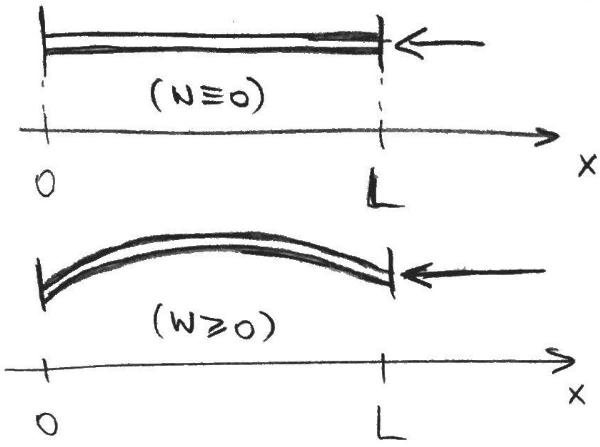
LECTURE 1: INTRODUCTION TO BIFURCATION THEORY

MOTIVATING PROBLEM: BUCKLING UNDER COMPRESSIVE STRESS



$x \in [0, L]$

$W(x)$ = DEFLECTION AT x

E = ELASTICITY CONSTANT

P = LOAD

EULER (1757):

$$\boxed{E \frac{d^2 W}{dx^2} + PW = 0}$$

- GENERAL SOLUTION:

$$W(x) = A \sin(\lambda x) + B \cos(\lambda x), \qquad \lambda = \sqrt{\frac{P}{E}}$$

- BOUNDARY CONDITIONS: PINNED ENDS

$$W(0) = 0 \implies B = 0$$

$$W(L) = 0 \implies A \sin(\lambda L) = 0 \overset{A \neq 0}{\implies} \lambda = \frac{n\pi}{L}, \quad n \in \mathbb{N}$$

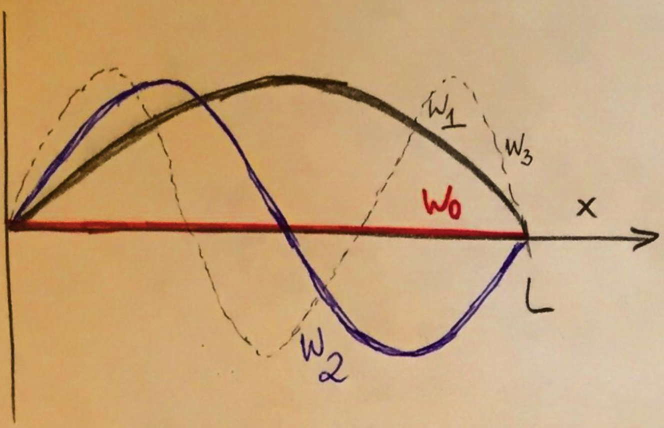
- SO THE NUMBER OF SOLUTIONS (BUCKLING MODES) DEPENDS ON HOW LARGE THE LOAD P IS:

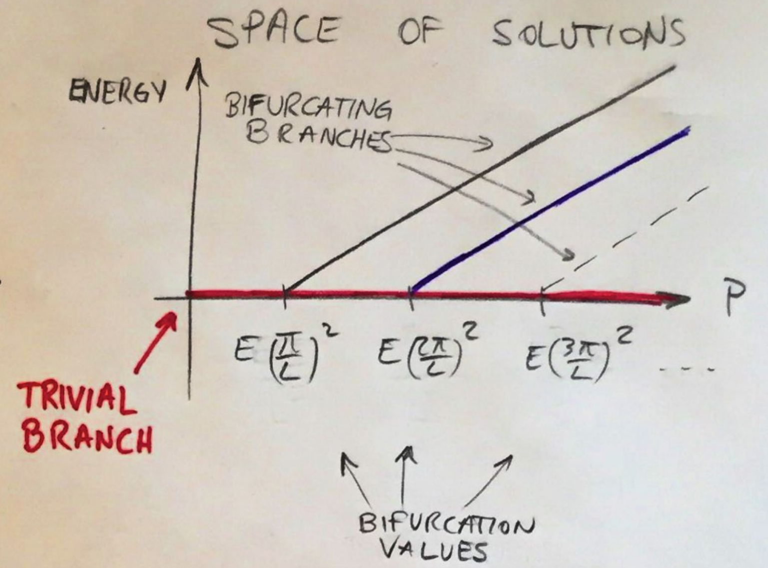
"CRITICAL LOAD"

$$P < E\left(\frac{\pi}{L}\right)^2 \implies \lambda = \sqrt{\frac{P}{E}} < \frac{\pi}{L} \implies W_0 \equiv 0 \quad \left(\begin{array}{l}\text{ONLY SOLUTION}\\ \text{IS TRIVIAL}\end{array}\right)$$

$$P \geq E\left(\frac{n\pi}{L}\right)^2 \implies \boxed{W_j(x) = A \sin(\lambda_j x), \quad \lambda_j = \frac{j\pi}{L}, \quad 0 \leq j \leq n}$$

(n NONTRIVIAL SOLUTIONS)





POINCARÈ (1885): "TOPOLOGICAL CHANGE IN THE STRUCTURE OF A DYNAMICAL SYSTEM WHEN A PARAMETER CROSSES A BIFURCATION VALUE".

IN APPLIED SCIENCES:

- BUCKLING OF STRUCTURES
- CURRENT OSCILLATIONS IN ELECTRIC CIRCUITS
- VORTICES & TURBULENCE IN FLUID DYNAMICS
- PHASE SEPARATION IN FLUIDS

IN MATHEMATICS:

$$\boxed{\text{TRIVIAL SOLUTIONS}} \xrightarrow{\text{"INSTABILITY"}} \boxed{\text{NEW SOLUTIONS}}$$

W_0 $\qquad$ $W_1, W_2, \dots$

- $X = \{\text{"STATES"}\}$ OR $\{\text{"CONFIGURATIONS"}\}$
- $f_t : X \longrightarrow \mathbb{R}$ 1-PARAMETER FAMILY OF FUNCTIONALS
- EULER-LAGRANGE EQUATION $df_t(x) = 0$
- $x_t \in X$ TRIVIAL BRANCH OF SOLUTIONS: $df_t(x_t) = 0$

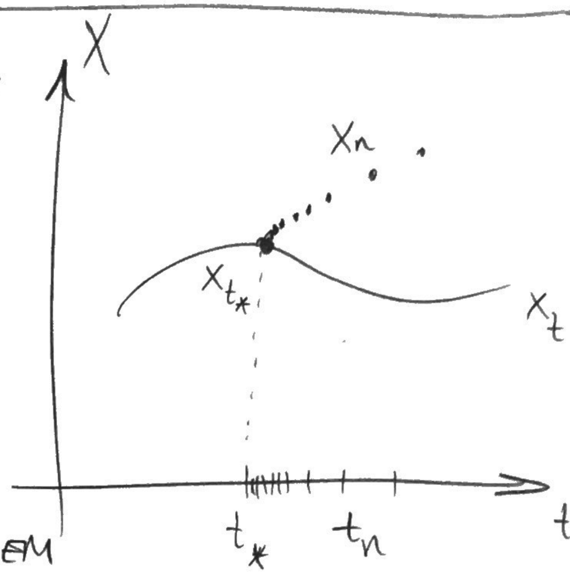
"GROUND STATE", TYPICALLY MINIMIZES ENERGY AND IS THE STATE OBSERVED IN NATURE ("PRINCIPLE OF LEAST ACTION")

DEF: BIFURCATION OCCURS AT t_* IF:

$\exists\, t_n \to t_*$, $\exists\, x_n \to x_{t_*}$ SUCH THAT

- $df_{t_n}(x_n) = 0$
- $x_n \neq x_{t_n}$

I.E. THE IMPLICIT FUNCTION THEOREM FAILS AT x_{t_*}.



