Pinched 4-manifolds

Renato G. Bettiol

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A very naive approach to a very big problem

$0 \quad \dots \quad \delta$ 

Conjecture (Folklore)

(M^4, g) *closed,*
$\pi_1(M) = \{1\}$, $\sec_M > 0$ $\implies$ $M^4 \cong_{\textit{diffeo}}$ $\underline{S^4 \text{ or } \mathbb{C}P^2}$.

Definition

(M^n, g) is δ-pinched if $\underline{\delta \leq \sec_M \leq 1}$.

Question

How small can we make $\delta > 0$ and prove:

(M^4, g) *closed,*
$\pi_1(M) = \{1\}$, *δ-pinched* $\implies$ $M^4 \cong_{\substack{\textit{diffeo?} \\ \textit{homeo?}}}$ $\underline{S^4 \text{ or } \mathbb{C}P^2}$.

Pinching theorems

Let (M^4, g) be closed, $\pi_1(M) = \{1\}$, and δ-pinched.

Theorem (Berger 1960, Klingenberg 1961)

$\delta > \frac{1}{4} \implies \underline{M \cong_{homeo} S^4}$

Theorem (Brendle–Schoen, 2009)

$\delta > \frac{1}{4} \implies \underline{M \cong_{diffeo} S^4}$

Theorem (Berger 1983; Petersen–Tao, 2009)

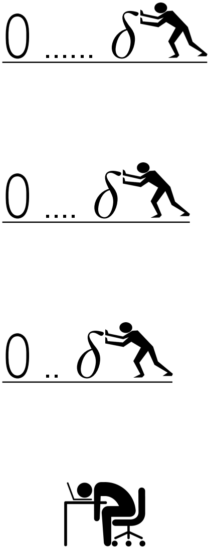
$\exists \varepsilon > 0, \quad \delta > \frac{1}{4} - \varepsilon \implies \underline{M \cong_{diffeo} S^4 \text{ or } \mathbb{C}P^2}$

Theorem (Ville, 1989)

$\delta \geq \frac{4}{19} \cong 0.2105 \implies \underline{M \cong_{homeo} S^4 \text{ or } \mathbb{C}P^2.}$

Theorem (Seaman, 1989)

$\delta \geq \frac{1}{3\sqrt{1+(2^{5/4}/5^{1/2})}+1} \cong 0.1883 \implies \underline{M \cong_{homeo} S^4 \text{ or } \mathbb{C}P^2.}$



0 ...... δ
0 .... δ
0 .. δ

Theorem A (B., Kummer, Mendes, 2020)

Let (M^4, g) be closed, $\pi_1(M) = \{1\}$, and δ-pinched, with

$$\delta \geq \frac{1}{1+3\sqrt{3}} \cong 0.16139$$

Then $\underline{\ M \cong_{homeo} S^4 \text{ or } \mathbb{C}P^2\ }$.

MATHEMATICIANS ARE WEIRD
YOU KNOW THAT THING THAT WAS 2.3728642?
YES.?
I GOT IT DOWN TO 2.3728639.
Thunderous applause
smbc-comics.com

Actually, there's a bit more to it

Throughout proofs: **sharp pointwise algebraic bounds**, only possible slack is **global**. Fits in larger context:

- Convex algebraic geometry of curvature operators
  Spectrahedral shadows, Lasserre relaxations, ...
- Long-term projects using optimization in geometry
  Semidefinite programming
- Other applications on the horizon
  Quadratic curvature funcionals, Ricci solitons, ...

And also:

Theorem B (B., Kummer, Mendes, 2020)

Given $\delta > 0$*, if* (M^4, g)*,* $\pi_1(M) = \{1\}$*, is* δ*-pinched, then:*

$$\chi(M) \leq \underline{\frac{8}{9}\left(\frac{1}{\delta} - 1\right)^2} \quad \textit{and} \quad |\sigma(M)| \leq \underline{\frac{8}{27}\left(\frac{1}{\delta} - 1\right)^2}.$$

cf. Berger, 1962: $\chi(M) \leq \underline{\frac{8}{9}\left(\frac{1}{\delta} - 1\right)^2} + \frac{11}{27\delta^2} + \frac{16}{27}\left(\frac{2}{\delta} - 1\right)$.

