{aligned} 8\big(\underline{\chi}(R) - 2\,\underline{\sigma}(R)\big) &= 6u^2 - \tfrac{1}{3}|W_+|^2 + \tfrac{7}{3}|W_-|^2 - 2|C|^2 \\ &\geq \underbrace{-\tfrac{1}{3}|W_+|^2 + \tfrac{7}{3}|W_-|^2 - 2\langle W_+, W_-\rangle + 6\alpha^2 + 12u\delta - 6\delta^2}_{q(\mathrm{proj}(R),\alpha)} \end{aligned}$$

$q \colon \widetilde{E}_\delta \longrightarrow \mathbb{R}$ indefinite quadratic form, $\mathrm{ind}(A_q) = 3$.

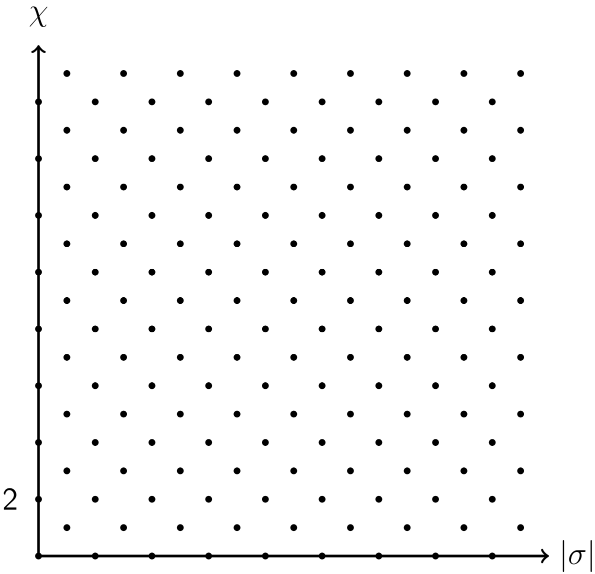
- Optimize q on the 6-simplex $\widetilde{E}_\delta$ as before, obtaining

$$8 \min_{R \in \Omega_\delta} \underline{\chi}(R) - 2\,\underline{\sigma}(R) \;\geq\; \min_{\widetilde{E}_\delta} q \;>\; 0.$$

□



Thank you for your attention!


χ
2
|σ|

Bonus: Hopf Conjecture

Conjecture (Hopf)

M^{2d} compact: $\pm\sec_M \geq 0 \implies \underline{(\pm 1)^d \chi(M) \geq 0}$

Algebraic version: $\pm\sec_R \geq 0 \implies \underline{\underline{\chi}(R) \geq 0}$

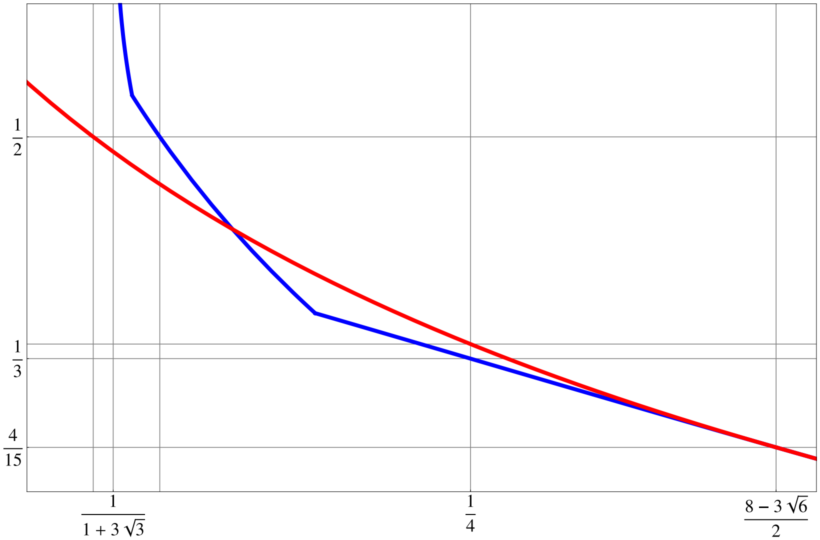
- **True** if $2d = 4$ (Milnor / Chern, 1955)
- **False** if $2d \geq 6$ (Geroch, 1976)

Proof ($2d = 4$).