$\Longrightarrow$ NECESSARY CONDITION (BUT NOT SUFFICIENT): x_{t_*} IS A DEGENERATE CRITICAL PT

$$\left(\operatorname{Ker} d^2 f_{t_*}(x_{t_*}) \neq \{0\}\right)$$

TO STATE A SUFFICIENT CONDITION:

$$i_{\text{Morse}}(x) = \# \operatorname{Spec}\left(d^2 f_t(x)\right) \cap (-\infty, 0) \qquad \text{MORSE INDEX}$$

THM (KRASNOSEL'SKII). IF $d^2 f_t$ IS FREDHOLM (OF INDEX 0) AND $\exists\, a < b$ SUCH THAT x_a AND x_b ARE NONDEGENERATE AND

$$i_{\text{Morse}}(x_a) \neq i_{\text{Morse}}(x_b)$$

THEN $\exists\, t_* \in (a,b)$ A BIFURCATION INSTANT.

PROOF RELIES ON DEGREE THEORY (TOPOLOGICAL METHODS)

REVISITING BUCKLING EXAMPLE:

- $X = W_0^{1,2}([0,L]) = \overline{C_c^\infty([0,L])}^{W^{1,2}}$
- $f_P(w) = \frac{1}{2}\int_0^L E(w')^2 - Pw^2 \, dx$
- $df_P(w)u = \int_0^L Ew'u' - Pwu \, dx = -\int_0^L (Ew'' + Pw)\, u \, dx$

- $df_P(w) = 0 \iff Ew'' + Pw = 0 \longleftarrow$ NOTE: EULER-LAGRANGE EQUATION IS ALREADY LINEAR (USUALLY NOT!)

- $d^2 f_P(w)(\phi,\phi) = -\int_0^L (E\phi'' + P\phi)\,\phi\,dx = \langle J_P(\phi), \phi \rangle_{L^2}$

$$J_P = -E\frac{d^2}{dx^2} - P \qquad \text{"JACOBI OPERATOR"}$$

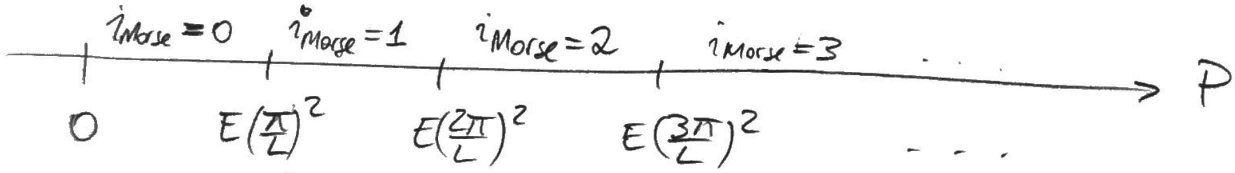
USE L^2-PAIRING TO COMPUTE OPERATOR, AND INDEX AND NULLITY AGREE WITH OPERATOR USING $W_0^{1,2}$-PAIRING

- $i_{Morse}(w_0) = \#\, \mathrm{Spec}(J_P) \cap (-\infty, 0)$

$w_0 \equiv 0$ IS THE TRIVIAL SOLUTION

$$= \# \left\{ E\left(\frac{j\pi}{L}\right)^2 - P : j \in \mathbb{N} \right\} \cap (-\infty, 0)$$

$$= \# \left\{ j \in \mathbb{N} : P \geq E\left(\frac{j\pi}{L}\right)^2 \right\}$$



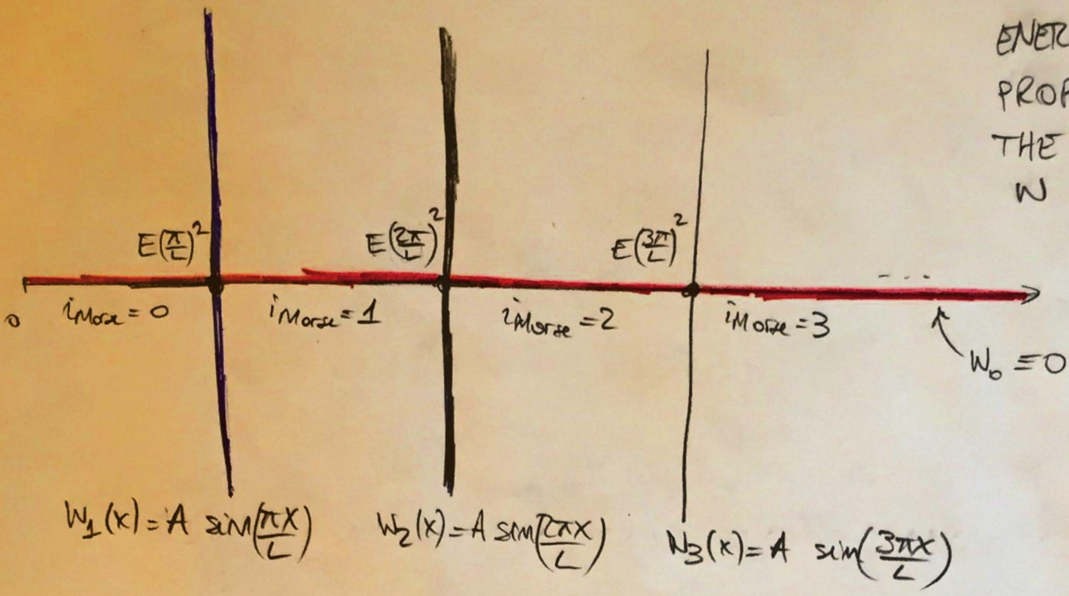
BIFURCATION INSTANTS $P_j = E\left(\frac{j\pi}{L}\right)^2$