Corollary

Explicit (finite) list *of homeomorphism types for each* $\delta > 0$*:*

(M^4, g), $\pi_1(M) = \{1\}$,
δ*-pinched*

$$\implies \quad M^4 \cong_{homeo} \begin{cases} \#^r \mathbb{C}P^2 \#^s \overline{\mathbb{C}P^2}, \\ \#^k S^2 \times S^2, \end{cases}$$

where $r, s, k \in \mathbb{N}_0$ *satisfy*

$$\begin{cases} r + s + 2 \leq \min\left\{ \underline{\tfrac{8}{9}\left(\tfrac{1}{\delta} - 1\right)^2}, 10^{1440} \right\} \\ |r - s| \leq \underline{\tfrac{8}{27}\left(\tfrac{1}{\delta} - 1\right)^2} \\ 2k + 2 \leq \min\left\{ \underline{\tfrac{8}{9}\left(\tfrac{1}{\delta} - 1\right)^2}, 10^{1440} \right\} \end{cases}$$

Recall that, conjecturally,

$$(r, s) \in \left\{ \underline{(0,0),\ (0,1),\ (1,0)} \right\} \text{ and } k = \underline{0}.$$

Topology of (smooth) 4-manifolds

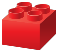
$$M^4 \text{ closed, } \pi_1(M) = \{1\} \implies \begin{array}{l} b_1(M) = b_3(M) = 0 \\ \text{All information in } H_2(M) \\ \text{"intersection form"} \end{array}$$

Hodge Theory:

$$b_2(M) = b_+(M) + b_-(M)$$
$$b_+(M) = \dim\{\alpha \in \Omega^2(M) : \Delta\alpha = 0, *\alpha = \alpha\}$$
$$b_-(M) = \dim\{\alpha \in \Omega^2(M) : \Delta\alpha = 0, *\alpha = -\alpha\}$$

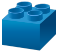
$$\chi(M) = \underline{2 + b_+(M) + b_-(M)}, \qquad \sigma(M) = \underline{b_+(M) - b_-(M)}$$

Topological building blocks



$\mathbb{C}P^2$

$b_+ = 1$
$b_- = 0$

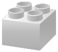


$S^2 \times S^2$

$b_+ = 1$
$b_- = 1$

$\overline{\mathbb{C}P^2}$

$b_+ = 0$
$b_- = 1$



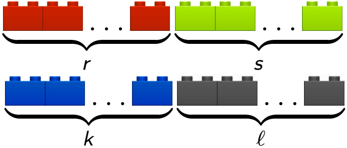
M_{E_8}

$b_+ = 8$
$b_- = 0$
(not smooth!)

Theorem (Freedman 1982; Donaldson 1983)

If M^4 is closed, smooth, $\pi_1(M) = \{1\}$, then

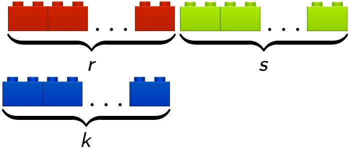
$$M^4 \cong_{homeo} \begin{cases} \#^r \mathbb{C}P^2 \#^s \overline{\mathbb{C}P^2} \\ \#^k (S^2 \times S^2) \#^\ell M_{E_8} \end{cases}$$



Corollary (Lichnerowicz, Atiyah-Singer, Hirzebruch)

If (M^4, g) *is closed,* $\pi_1(M) = \{1\}$*, and* <u>scal > 0</u> *then*

$$M^4 \cong_{homeo} \begin{cases} \#^r \mathbb{C}P^2 \#^s \overline{\mathbb{C}P^2} \\ \#^k S^2 \times S^2 \end{cases}$$



Conversely...

Theorem (Gromov–Lawson 1980; Schoen–Yau 1979)

Any connected sum of $\mathbb{C}P^2$*'s,* $\overline{\mathbb{C}P^2}$*'s,* $S^2 \times S^2$*'s has* <u>scal > 0</u>.

Theorem (Sha–Yang 1993; Perelman 1997)

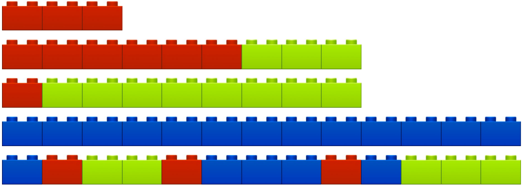
Any connected sum of $\mathbb{C}P^2$*'s,* $\overline{\mathbb{C}P^2}$*'s,* $S^2 \times S^2$*'s has* <u>Ric > 0</u>.