If $\pm\sec_R \geq 0$, then $\pm R + \alpha * \succeq 0$ for some $\alpha \in \mathbb{R}$. Thus:

$$\begin{aligned} 8\underline{\chi}(\pm R) &= 6u^2 + |W_+|^2 + |W_-|^2 - 2|C|^2 \\ \text{(Corollary)} \quad &\geq 6\alpha^2 + |W_+|^2 + |W_-|^2 - 2\langle W_+, W_- \rangle \\ &= 6\alpha^2 + \left| W_+ - W_- \right|^2 \\ &\geq 0. \end{aligned}$$

□



1/2
1/3
4/15
1/(1 + 3√3)
1/4
(8 − 3√6)/2

Vertices of Einstein simplices E_δ and $\widetilde{E}_\delta$

The Einstein simplex $E_\delta \subset \mathbb{R}^5$ is the convex hull of the rows

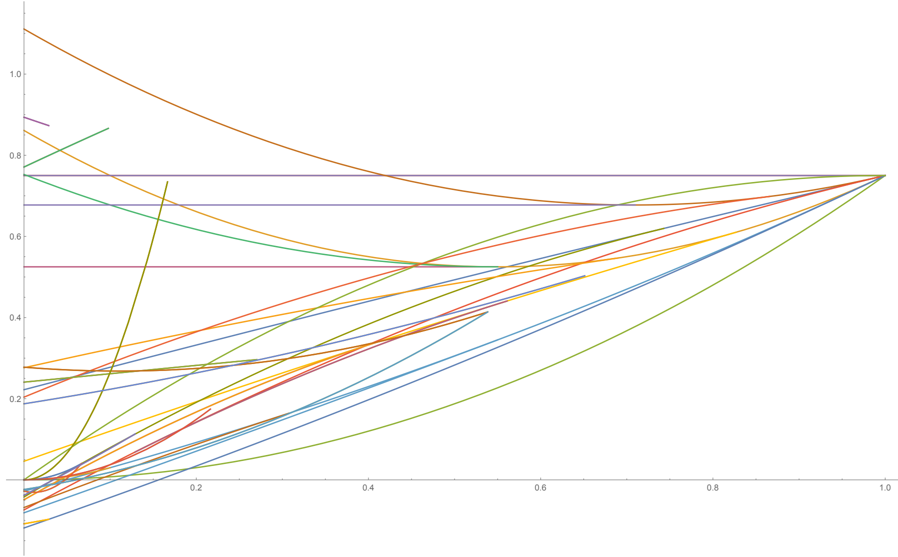
w_1^+	w_2^+	w_3^+	w_1^-	w_2^-	w_3^-	u
$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{4}{3}-\frac{4}{3}\delta$	0	0	0	$\frac{2}{3}\delta+\frac{1}{3}$
$\frac{4}{3}\delta-\frac{4}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$\frac{2}{3}-\frac{2}{3}\delta$	0	0	0	$\frac{1}{3}\delta+\frac{2}{3}$
0	0	0	$\frac{4}{3}\delta-\frac{4}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$\frac{1}{3}\delta+\frac{2}{3}$
0	0	0	$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{2}{3}\delta-\frac{2}{3}$	$-\frac{4}{3}\delta+\frac{4}{3}$	$\frac{2}{3}\delta+\frac{1}{3}$
0	0	0	0	0	0	δ
0	0	0	0	0	0	1

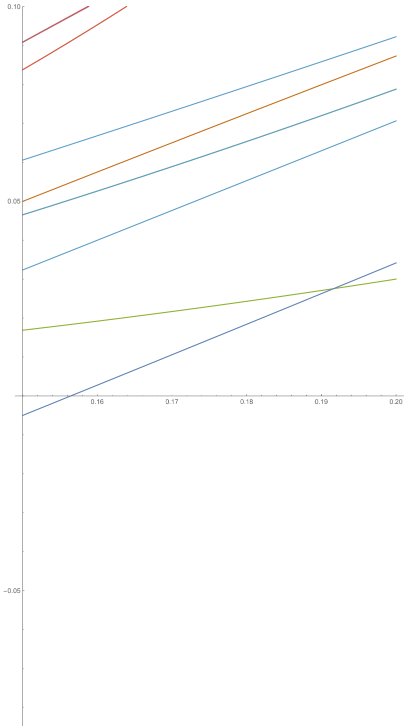
The simplex $\widetilde{E}_\delta \subset \mathbb{R}^6$ is the convex hull of the rows of

w_1^+	w_2^+	w_3^+	w_1^-	w_2^-	w_3^-	u	α
$\frac{4}{3}\delta-\frac{4}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$\frac{2}{3}-\frac{2}{3}\delta$	0	0	0	$\frac{1}{3}\delta+\frac{2}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$
$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{4}{3}-\frac{4}{3}\delta$	0	0	0	$\frac{2}{3}\delta+\frac{1}{3}$	$-\frac{1}{3}\delta+\frac{1}{3}$
0	0	0	$\frac{4}{3}\delta-\frac{4}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$\frac{1}{3}\delta+\frac{2}{3}$	$\frac{2}{3}\delta-\frac{2}{3}$
0	0	0	$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{2}{3}\delta-\frac{2}{3}$	$-\frac{4}{3}\delta+\frac{4}{3}$	$\frac{2}{3}\delta+\frac{1}{3}$	$\frac{1}{3}\delta-\frac{1}{3}$
0	0	0	0	0	0	δ	0
0	0	0	0	0	0	1	$\delta-1$
0	0	0	0	0	0	1	$-\delta+1$