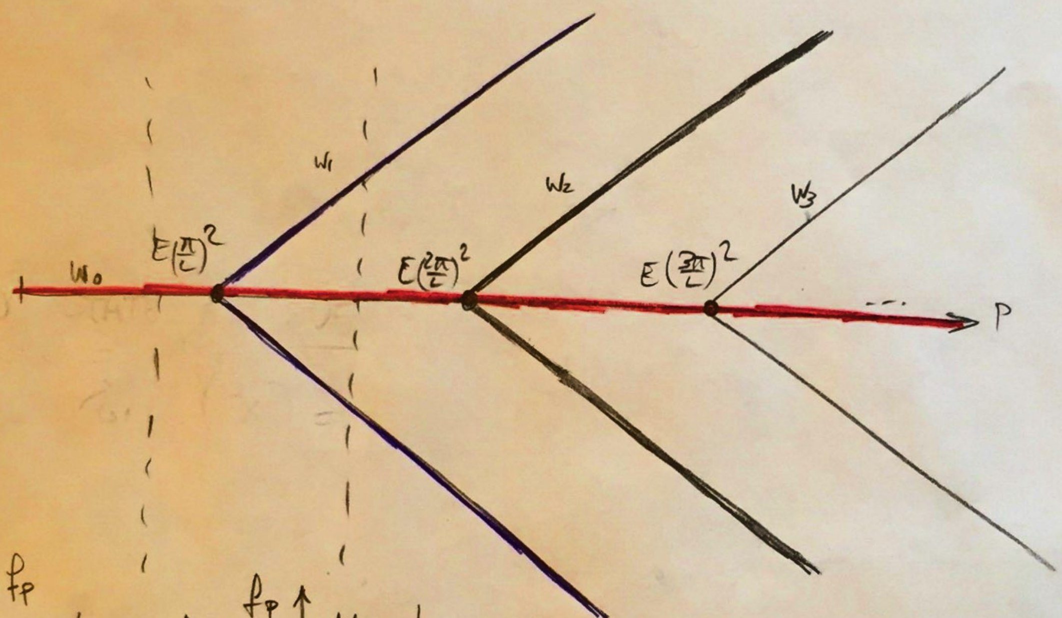
BIFURCATING BRANCHES: $w_j(x) = \pm \sin\left(\frac{j\pi x}{L}\right)$

NOTE: BIFURCATING BRANCHES ARE <u>CONTINUOUS CURVES</u>, NOT JUST SEQUENCES ACCUMULATING AT $P_j = E\left(\frac{j\pi}{L}\right)^2$.

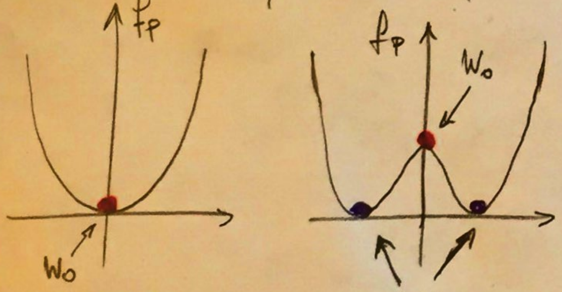
<u>THM</u> (CRANDALL-RABINOWITZ). IF $d^2 f_t$ IS FREDHOLM (OF INDEX ZERO) x_t, $t \in [t_* - \varepsilon, t_* + \varepsilon]$, IS NONDEGENERATE IF $t \neq t_*$, $\operatorname{Ker} d^2 f_{t_*}(x_{t_*}) = \operatorname{span}\{v_*\}$ AND $\frac{d}{dt} d^2 f_t(x_t)(v_*, v_*)\Big|_{t = t_*} \neq 0$, THEN THE SOLUTIONS OF $df_t(x) = 0$ ON A NEIGHBORHOOD $U \ni (t_*, x_{t_*})$ FORM 2 CURVES INTERSECTING ONLY AT (t_*, x_{t_*}).

ENERGY $f_P(w)$ IS PROPORTIONAL TO THE DISTANCE FROM w TO TRIVIAL BRANCH





VERTICAL LINES ARE LEVELSETS OF THE PARAMETER P (LOAD)



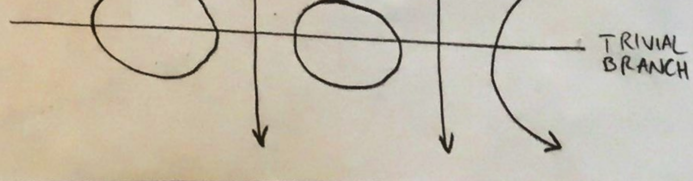
$P < E\left(\frac{\pi}{L}\right)^2$ $\qquad$ $P > E\left(\frac{\pi}{L}\right)^2$

BIFURCATIONS FROM SIMPLE EIGENVALUE CROSSINGS ARE "GLOBAL": EITHER

(1) BIFURCATING BRANCH IS UNBOUNDED

(2) BIFURCATING BRANCH REATTACHES TO TRIVIAL BRANCH

E.G.



APPLICATION TO CONJUGATE POINTS ALONG GEODESICS:

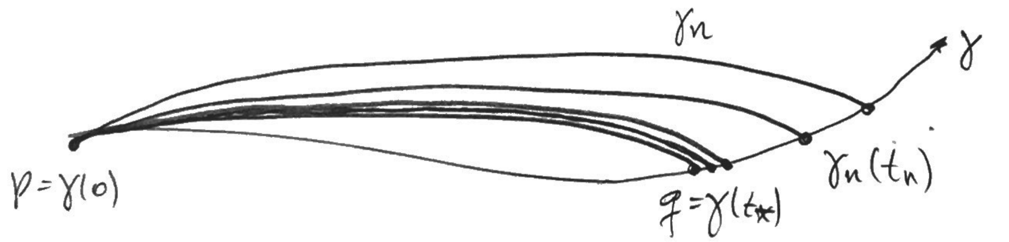
(M,g) RIEM. MFLD, $\gamma:[0,L]\to M$ GEODESIC, $|\dot\gamma|=1$.

$p=\gamma(0)$ AND $q=\gamma(t_*)$ CONJUGATE ALONG γ; THAT IS, $\exists J$ JACOBI FIELD ALONG γ, WITH $J\not\equiv 0$, $J(0)=0$ AND $J(t_*)=0$

I.E., $d(\exp_p)$ NOT INJECTIVE AT $\exp_p^{-1}(q)$

<u>Q</u>: IS $\exp_p$ NOT INJECTIVE NEAR $t_*\dot\gamma(0)$?

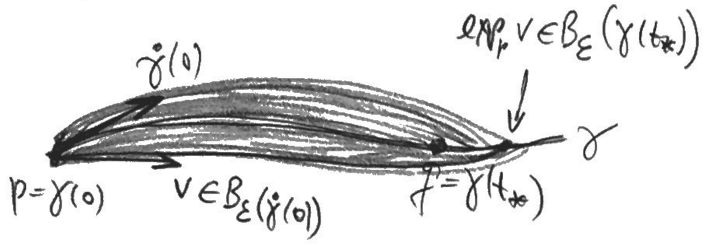
<u>THM</u> (MORSE – LITTAUER 1932, WARNER 1965). YES! MORE PRECISELY:
$\exists\, \gamma_n:[0,L]\to M$ SEQUENCE OF GEODESICS, $\gamma_n\to\gamma$, AND $\exists\, t_n\in[0,L]$, $t_n\to t_*$, S.T. $\gamma_n(0)=\gamma(0)=p$, $\gamma_n(t_n)=\gamma(t_n)\to\gamma(t_*)=q$