What about  ?

$$\mathbb{C}P^2\#(S^2 \times S^2) \cong_{\text{diffeo}} \underline{\#^2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}}$$

 $\cong_{\text{diffeo}}$ 

Similarly for , since $\overline{S^2 \times S^2} \cong_{\text{diffeo}} S^2 \times S^2$.



All have Ric > 0!

But recall that, conjecturally, very few have sec > 0...

S^4 $\qquad$ $\mathbb{C}P^2$ $\qquad$ $\overline{\mathbb{C}P^2}$

Curvature operator

Eigenspaces of Hodge star $*$

$$\wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM$$

Curvature operator canonical form, using $\mathrm{O}(4) \curvearrowright \wedge^2 TM$

$$R\colon\ \wedge^2 TM \longrightarrow \wedge^2 TM$$

$$R = \begin{pmatrix} u\mathrm{Id} + W_+ & C \\ C^t & u\mathrm{Id} + W_- \end{pmatrix} \in \mathrm{Sym}^2_b(\wedge^2 TM)$$

$u = \frac{1}{12}\,\mathrm{scal}$

$W_\pm = \mathrm{diag}(w_1^\pm, w_2^\pm, w_3^\pm)$ Weyl tensor

$C = \overset{\circ}{\mathrm{Ric}}$ traceless Ricci

R diagonal $\iff$ $C =$ 0 $\iff$ M is Einstein

Chern–Gauss–Bonnet, Hirzebruch

Integral formulas for topological invariants:

$$\chi(M) = \frac{1}{\pi^2}\int_M \underline{\chi}(R) \qquad \sigma(M) = \frac{1}{\pi^2}\int_M \underline{\sigma}(R)$$

Integrands are indefinite quadratic forms on $R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$, and SO(4)-invariant:

$$\underline{\chi},\ \underline{\sigma}\colon\ \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4) \longrightarrow \mathbb{R}$$

$$\underline{\chi}(R) = \frac{1}{8}\left(6u^2 + |W_+|^2 + |W_-|^2 - 2|C|^2\right)$$

$$\underline{\sigma}(R) = \frac{1}{12}\left(|W_+|^2 - |W_-|^2\right)$$

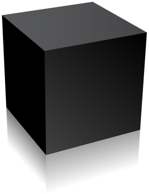
Blackbox optimization lemmas (more on this later)

Definition

$\Omega_\delta := \underline{\left\{R \in \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4) : \delta \leq \sec_R \leq 1\right\}}$

Lemma A

If $\delta > \frac{1}{1+3\sqrt{3}}$, *then* $\min\limits_{R\in\Omega_\delta} \underline{\chi}(R) - 2\,\underline{\sigma}(R) > 0$.



Lemma B

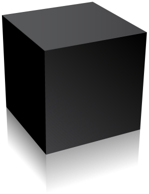
For all $\delta > 0$, *and* $t \in [-1,1]$,

$$\max_{R\in\Omega_\delta} |W_+|^2 + t\,|W_-|^2 = \tfrac{8}{3}(1-\delta)^2.$$

In particular,

$(t = +1)$ $\max\limits_{R\in\Omega_\delta} |W|^2 = \frac{8}{3}(1-\delta)^2$,

$(t = -1)$ $\max\limits_{R\in\Omega_\delta} \underline{\sigma}(R) = \frac{2}{9}(1-\delta)^2$.



Proof of Theorem A

Let (M^4, g) be closed, $\pi_1(M) = \{1\}$, and δ-pinched, with

$$\delta \geq \frac{1}{1+3\sqrt{3}} \cong 0.16139$$

Up to reversing orientation, assume $\sigma(M) \geq 0$.

Theorem (Diógenes–Ribeiro, 2019)

M^4 is definite, i.e., $b_-(M) = 0$.

Thus: $M^4 \cong_{\text{homeo}} \#^r \mathbb{C}P^2, \; r = b_+(M) \geq 0, \; b_-(M) = 0.$

...

$$\text{Lemma A} \implies \chi(M) - 2\,\sigma(M) > 0$$

$$\implies 2 + b_+(M) - 2b_+(M) > 0$$

$$\implies b_+(M) \leq 1$$

□

Proof of Theorem B (Part 1)

Let (M^4, g) be closed, $\pi_1(M) = \{1\}$, and δ-pinched, $\delta > 0$.

