# Sectional Curvature Bounds from the Perspective of Convex Algebraic Geometry

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## §1. Convex Algebraic Geometry

### §1.1. Hierarchy of convex sets (in $\mathbb{R}^N$)

$$\left\{\begin{matrix}\text{convex}\\ \text{semi-algebraic sets}\end{matrix}\right\} \underset{?}{\supseteq} \left\{\begin{matrix}\text{spectrahedral}\\ \text{shadows}\end{matrix}\right\} \supset \{\text{spectrahedra}\} \supset \{\text{polyhedra}\}$$

Finitely many polynomial equalities and inequalities

Linear projections of spectrahedra

"Semidefinite representation" "Linear matrix inequality"

Intersection of finitely many half-spaces

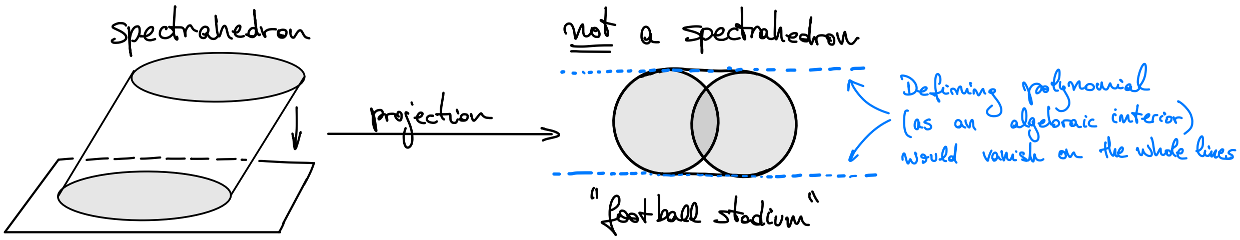
Helton-Nie Conjecture

Preserved under linear projections: ✓ (Tarski – Seidenberg Thm) ✓ ✗ ✓

**Def.** A spectrahedron is a set $S \subset \mathbb{R}^N$ of the form

$$S = \left\{ x \in \mathbb{R}^N : A_0 + \sum_{i=1}^{N} x_i A_i \succeq 0 \right\} \text{ where } A_0, A_1, \dots, A_N \in \mathrm{Sym}^2(\mathbb{R}^d)$$

**Note:** If $A_i$ are all diagonal, then $S$ is a polyhedron. (Intersection of $d$ half-spaces)




**Def:** A spectrahedral shadow is a set $S \subset \mathbb{R}^N$ of the form

$$S = \left\{ x \in \mathbb{R}^N : \exists y \in \mathbb{R}^M,\ A_0 + \sum_{i=1}^{N} x_i A_i + \sum_{j=1}^{M} y_j B_j \succeq 0 \right\}$$

where $A_i, B_j \in \mathrm{Sym}^2(\mathbb{R}^d)$.

**Thm** (Scheiderer, 2018). The Helton-Nie Conjecture is

- TRUE if $N \leq 2$
- FALSE in general (counter-examples known for $N \geq 14$)

## §1.2 NONNEGATIVE v. SUM OF SQUARES

Replacing $X$ with $\mathbb{R}^n$, this problem dates back to Hilbert.

$X$ real projective variety

$X \subset \mathbb{CP}^n$ real, irreducible, full, $X(\mathbb{R})$ Zariski-dense

$$\Sigma_X := \left\{ p \in \mathbb{R}[X]_2 : p = \sum q_i^2,\ q_i \in \mathbb{R}[X]_1 \right\}$$

Always a spectrahedral shadow!

$$P_X := \left\{ p \in \mathbb{R}[X]_2 : p(x) \geq 0 \quad \forall x \in X \right\}$$

In principle, just a convex semialgebraic set.

$$\boxed{P_X = \Sigma_X\ ?}$$

$\Sigma_X \subsetneq P_X$ is first step to show $P_X$ is not a spectrahedral shadow!

**Finsler's Lemma** (1936). If $X$ is a quadric, then $P_X = \Sigma_X$.

**Thm** (Blekherman-Smith-Velasco, 2016). $P_X = \Sigma_X \iff X$ has minimal degree $(\deg X = \operatorname{codim} X + 1)$

## §2. SECTIONAL CURVATURE BOUNDS

$\forall \sigma \in Gr_2 \mathbb{R}^n$ $\sec_R(\sigma) = \langle R\sigma, \sigma\rangle \geq 0$

$$\mathcal{R}_{\sec \geq 0}(n) := \left\{ R \in \mathrm{Sym}^2_b(\Lambda^2 \mathbb{R}^n) : \sec_R \geq 0 \right\}$$

**Thm** (A. Weinstein, 1971). $\mathcal{R}_{\sec\geq 0}(n)$ is a convex semialgebraic set.

Letting $X = \mathrm{Gr}_2\mathbb{R}^n \subset \Lambda^2\mathbb{R}^n$, we have $\mathcal{R}_{\sec\geq 0}(n) = P_X$.