<u>PF</u>:
- $X_t=\{\alpha:[0,t]\to M : \alpha(0)=\gamma(0)=p,\ \alpha(t)=\gamma(t),\ \alpha\in W^{1,2}\}$
- $f_t:X_t\to\mathbb{R}$, $f_t(\alpha)=\frac{1}{2}\int_0^t g(\dot\alpha,\dot\alpha)\,ds$
- $\alpha\in X_t$, $df_t(\alpha)=0 \iff \alpha$ GEODESIC FROM $\alpha(0)=\gamma(0)=p$ TO $\alpha(t)=\gamma(t)$.
- α NONDEGENERATE $\iff \alpha(0)=p$ NOT CONJUGATE TO $\alpha(t)$ ALONG α
- $i_{Morse}(\alpha)=\#\{$CONJUGATE POINTS ALONG α, COUNTED W/ MULTIPLICITY$\}$
- $i_{Morse}(\gamma|_{[0,t]})$ JUMPS AT $t=t_*$ (BY THE MULTIPLICITY OF $q=\gamma(t_*)$)

• CONCLUSION FOLLOWS FROM KRASNOSEL'SKII'S THM. $\square$

REMARK: IF $\dim M = 2$, THEN ALL CONJUGATE POINTS HAVE MULTIPLICITY 1; SO CRANDALL-RABINOWITZ APPLIES:



$\exists H(s,t)$ HOMOTOPY:

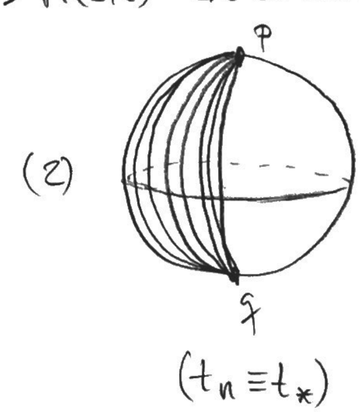
- $H(0,t) = \gamma(t)$
- $t \mapsto H(s,t)$ GEODESIC
- $H(s,0) = p$, $\forall s$
- $H(s,t(s)) = \gamma(t)$, $\forall s$

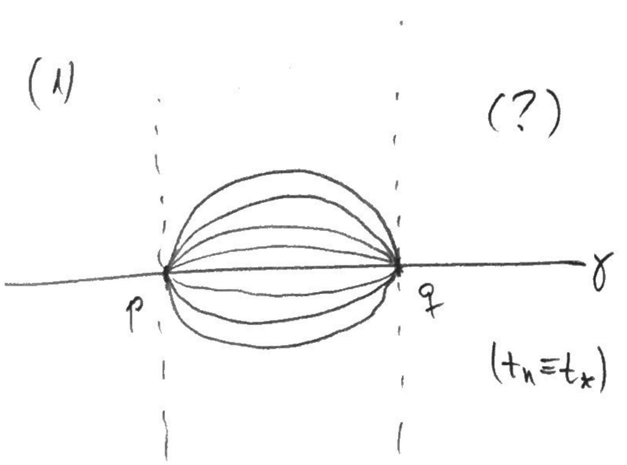
CONSEQUENCE OF GLOBAL BIFURCATION: EITHER

(1) $H(s,t)$ EXISTS FOR ALL $s \in \mathbb{R}$ (UNBOUNDED BRANCH), I.E., $\exists H : \mathbb{R}^2 \to M$ IMMERSION, $t \mapsto H(s,t)$ GEODESICS.

OR:

(2) $H(s,t)$ EVENTUALLY REATTACHES TO TRIVIAL BRANCH γ, I.E., $\exists H : S^2 \to M$ IMMERSION, $t \mapsto H(s,t)$ GEODESICS.
(S^2 OR $\mathbb{RP}^2$)





APPLICATION TO SEMILINEAR ELLIPTIC PDEs:

$$(\mathrm{PDE}) \quad \begin{cases} \Delta u(x) + t f(u(x)) = 0 & \text{in } B^n \\ \alpha u(x) - t\beta \dfrac{\partial u}{\partial \nu}(x) = 0 & \text{on } \partial B^n \end{cases}$$

- LOOK FOR <u>RADIAL SOLUTIONS</u> $u = u(r)$, $r = |x|$, I.E. INVARIANT UNDER $O(n) \curvearrowright B^n$.

$$(\mathrm{ODE}) \quad \begin{cases} u''(r) + \dfrac{n-1}{r} u'(r) + t f(u(r)) = 0 \\ u'(0) = 0 = \alpha u(1) - \beta u'(1) \end{cases}$$

- IF $f(u)$ SATISFIES CERTAIN CONDITIONS (E.G., $f(u) = \sin u$), THEN $\exists\, u_t$ SOLUTION OF (ODE) $\forall t \geq t_0$

> <u>THM</u> (SMOLLER-WASSERMAN, 1990). THERE ARE INFINITELY MANY BIFURCATIONS FROM u_t AS $t \nearrow +\infty$ BY <u>NONRADIAL SOLUTIONS</u>

"SYMMETRY-BREAKING"

<u>EQUIVARIANT BIFURCATION</u> [SMOLLER-WASSERMAN, 1990; B.-PICCIONE-SICILIANO, 2014]

- $G \curvearrowright X$, $f: X \to \mathbb{R}$ G-INVARIANT, x_t HAS CONSTANT ISOTROPY $H < G$ ([SW] REQUIRE $H = G$)
- $H \curvearrowright T_{x_t} X$ ISOTROPY REPRESENTATION

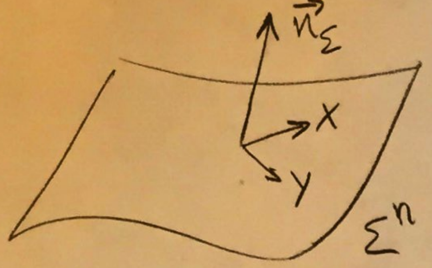
NOTE: $i_{Morse}(x_t) = \dim(E_t^-)$

$E_t^\lambda = \{\lambda\text{-EIGENSPACE OF } d^2 f_t(x_t)\}$, $\quad E_t^- = \bigoplus_{\lambda < 0} E_t^\lambda \subset T_{x_t} X$

> <u>THM</u>: IF $\exists\, a < b$ SUCH THAT x_a AND x_b ARE EQUIVARIANTLY NONDEGENERATE (I.E., $\operatorname{Ker} d^2 f_t(x_t) = T_{x_t} G(x_t)$ FOR $t = a$, $t = b$) AND $H \curvearrowright E_a^-$ NOT ISOMORPHIC TO $H \curvearrowright E_b^-$, THEN $\exists\, t_* \in (a,b)$ BIFURCATION INSTANT FOR x_t.

LECTURE 2: BIFURCATING CONSTANT MEAN CURVATURE HYPERSURFACES

RECALL: $\Sigma^n \hookrightarrow M^{n+1}$



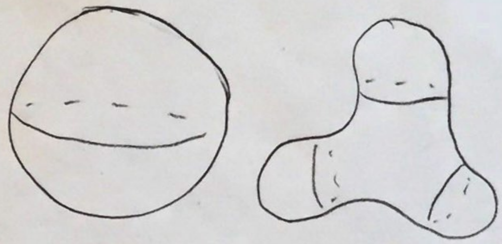
$$A(X,Y) = -g(\nabla_X \vec{n}_\Sigma, Y)$$

$$H = \operatorname{tr} A$$

DEF: Σ^n HAS CONSTANT MEAN CURVATURE (CMC) IF $H \equiv$ const.

MOTIVATION (ISOPERIMETRIC PROBLEM). WHAT HYPERSURFACE $\Sigma^n \subset M^{n+1}$ HAS LEAST AREA AMONG THOSE ENCLOSING A FIXED VOLUME?