Theorem (Chang–Gurksy–Yang, 2003)

If $\int_M |W|^2 < 4\pi^2 \chi(M)$, *then* $\underline{M \cong_{\mathit{diffeo}} S^4}$.

Suppose $\underline{M \ncong_{\mathrm{homeo}} S^4}$. Then:

$$\chi(M) \leq \frac{1}{4\pi^2} \int_M |W|^2$$

Lemma B

$$\leq \underline{\frac{1}{4\pi^2} \frac{8}{3} (1-\delta)^2 \ \mathrm{Vol}(M)}$$

Bishop–Gromov
Diameter Sphere Theorem

$$\leq \underline{\frac{1}{4\pi^2} \frac{8}{3} (1-\delta)^2 \ \mathrm{Vol}\left(S^4_+\left(\tfrac{1}{\sqrt{\delta}}\right)\right)}$$

$$\boxed{\mathrm{Vol}\left(S^4_+\left(\tfrac{1}{\sqrt{\delta}}\right)\right) = \frac{4\pi^2}{3\delta^2}}$$

$$= \frac{8}{9}\left(\frac{1}{\delta} - 1\right)^2.$$

Proof of Theorem B (Part 2)

Let (M^4, g) be closed, $\pi_1(M) = \{1\}$, $M \not\cong_{\text{homeo}} S^4$, δ-pinched.

$$\sigma(M) = \frac{1}{\pi^2} \int_M \underline{\sigma}(R)$$

Lemma B

$$\leq \underline{\frac{1}{\pi^2} \frac{2}{9} (1-\delta)^2 \operatorname{Vol}(M)}$$

Bishop–Gromov
Diameter Sphere Theorem

$$\leq \underline{\frac{1}{\pi^2} \frac{2}{9} (1-\delta)^2 \operatorname{Vol}\left(S^4_+\left(\tfrac{1}{\sqrt{\delta}}\right)\right)}$$

$$\boxed{\operatorname{Vol}\left(S^4_+\left(\tfrac{1}{\sqrt{\delta}}\right)\right) = \frac{4\pi^2}{3\delta^2}}$$

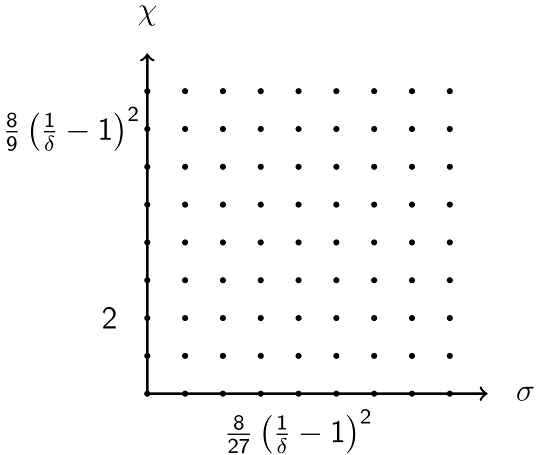
$$= \frac{8}{27} \left(\frac{1}{\delta} - 1\right)^2.$$

$\square$

Remark: Nothing else to squeeze from $\boldsymbol{a}\,\underline{\chi} + \boldsymbol{b}\,\underline{\sigma}$.

Lemma B: $\quad \forall t \in [-1, 1], \quad \max_{R \in \Omega_\delta} |W_+|^2 + t\,|W_-|^2 = \frac{8}{3}(1-\delta)^2$

$$\implies \forall s \geq 0, \quad \chi(M) \leq \frac{8}{27}\left(\frac{1}{\delta} - 1\right)^2 (s+3) - s\,\sigma(M)$$



Inside the blackbox

- $q \colon \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4) \longrightarrow \mathbb{R}$ quadratic form

$$q(R) = R^t \cdot A_q \cdot R + b_q \cdot R + c_q$$

- $\Omega_\delta \subset \mathrm{Sym}^2_b(\wedge^2\mathbb{R}^4)$ is a compact *spectrahedral shadow* (Thorpe's trick / Finsler's Lemma)

$$R \in \Omega_\delta \iff \exists\, \alpha, \beta \in \mathbb{R}, \; \underline{\begin{array}{l} R - \delta \mathrm{Id} + \alpha * \succeq 0 \\ \mathrm{Id} - R + \beta * \;\; \succeq 0 \end{array}}$$