$$\begin{tikzcd}
\Lambda^4\mathbb{R}^n \arrow[r] \arrow[d, equal] & \mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \arrow[r, "\pi"] \arrow[d, equal] & \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^n) \arrow[d, equal] & \supset & \mathcal{R}_{\sec\geq 0}(n) \arrow[d, equal] & \\
I_2 \arrow[r] & \mathbb{R}[x_{ij}]_2 \arrow[r] & \mathbb{R}[X]_2 \arrow[d, equal] & \supset & P_X & \supset \Sigma_X \\
& & \mathbb{R}[X]_2/I_2 & & &
\end{tikzcd}$$

$\mathrm{Gr}_2\mathbb{R}^n = \{\sigma \in \Lambda^2\mathbb{R}^n : \sigma\wedge\sigma = 0,\ |\sigma| = 1\}$

$\sigma\wedge\sigma = 0 \iff \langle \omega(\sigma), \sigma\rangle = 0 \quad \forall \omega \in \Lambda^4\mathbb{R}^n$

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**Thm.** (B.–Kummer–Mendes, 2021). The set $\mathcal{R}_{\sec\geq 0}(n)$ is

(i) not a spectrahedral shadow, if $n \geq 5$

(ii) a spectrahedral shadow, but not a spectrahedron, if $n = 4$

(iii) a spectrahedron, if $n \leq 3$.

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Sketch: (iii) $\sec_R \geq 0 \iff R \succeq 0$

(ii) Finsler's Lemma: $X = \mathrm{Gr}_2\mathbb{R}^4 \Rightarrow \mathcal{R}_{\sec\geq 0}(4) = P_X = \Sigma_X$ is a spectrahedral shadow.

$X$ is a quadric/has minimal degree

(i) Refinement of Scheiderer's criterion for $X = \mathrm{Gr}_2\mathbb{R}^n$:

$$\boxed{P_X = \Sigma_X \iff P_X \text{ is a spectrahedral shadow.}}$$

[Zoltek'1979]: $\exists R \in P_X \setminus \Sigma_X$ if $n \geq 5$.

or from [BSV'2016]: $X$ has minimal degree $\iff n \leq 4$. $\square$

## §3. GEOMETRIC APPLICATIONS

### §3.1. DIMENSION $n=4$.

$\pi : \mathrm{Sym}^2(\Lambda^2\mathbb{R}^4) \to \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^4)$

orthogonal projection ($\ker\pi = \mathrm{span}\, * \cong \Lambda^4\mathbb{R}^4$).

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Finsler–Thorpe Trick. $\underbrace{\mathcal{R}_{\sec\geq 0}(4)}_{\text{spectrahedral shadow}} = \pi\Big(\underbrace{\{R \in \mathrm{Sym}^2(\Lambda^2\mathbb{R}^4) : R \succeq 0\}}_{\text{spectrahedron}}\Big)$

$= \{R \in \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^4) : \exists\, a \in \mathbb{R},\ R + a\,* \succeq 0\}$

Hodge star

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$\boxed{\underline{\text{Cor}}: \ (M^4,g) \text{ has } \sec \geq 0 \iff \exists f: M \to \mathcal{R}, \quad R + f * \geq 0.}$

$\underline{\text{Thm}}$ (B.–Kummer–Mendes, 2021). If $(M^4,g)$ is oriented and has $\delta \leq \sec \leq 1$ or $-1 \leq \sec \leq -\delta$ and finite volume, then

$$|\sigma(M^4)| \leq \lambda(\delta) \cdot \chi(M^4)$$

where $\lambda : (0,1] \to [0,+\infty)$ is an $\underline{\text{explicit}}$ function of $\delta$.

$\underline{\text{Notable values}}$: $\lambda\left(\frac{1}{4}\right) = \frac{1}{3}$ [Ville, 80's], $\lambda\left(\frac{1}{1+3\sqrt{3}}\right) < \frac{1}{2}$, $\lambda(1) = 0$.

Sharp: $\mathbb{CP}^2$, $\mathbb{H}^2_{\mathbb{C}}/\Gamma$

$\sigma(M) \neq 0$ and $\delta \nearrow 1$ ($\sec \approx -1$)
$\Rightarrow \chi(M) \nearrow +\infty$

$\underline{\text{Sketch}}$: Let $\varphi_\lambda(R) = \lambda \cdot \underbrace{\chi(R)}_{\substack{\text{Chern–Gauss–Bonnet} \\ \text{integrand}}} - \underbrace{\sigma(R)}_{\substack{\text{Signature} \\ \text{integrand}}}$, so that $\displaystyle\int_M \varphi_\lambda(R) = \lambda \chi(M) - \sigma(M)$.

Optimize $\varphi_\lambda : \underbrace{\mathcal{R}_{\delta \leq \sec \leq 1}(4)}_{\text{Spectrahedral shadow}} \to \mathbb{R}$ to get $\Omega = \left\{(\delta,\lambda) : \min\limits_{R \in \mathcal{R}_{\delta \leq \sec \leq 1}(4)} \varphi_\lambda \geq 0\right\}$

Use cylindrical algebraic decomposition to write $\Omega = \{(\delta,\lambda) : \lambda \geq \underline{\lambda(\delta)}\}$. $\square$

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Q: Which $(M^4,g)$ have $\sec > 0$? / $\underline{\text{Hopf Question (1932)}}$: Does $S^2 \times S^2$ have $\sec > 0$?