$X = \mathrm{Emb}(\Sigma, M)$, E.G., $X = \mathrm{Emb}(S^2, \mathbb{R}^3)$



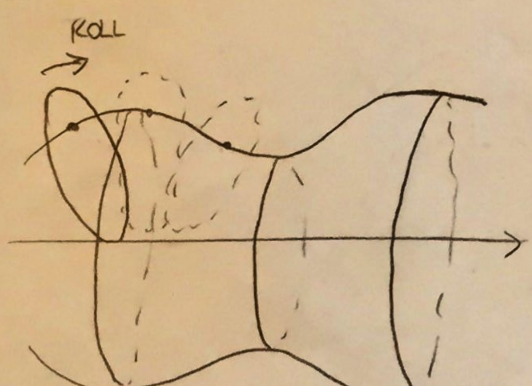
$f_H : X \to \mathbb{R}$

LAGRANGE MULTIPLIER

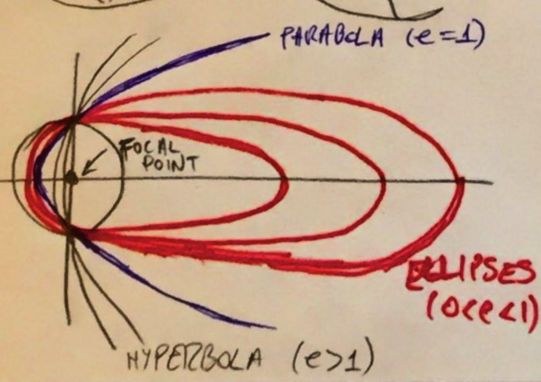
$f_H(x) = \mathrm{Area}(x) + H \cdot \mathrm{Vol}(x)$

$df_H(x) = 0 \iff x : \Sigma \hookrightarrow M$ HAS CONSTANT MEAN CURVATURE H.

THM (DELAUNAY, 1841). A SURFACE OF REVOLUTION $\Sigma^2 \subset \mathbb{R}^3$ HAS CMC IF AND ONLY IF ITS PROFILE CURVE IS THE ROULETTE OF A CONIC SECTION.



CONIC	SURFACE
ELLIPSE	UNDULOID
PARABOLA	CATENOID
HYPERBOLA	NODOID
—	SPHERE
CIRCLE	CYLINDER



CONIC OF ECCENTRICITY e: $r = \dfrac{\ell}{1 + e\cos\theta}$

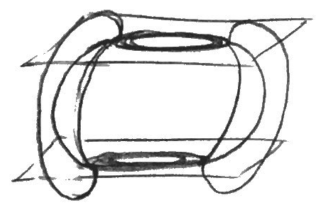
THM (MAZZEO-PACARD, 2002) THERE ARE INFINITELY MANY FAMILIES OF CMC SURFACES IN $\mathbb{R}^3$ THAT BIFURCATE FROM NODOIDS AS THEIR ECCENTRICITY GOES TO $+\infty$.

(ISOMETRY OF TRIVIAL BRANCH: $S^1 \times \mathbb{Z}$
ISOMETRY OF BIFURCATING BRANCH: $\mathbb{Z}_m \times \mathbb{Z}$)

NOTE: (SYMMETRY-BREAKING) BIFURCATING SURFACES ARE NOT OF REVOLUTION!

THM (KOISO-PALMER-PICCIONE, 2015) THERE ARE INFINITELY MANY FAMILIES OF CMC SURFACES IN $\mathbb{R}^3$, WITH BOUNDARY ON TWO FIXED COAXIAL CIRCLES, THAT BIFURCATE FROM (PORTIONS OF) NODOIDS AS THEIR CONORMAL ANGLE VARIES.

OTHER WORK BY:

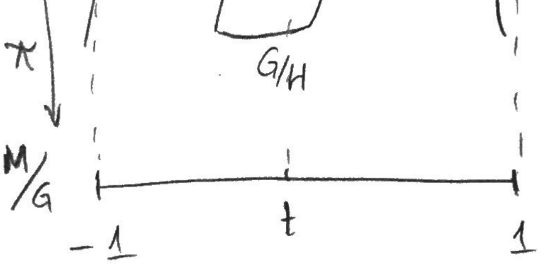


- K. GROSSE-BRAUCKMANN - HE [2007]
- W. ROSSMAN [2005, 2008].

Q. DELAUNAY HYPERSURFACES IN AMBIENT SPACES OTHER THAN $\mathbb{R}^3$?

- CLASSICAL CONSTRUCTION IN S^n, $\mathbb{R}^n$, $\mathbb{H}^n$.
- MORE GENERAL CONSTRUCTION USING BIFURCATION FROM ORBITS!

INTERMEZZO ON COHOMOGENEITY ONE MANIFOLDS:



$M = (G \times_{K_-} D_-) \cup_{G/H} (G \times_{K_+} D_+)$

- $G \curvearrowright M$ ISOMETRIC ACTION
- $\dim M/G = 1 \Rightarrow M/G = \begin{cases} [-1,1] \ * \\ S^1 \\ [0,+\infty) \\ \mathbb{R} \end{cases}$
- $\gamma : [-1,1] \to M$ HORIZONTAL GEODESIC
  $\pi(\gamma(t)) = t$
- $K_\pm = G_{\gamma(\pm 1)}$; $H = G_{\gamma(t)}$, $t \in (-1,1)$
- $\Sigma_\pm = G/K_\pm$ "SINGULAR ORBITS" ; $\Sigma_t = \pi^{-1}(t) \cong G/H$ "PRINCIPAL ORBITS"

G-EQUIVARIANCE $\Longrightarrow$ $\begin{cases} \Sigma_t \text{ ARE } \underline{\text{CMC HYPERSURFACES}} \\ (\Sigma_\pm \text{ ARE } \underline{\text{MINIMAL}}) \end{cases}$

"TRIVIAL BRANCH"

$\boxed{\text{}}$

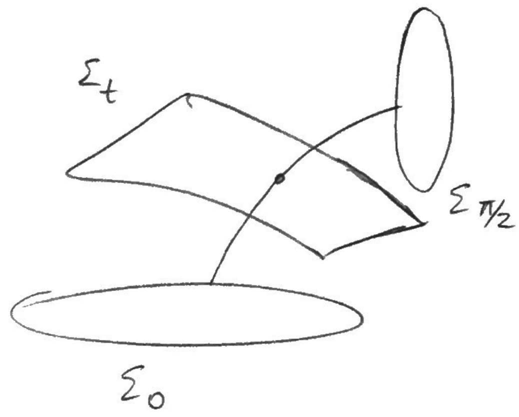
<u>THM</u> (B. – PICCIONE, 2015). IF $\Sigma_+ = G/K_+$ IS A SINGULAR ORBIT ON A COHOMOGENEITY ONE MANIFOLD M, WITH $\Sigma_+ \neq \{pt\}$ AND EITHER $H \triangleleft K_+$ OR $K_+ \triangleleft G$, THEN THERE ARE INFINITELY MANY FAMILIES OF CMC HYPERSURFACES IN M (DIFFEOMORPHIC TO G/H) BIFURCATING FROM Σ_t AS $t \to 1$

<u>NOTE</u>: (PARTIAL SYMMETRY BREAK): BIFURCATING HYPERSURFACES ARE NOT G-INVARIANT BUT REMAIN K_+-INVARIANT.