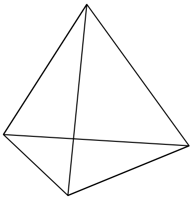
- If A_q is *indefinite*, "brute force" semidefinite programming on Ω_δ does not work, nor do other *convex* methods

$$\min_{R \in \Omega_\delta} q(R) = ? \qquad \max_{R \in \Omega_\delta} q(R) = ?$$

The Einstein simplex

Proposition

The set of δ-pinched Einstein curvature operators $E_\delta := \mathrm{Diag} \cap \Omega_\delta$ is a <u>5-*simplex*</u>, *and* <u>$\mathrm{proj}(\Omega_\delta) = E_\delta$</u>, *where* $\mathrm{proj}\colon \mathrm{Sym}^2_b(\wedge^2 \mathbb{R}^4) \longrightarrow \mathrm{Diag}$.



Proposition

The set of modified δ-pinched Einstein operators $\widetilde{E}_\delta := \left\{ (R, \alpha) \in E_\delta \times \mathbb{R} : R - \delta \mathrm{Id} + \alpha * \succeq 0 \right\}$ *is a* <u>6-*simplex*</u>.

Note: All vertices are explicit *affine functions* of δ.

Quadratic optimization on simplices

Lemma

$\Delta^n = \mathrm{conv}(V) \subset \mathbb{R}^n$ *simplex,* $q\colon \mathbb{R}^n \to \mathbb{R}$ *quadratic form,* $A_q = \mathrm{Hess}\, q$

$$A_q \quad \underline{\textit{indefinite}} \quad \Longrightarrow \quad \max_{\Delta^n} q = \max_{\partial\Delta^n} q$$

$$A_q \quad \underline{\textit{positive-semidefinite}} \quad \Longrightarrow \quad \max_{\Delta^n} q = \max_{V} q$$

$$A_q \quad \underline{\textit{negative-definite}} \quad \Longrightarrow \quad \textit{Use Calculus to find } \max_{\mathbb{R}^n} q$$

Proof of Lemma B.

For all $\delta > 0$, and $t \in [-1, 1]$, since $\mathrm{proj}(\Omega_\delta) = E_\delta$,

$$\max_{R \in \Omega_\delta} |W_+|^2 + t\,|W_-|^2 \overset{\text{Prop}}{=} \max_{R \in E_\delta} \underbrace{|W_+|^2 + t\,|W_-|^2}_{q_t(R)} \overset{\text{Lemma}}{=} \tfrac{8}{3}(1-\delta)^2.$$

Need to inspect faces of dimension $\leq \mathrm{ind}(A_{q_t}) = 2$ if $t < 0$. □

Integrand in Lemma A depends on $|C|^2$...

Lemma

Given $\lambda_i \geq 0$, $\mu_i \geq 0$,

$$\begin{pmatrix} \mathrm{diag}(\lambda_i) & C \\ C^t & \mathrm{diag}(\mu_i) \end{pmatrix} \succeq 0 \implies |C|^2 \leq \sum_{i=1}^{3} \lambda_i \mu_i .$$

Proof.

- Schur complements
- $\mathrm{O}(3) \subset B_1^{\mathrm{spec}} \subset \mathrm{Mat}_{3\times 3}(\mathbb{R})$ are its extreme points
- Birkhoff–von Neumann Theorem □

Corollary

If $R \in \Omega_\delta$ *is such that* $R - \delta \mathrm{Id} + \alpha * \succeq 0$, *then*

$$|C|^2 \leq \sum_{i=1}^{3} (u - \delta + w_i^+ + \alpha)(u - \delta + w_i^- - \alpha)$$
$$= 3(u-\delta)^2 - 3\alpha^2 + \langle W_+, W_- \rangle .$$

Proof of Lemma A

- Let $\delta > \frac{1}{1+3\sqrt{3}}$, $R \in \Omega_\delta$, and $\alpha \in \mathbb{R}$ s.t. $R - \delta \mathrm{Id} + \alpha * \succeq 0$.
- Recall that $\mathrm{proj}(\Omega_\delta) = E_\delta$, hence $(\mathrm{proj}(R), \alpha) \in \widetilde{E}_\delta$.
- Use Corollary to eliminate $|C|^2$:

$$\begin{aligned} 8\big(\underline{\chi}(R) - 2\,\underline{\sigma}(R)\big) &= 6u^2 - \tfrac{1}{3}|W_+|^2 + \tfrac{7}{3}|W_-|^2 - 2|C|^2 \\ &\geq \underbrace{-\tfrac{1}{3}|W_+|^2 + \tfrac{7}{3}|W_-|^2 - 2\langle W_+, W_-\rangle + 6\alpha^2 + 12u\delta - 6\delta^2}_{q(\mathrm{proj}(R),\alpha)} \end{aligned}$$

$q \colon \widetilde{E}_\delta \longrightarrow \mathbb{R}$ indefinite quadratic form, $\mathrm{ind}(A_q) = 3$.