Recall only need columns $w_1^\pm, w_2^\pm, u$, since $w_3^\pm = -w_1^\pm - w_2^\pm$

Values of $q\colon \widetilde{E}_\delta \to \mathbb{R}$ at vertices and critical points in interior of faces of $\widetilde{E}_\delta$, as functions of $\delta > 0$.





Zoom near $\delta \cong 0.16139$

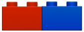
Change of sign at:

$$\delta = \frac{1}{71}\left(9\sqrt{545} - 199\right)$$
$$\cong 0.1564$$

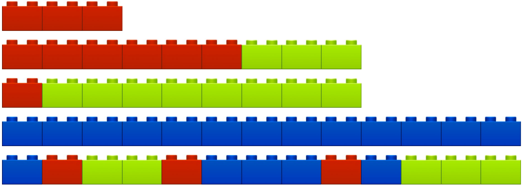
Mixing up red, green, blue

What about  ?

$$\mathbb{C}P^2\#(S^2 \times S^2) \cong_{\text{diffeo}} \underline{\#^2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}}$$

 $\cong_{\text{diffeo}}$ 

Similarly for , since $\overline{S^2 \times S^2} \cong_{\text{diffeo}} S^2 \times S^2$.



All have $\text{Ric} > 0$!

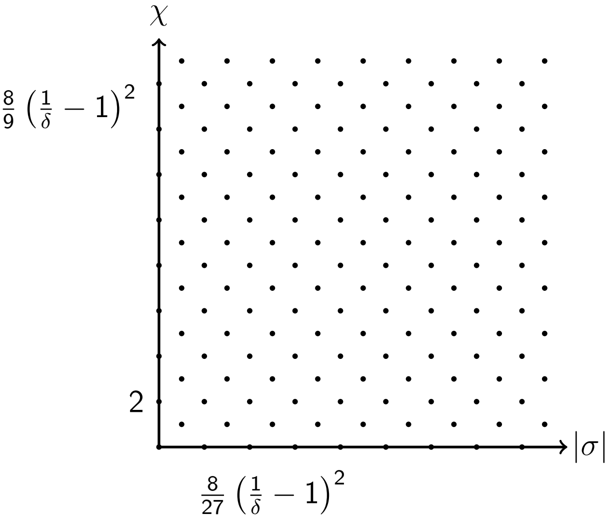
Recall that, conjecturally, very few have $\sec > 0$...

S^4 $\mathbb{C}P^2$ $\overline{\mathbb{C}P^2}$

Remark: Nothing else to squeeze from $a\,\underline{\chi} + b\,\underline{\sigma}$.

Lemma B: $\forall s \geq 0, \quad \chi(M) \leq \frac{8}{27}\left(\frac{1}{\delta} - 1\right)^2 (s+3) - s\,\sigma(M)$



A walk on the wild side: $\mathbf{sec}_M < 0$, $|\pi_1(M)| = \infty$

Theorem (Ville)

If M^4 is $\frac{1}{4}$-pinched, $\pi_1(M) \cong \pi_1(\mathbb{C}H^2/\Gamma)$, then $\underline{\;M = \mathbb{C}H^2/\Gamma\;}$.

Theorem (Gromov)

If M is $\delta(V)$-pinched with $\mathrm{Vol}(M) \leq V$, then M has $\underline{\sec \equiv -1}$.

Gromov–Thurston examples ($\sigma = 0$, non-explicit)

$\forall \delta \nearrow 1$, $\exists M^4$ δ-pinched that cannot have $\underline{\;\sec \equiv -1\;}$.

$\forall \delta \searrow 0$, $\exists M^4$ with $\sec < 0$ that cannot be $\underline{\;\delta\text{-pinched}\;}$.

Examples of Mostow–Siu, Ontaneda (cf. $\lambda(1) = 0$)

(Explicit) compact Kähler mfld, $\sec < 0$, $\frac{\sigma}{\chi} = \frac{128}{447} = \lambda(0.303)$.
$\rightsquigarrow$ "unpinchable" past $\delta = 0.303$

$\forall \delta \nearrow 1$, $\exists M^4$ δ-pinched manifolds with $\sigma \neq 0$.
$\rightsquigarrow \lambda(\delta) \searrow 0 \implies \chi \nearrow +\infty$