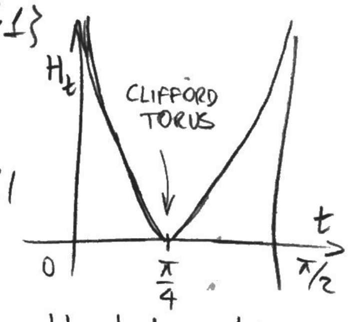
<u>EXAMPLES</u>: $M = S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$

$G = T^2 \curvearrowright S^3$: $(e^{i\theta}, e^{i\varphi})(z,w) = (e^{i\theta} z, e^{i\varphi} w)$

$\Sigma_t = \{(z,w) \in S^3 : |z| = \cos t,\ |w| = \sin t\}$, $t \in [0, \frac{\pi}{2}]$



- $K_0 = \{1\} \times S^1$, $K_{\pi/2} = S^1 \times \{1\}$
  $H = \{1\}$



$H_t = \tan t - \cot t$

- DELAUNAY-TYPE TORI ("UNDULOIDS") BIFURCATE FROM Σ_t AS $t \searrow 0$ AND $t \nearrow \pi/2$

<u>NOTE</u>: [ANDREWS – LI, 2015] PROVED THESE ARE THE ONLY EMBEDDED CMC TORI IN S^3.

- ISOMETRY GROUP OF BIFURCATING SURFACES IS $S^1 \times \mathbb{Z}_m$, WHERE

$$\cot \frac{\pi}{m} < H < \frac{m^2 - 2}{2\sqrt{m^2 - 1}}$$

MORE GENERALLY: $M = S^{n+1} \subset \mathbb{R}^{n+2} = \mathbb{R}^{k+1} \oplus \mathbb{R}^{n-k+1}$

$$G = SO(k+1) \times SO(n-k+1)$$

$$\Sigma_t = S^k \times S^{n-k}$$

- NORMALITY ASSUMPTION SATISFIED IF $k=1$ OR $k=n-1$
- RESULT REMAINS VALID FOR ALL k BY [ALIAS–PICCIONE, 2013]

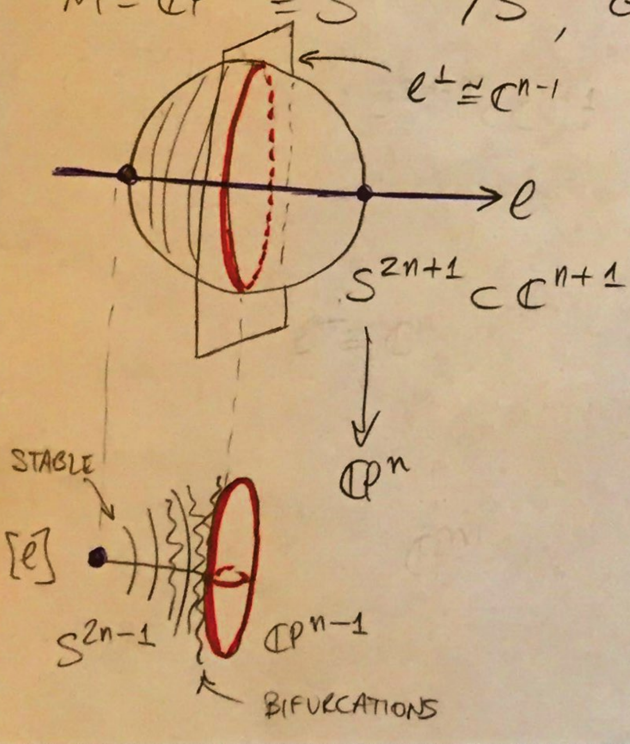
NEW FAMILIES OF DELAUNAY-TYPE HYPERSURFACES ON SPHERES:

(MAYBE SKIP?)

M	G	Σ	
S^{2k+3}	$S^1 \cdot SO(k+2)$	$(S^1 \times T_1 S^{k+1})/\mathbb{Z}_2$	"ESSENTIAL ACTIONS"
S^{15}	$S^1 \cdot Spin(7)$	$(S^1 \times T_1 S^7)/\mathbb{Z}_2$	
S^{13}	$S^1 \cdot G_2$	$(S^1 \times T_1 S^6)/\mathbb{Z}_2$	
S^{k+1}	$S^1 \cdot SO(k)$	$S^1 \times S^{k-1}$	"NON-ESSENTIAL ACTIONS"
S^{2k+1}	$U(1) \cdot U(k)$	$S^1 \times S^{2k-1}$	
S^{4k+3}	$Sp(1) \cdot Sp(k)$	$S^3 \times S^{4k-1}$	

DELAUNAY-TYPE SPHERES IN PROJECTIVE SPACES $\mathbb{C}P^n$ AND $\mathbb{H}P^n$:

$M = \mathbb{C}P^n = S^{2n+1}/S^1, \quad G = U(n)$



- PRINCIPAL ORBITS:

  $\Sigma_t = S^{2n-1}$ ARE DISTANCE SPHERES AROUND THE FIXED POINT OF THE $U(n)$-ACTION ON $\mathbb{C}P^n$

- BIFURCATING CMC HYPERSURFACES:

  $S^{2n-1} \subset \mathbb{C}P^n$, BRANCHES ISSUE NEAR $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$
- RECALL ISOPERIMETRIC PROBLEM IS OPEN ON $\mathbb{C}P^n$, $n \geq 2$.

DELAUNAY-TYPE HYPERSURFACES IN KERVAIRE SPHERES.

$$M_d^{2n-1} = \{ z \in \mathbb{C}^{n+1} : z_0^d + z_1^2 + \dots + z_n^2 = 0 \text{ AND } |z_0|^2 + \dots + |z_n|^2 = 1 \}$$

- n AND d ODD $\Rightarrow M_d^{2n-1} \underset{\text{HOMEO}}{\cong} S^{2n-1}$
- $2n-1 \equiv 1 \mod 8 \Rightarrow M_d^{2n-1} \underset{\text{DIFFEO}}{\not\cong} S^{2n-1}$ (EXOTIC SPHERE)

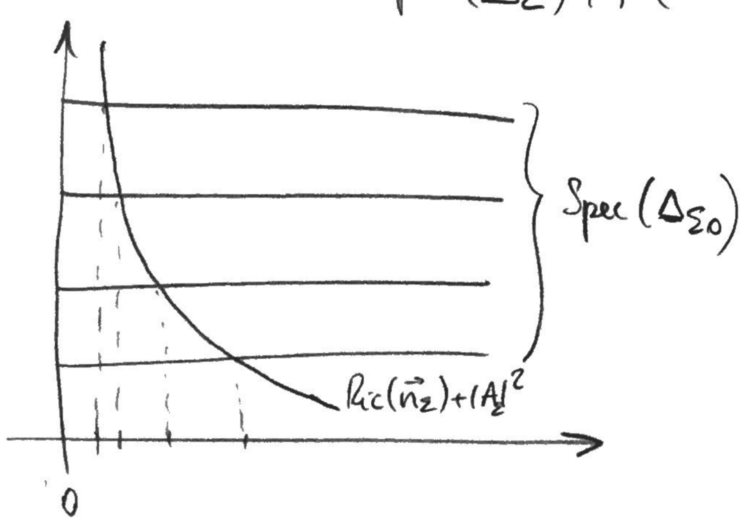
$G = S^1 \cdot SO(n)$, $\quad (e^{i\theta}, A)(z_0, \dots, z_n) = (e^{2i\theta} z_0, A(z_1, \dots, z_n))$