- Optimize q on the 6-simplex $\widetilde{E}_\delta$ as before, obtaining

$$8 \min_{\Omega_\delta} \underline{\chi} - 2\,\underline{\sigma} \;\geq\; \min_{\widetilde{E}_\delta} q \;>\; 0.$$

□

Bonus: Hopf Conjecture

M compact, even-dimensional, $\sec_M > 0 \implies \underline{\chi(M) > 0}$

Algebraic version: $\sec_R > 0 \implies \underline{\underline{\chi}(R) > 0}$

- **True** if $n = 4$ (Milnor / Chern, 1955)
- **False** if $n \geq 6$ (Geroch, 1976)

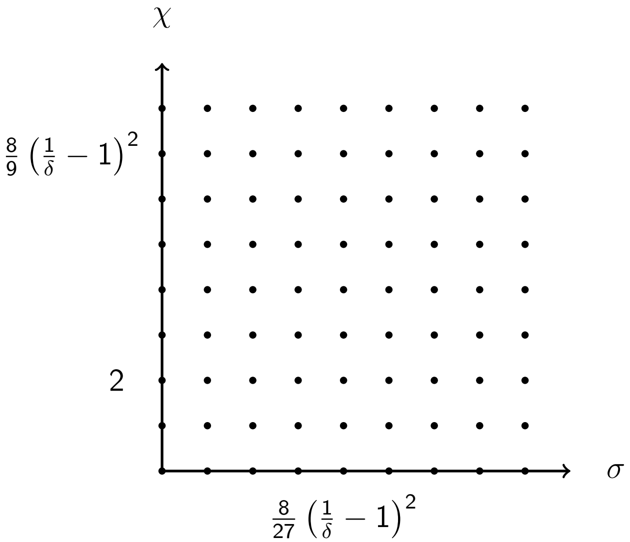
Proof ($n = 4$).

If $\sec_R > 0$, then $R + \alpha * \succ 0$ for some $\alpha \in \mathbb{R}$. Thus:

$$\begin{aligned} 8\underline{\chi}(R) &= 6u^2 + |W_+|^2 + |W_-|^2 - 2|C|^2 \\ \text{(Corollary)} \qquad &> 6\alpha^2 + |W_+|^2 + |W_-|^2 - 2\langle W_+, W_- \rangle \\ &= 6\alpha^2 + \left| W_+ - W_- \right|^2 \\ &\geq 0. \end{aligned}$$

□

Thank you for your attention!


χ
$\frac{8}{9}\left(\frac{1}{\delta}-1\right)^2$
2
σ
$\frac{8}{27}\left(\frac{1}{\delta}-1\right)^2$

Vertices of Einstein simplices E_δ and $\widetilde{E}_\delta$

The Einstein simplex $E_\delta \subset \mathbb{R}^5$ is the convex hull of the rows