$\underline{\text{Cor}}$: If $(M^4,g)$ is simply-connected and $\underbrace{\frac{1}{1+3\sqrt{3}}}_{0.161\ldots} \leq \sec \leq 1$,

then $M^4 \underset{\text{homeo}}{\cong} S^4$ or $\mathbb{CP}^2$.

$\underline{\text{Pf}}$: By [Diógenes–Ribeiro, 2019], $M^4$ has definite intersection form:

$$b_2(M) = b_+(M) + \underbrace{b_-(M)}_{=0}.$$

so $\sigma(M) = b_+(M)$, and $\chi(M) = 2 + b_+(M)$.

By Thm, $|\sigma| < \frac{1}{2}\chi$ hence $|b_+| < 1 + \frac{1}{2} b_+$ so $b_+ \leq 1$.

Donaldson–Freedman: $b_+ = 0 \Rightarrow M \cong S^4$, $b_+ = 1 \Rightarrow M \cong \mathbb{CP}^2$. $\square$

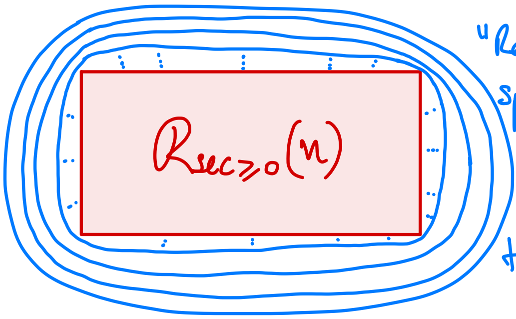
## §3.1 DIMENSIONS $n \geq 5$

"No Finsler–Thorpe trick, but..."

...convex algebro-geometric point of view is still fruitful.

**Thm** (B.–Mendes, 2017)

$$\mathcal{R}_{\sec\geq 0}(n) = \bigcap_{p\geq 2} \underbrace{\left\{R\in \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^n) : K\left(R, \mathrm{Sym}^p_0(\mathbb{R}^n)\right) \geq 0\right\}}_{\text{these are spectrahedra, for each } p\geq 2}$$




"Relaxation" by spectrahedra

$\rightsquigarrow$ Prove thm's by taking limits as $p \nearrow +\infty$.

$(M^n, g)$, $\sec \geq 0$, compact

$$\Longrightarrow \underbrace{\exists N\in\mathbb{N}}_{\text{likely impossible to compute!}} \text{ s.t. } \forall x\in M,\ R_x \in \bigcap_{p=2}^{N} \dots$$

Or switch to other curvature conditions... e.g:

**Def:** $R\in \mathrm{Sym}^2_b(\Lambda^2\mathbb{R}^n)$ is <u>$k$-positive</u> if $\lambda_1+\dots+\lambda_k > 0$

$$\begin{array}{ccccccc}
k=1 & \Rightarrow \dots \Rightarrow & k=n-1 & \Rightarrow \dots \Rightarrow & k=\binom{n}{2} \\
\Updownarrow & \xleftarrow{\text{stronger}} & \Downarrow & \xrightarrow{\text{weaker}} & \Updownarrow \\
R>0 & \Rightarrow \dots \Rightarrow & \mathrm{Ric}>0 & \Rightarrow \dots \Rightarrow & \mathrm{scal}>0
\end{array}$$

Note: For all $1\leq k\leq \binom{n}{2}$, this defines a spectrahedron!

**Thm** (Petersen–Wink, 2021). $(M^n, g)$ closed, with $(n-p)$-positive $R$, then
$b_1(M) = \dots = b_p(M) = 0$ and $b_{n-p}(M) = \dots = b_n(M) = 0$.
In particular, if $\lceil \frac{n}{2} \rceil$-positive, then $M^n$ is a rational homology sphere.

**Thm** (B.—Goodman, 2021). If $(M^{2m}, g)$ is closed and spin, with $k$-positive $R$ where $k \leq \frac{m(2m+7)}{m+8}$, and $\frac{\mathrm{scal}}{8} - \mathrm{Ric} \geq 0$, then:

$$\underbrace{\langle \hat{A}(TM).\mathrm{ch}(TM_{\mathbb{C}}), [M]\rangle}_{\text{Elliptic genus associated to } D_{TM}: \$\otimes TM \to \$\otimes TM} = 0.$$

**Cor:** If $(M^8, g)$ is spin, Einstein, and has 5-positive $R$, then $M^8$ is null-cobordant: $\hat{A}(M^8) = 0$ and $\sigma(M^8) = 0$.

In particular, $\mathbb{HP}^2$ does not have an Einstein metric w/ 5-positive $R$.