PRINCIPAL ORBITS: $\quad \Sigma \cong (S^1 \times T_1 S^{n-1}) / \mathbb{Z}_2$

BIFURCATING CMC HYPERSURFACES: WHEN Σ COLLAPSE TO $T_1 S^{n-1}$

SKETCH OF PROOF:

- $f_H(x) = \mathrm{Area}(x) + H \cdot \mathrm{Vol}(x)$
- $df_H(x) = 0 \iff x(\Sigma) \subset M$ HAS CONSTANT MEAN CURVATURE $= H$.
- $d^2 f_H(x)(\psi, \psi) = \displaystyle\int_\Sigma \Big(\underbrace{\Delta_\Sigma \psi - \big(\mathrm{Ric}(\vec{n}_\Sigma) + |A_\Sigma|^2 \big) \psi}_{J_\Sigma \psi} \Big) \psi \, d\Sigma$
- $i_{\mathrm{Morse}}(x) = \# \, \mathrm{Spec}(\Delta_\Sigma) \cap (-\infty, \mathrm{Ric}(\vec{n}_\Sigma) + |A_\Sigma|^2)$

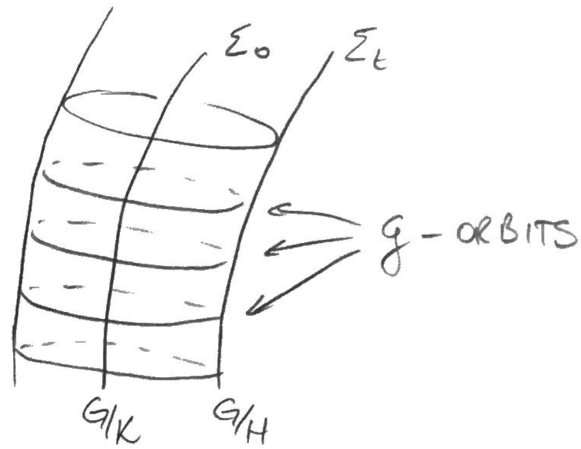


- Σ_0 = SINGULAR ORBIT

  $(\Sigma_t)_{t>0}$ = PRINCIPAL ORBITS
- $\mathrm{Spec}(\Delta_{\Sigma_0}) \subset \mathrm{Spec}(\Delta_{\Sigma_t})$, $t > 0$
- $\Sigma_t \to \Sigma_0$ COLLAPSE AS $t \downarrow 0$

  $\Rightarrow H_t \nearrow +\infty \Rightarrow |A_{\Sigma_t}|^2 \nearrow +\infty$

PROBLEM: HOW TO RULE OUT COMPENSATIONS/DEGENERACY?



A: USE EQUIVARIANT BIFURCATION

$$H \triangleleft K \Rightarrow \mathcal{G} = K/H \curvearrowright M$$

$$K \triangleleft G \Rightarrow \mathcal{G} = K \curvearrowright M$$

$\mathcal{G}$ - ORBITS ARE FIBERS OF

$$\left.\begin{matrix} K/H & \longrightarrow & G/H & \longrightarrow & G/K \\ \| & & \| & & \| \\ S^\perp & & \Sigma_t & & \Sigma_0 \end{matrix}\right\}$$

RIEMANNIAN SUBMERSION WITH TOTALLY GEODESIC FIBERS (WHEN METRIC IS "ADAPTED")

- $\mathrm{Emb}^{\mathcal{G}}(\Sigma, M) = \{ x \in \mathrm{Emb}(\Sigma, M) : \mathcal{G} \cdot x(\Sigma) = x(\Sigma) \}$
- PALAIS SYMMETRIC CRITICALITY PRINCIPLE:

$$d\left(f_H\big|_{\mathrm{Emb}^{\mathcal{G}}(\Sigma,M)}\right)(x) = 0 \iff df_H(x) = 0 \quad \text{AND} \quad x \in \mathrm{Emb}^{\mathcal{G}}(\Sigma, M).$$

- EQUIVARIANT MORSE INDEX:

INDEPENDENT OF t

$$i^{\mathcal{G}}_{Morse}(x) = \# \, \mathrm{Spec}(\Delta_{\Sigma_0}) \cap \left(-\infty, \mathrm{Ric}(\vec{n}_\Sigma) + |A_\Sigma|^2\right)$$

- $\boxed{\text{KRASNOSEL'SKII THM APPLIED TO } f_H : \mathrm{Emb}^{\mathcal{G}}(\Sigma,M) \to \mathbb{R}}$ $\Longrightarrow$ $\boxed{\exists\, t_n \searrow 0 \ (H_n \nearrow +\infty) \text{ SEQUENCE OF BIFURCATION INSTANTS FOR } \Sigma_t \text{ (AND BIFURCATING CMC HYPERSURFACES ARE } \mathcal{G}\text{-INVARIANT!)}}$

$\square$

LECTURE 3: BIFURCATING SOLUTIONS TO THE YAMABE PROBLEM

YAMABE PROBLEM: GIVEN A CLOSED RIEMANNIAN n-MANIFOLD (M^n, g_0), FIND A CONFORMAL METRIC $g \in [g_0]$ WITH $\mathrm{scal}_g \equiv \mathrm{const}$

PDE SETUP:

$$g = u^{\frac{4}{n-2}} g_0, \quad u : M \to \mathbb{R}, \ u > 0$$

$$\underbrace{4\frac{n-1}{n-2}\Delta_{g_0} u + \mathrm{scal}_{g_0} \cdot u}_{L_{g_0}(u)} = \mathrm{scal}_g \cdot u^{\frac{n+2}{n-2}}$$

$\frac{n+2}{n-2}$: CRITICAL SOBOLEV EXPONENT

$L_{g_0}(u)$: "CONFORMAL LAPLACIAN"

VARIATIONAL SETUP:

- $X = [g_0]_1 = \{ g \in [g_0] : \mathrm{Vol}(M, g) = 1 \} \stackrel{g = u^{\frac{4}{n-2}} g_0}{=} \{ u : M \to \mathbb{R}, u > 0, \int_M u^{\frac{2n}{n-2}} = 1 \}$.
- $f : X \longrightarrow \mathbb{R}$ HILBERT-EINSTEIN FUNCTIONAL

$$f(g) = \int_M \mathrm{scal}_g \cdot \mathrm{vol}_g = \int_M u \cdot L_{g_0} u \ \mathrm{vol}_{g_0}$$

- $df(g) = 0 \iff \mathrm{scal}_g \equiv \mathrm{const}.$
- $d^2 f(g)(\psi, \psi) = \frac{(n-1)(n-2)}{2} \int_M \underbrace{\left(\Delta_g \psi - \frac{\mathrm{scal}_g}{n-1} \psi\right)}_{J_g \psi} \psi \ \mathrm{vol}_g$