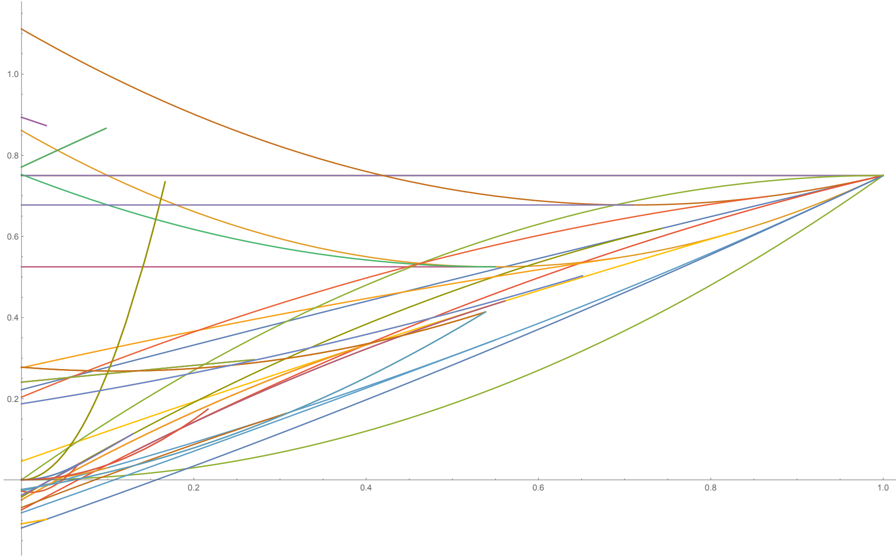
w_1^+	w_2^+	w_3^+	w_1^-	w_2^-	w_3^-	u
$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{4}{3}-\frac{4}{3}\delta$	0	0	0	$\frac{2}{3}\delta+\frac{1}{3}$
$\frac{4}{3}\delta-\frac{4}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$\frac{2}{3}-\frac{2}{3}\delta$	0	0	0	$\frac{1}{3}\delta+\frac{2}{3}$
0	0	0	$\frac{4}{3}\delta-\frac{4}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$\frac{1}{3}\delta+\frac{2}{3}$
0	0	0	$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{2}{3}\delta-\frac{2}{3}$	$-\frac{4}{3}\delta+\frac{4}{3}$	$\frac{2}{3}\delta+\frac{1}{3}$
0	0	0	0	0	0	δ
0	0	0	0	0	0	1

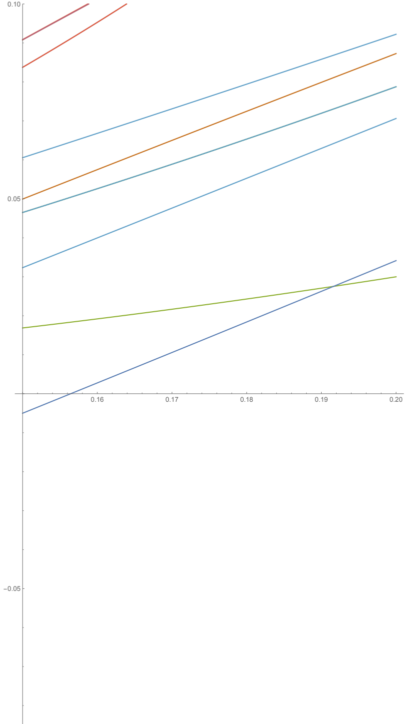
The simplex $\widetilde{E}_\delta \subset \mathbb{R}^6$ is the convex hull of the rows of

w_1^+	w_2^+	w_3^+	w_1^-	w_2^-	w_3^-	u	α
$\frac{4}{3}\delta-\frac{4}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$\frac{2}{3}-\frac{2}{3}\delta$	0	0	0	$\frac{1}{3}\delta+\frac{2}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$
$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{4}{3}-\frac{4}{3}\delta$	0	0	0	$\frac{2}{3}\delta+\frac{1}{3}$	$-\frac{1}{3}\delta+\frac{1}{3}$
0	0	0	$\frac{4}{3}\delta-\frac{4}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$-\frac{2}{3}\delta+\frac{2}{3}$	$\frac{1}{3}\delta+\frac{2}{3}$	$\frac{2}{3}\delta-\frac{2}{3}$
0	0	0	$\frac{2}{3}\delta-\frac{2}{3}$	$\frac{2}{3}\delta-\frac{2}{3}$	$-\frac{4}{3}\delta+\frac{4}{3}$	$\frac{2}{3}\delta+\frac{1}{3}$	$\frac{1}{3}\delta-\frac{1}{3}$
0	0	0	0	0	0	δ	0
0	0	0	0	0	0	1	$\delta-1$
0	0	0	0	0	0	1	$-\delta+1$

Recall only need columns $w_1^\pm, w_2^\pm, u$, since $w_3^\pm = -w_1^\pm - w_2^\pm$

Values of $q\colon \widetilde{E}_\delta \to \mathbb{R}$ at vertices and critical points in interior of faces of $\widetilde{E}_\delta$, as functions of $\delta > 0$.





Zoom near $\delta \cong 0.16139$

Change of sign at:

$$\delta = \frac{1}{71}\left(9\sqrt{545} - 199\right) \cong 0.1564$$