$J_g \psi$ "JACOBI OPERATOR"

- $i_{\mathrm{Morse}}(g) = \# \, \mathrm{Spec}(\Delta_g) \cap \left(-\infty, \frac{\mathrm{scal}_g}{n-1}\right)$

THM (YAMABE, 1960; TRUDINGER, 1968; AUBIN, 1976; SCHOEN, 1984). EVERY CLOSED MANIFOLD (M^n, g_0) ADMITS A CONFORMAL METRIC $g \in [g_0]$ WITH CONSTANT SCALAR CURVATURE $\mathrm{scal}_g \equiv \mathrm{const}$,

NOTE: PROOF ESTABLISHES EXISTENCE OF A MINIMIZER FOR $f: X \to \mathbb{R}$

$$Y(M,[g_0]) = \inf_{g \in [g_0]_1} f(g)$$ "YAMABE INVARIANT"

IS ACHIEVED AT SOME $g_Y \in [g_0]_1$, CALLED "YAMABE METRIC".

Q: IS THE SOLUTION UNIQUE? (UP TO HOMOTHETIES)

UNIQUENESS RESULTS:

1. MAXIMUM PRINCIPLE: SOLUTION IS UNIQUE IF $scal_{g_0} \leq 0$.

BASIC FACT: ALL SOLUTIONS HAVE scal OF THE SAME SIGN AS $Y(M,[g_0])$, AND $\lambda_1(L_{g_0})$. SET $R = scal_g = scal_{g_0}$, THEN $4\frac{n-1}{n-2}\Delta_{g_0} u = R \cdot u(u^{\frac{4}{n-2}} - 1)$

$R = 0 \Rightarrow \Delta_{g_0} u = 0 \rightarrow u \equiv const.$

$R < 0 \Rightarrow \begin{cases} \text{AT A MAXIMUM OF } u,\ \Delta_{g_0} u \geq 0 \Rightarrow u \leq 1 \\ \text{AT A MINIMUM OF } u,\ \Delta_{g_0} u \leq 0 \Rightarrow u \geq 1 \end{cases} \Rightarrow u \equiv 1.$

2. THM (OBATA, 1971). IF (M, g_0) IS EINSTEIN AND NOT CONFORMALLY EQUIVALENT TO (S^n, g_{round}), THEN SOLUTION IS UNIQUE.

NON-UNIQUENESS RESULTS:

1. $(M, g_0) = (S^n, g_{round})$

THM (OBATA, 1971). IF (S^n, g) HAS $scal_g \equiv const.$ AND $g \in [g_{round}]$, THEN $g = \phi^* g_{round}$, WHERE $\phi \in Conf(S^n, g_{round})$.

2. $(M, g_t) = (S^{n-1} \times S^1, t\, g_{round} \oplus d\theta^2)$, $t > 0$

THM (O. KOBAYASHI, 1985; R. SCHOEN, 1989). IF $\frac{n-2}{(\ell+1)^2} \leq t < \frac{n-2}{\ell^2}$, $\ell \in \mathbb{N}$, THEN THERE ARE $\ell+1$ SOLUTIONS TO THE YAMABE PROBLEM ON $(S^{n-1} \times S^1, [g_t])$. MORE PRECISELY, EVERY $t_\ell = \frac{n-2}{\ell^2}$ IS A BIFURCATION INSTANT FOR g_t

ANALOGY: $\#$(SOLUTIONS) $\simeq$ $\#$(BUCKLING MODES) $\nearrow +\infty$ AS $scal_{g_t} \approx$ (LOAD) $\nearrow +\infty$
(i.e., $t \searrow 0$)

SKETCH OF PROOF: $S^{n-1} \times \mathbb{R} \cong S^n \setminus \{\pm p\} \cong \mathbb{R}^n \setminus \{0\}$

$$\begin{array}{c} S^{n-1} \times \mathbb{R} \\ \downarrow \\ S^{n-1} \times S^1 \end{array}$$

[CAFFARELLI-GIDAS-SPRUCK] $\Longrightarrow$ ANY SOLUTION SINGULAR AT $\{0\}$ OR $\{\infty\}$ IS SINGULAR AT BOTH $\{0\}$ AND $\{\infty\}$ AND RADIAL: $u = u(r)$

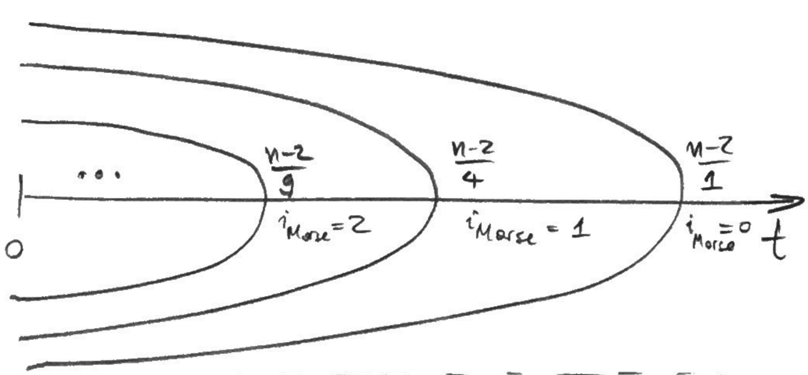
$\Longrightarrow$ YAMABE PDE REDUCES TO ODE:

$$u'' - \frac{(n-2)^2}{4} u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0$$

$\Longrightarrow$ REWRITE AS 1st ORDER AUTONOMOUS SYSTEM AND ANALYZE CRITICAL PTS, PERIODIC ORBITS, ...

BIFURCATION APPROACH:

- $i_{Morse}(g_t) = \# \, Spec(\Delta_{g_t}) \cap \left(-\infty, \frac{scal_{g_t}}{n-1}\right)$
- $scal_{g_t} = \dfrac{(n-1)(n-2)}{t}$
- $\Delta_{g_t} = \frac{1}{t} \Delta_{g_{round}} - \frac{d^2}{d\theta^2} \Longrightarrow Spec(\Delta_{g_t}) = \left\{ \frac{k(n+k-2)}{t} + \ell^2 : k \in \mathbb{N}, \ell \in \mathbb{N} \right\}$
- SIMPLE ANALYSIS: $i_{Morse}(g_t)$ INCREASES BY 1 EXACTLY WHEN $t = \frac{n-2}{\ell^2}$, $k=0$



- CRANDALL-RABINOWITZ: THE BIFURCATING BRANCHES ARE CONTINUOUS.
- TECHNICAL ISSUES: TO GET MULTIPLICITY 1, NEED TO WORK IN A SLICE FOR THE S^1-ACTION ON $[g_t]$, HENCE DO NOT IMMEDIATELY HAVE GLOBAL BIFURCATION

OTHER QUESTIONS: WHY DO BRANCHES EXIST FOR ALL SMALLER VALUES OF $t>0$? WHY DON'T THEY REATTACH? WHY 2 SIDES GIVE ISOMETRIC METRICS?

THM. (B. – PICCIONE, 2013). LET $H < K < G$ BE COMPACT LIE GROUPS WITH $H \triangleleft K$ OR $K \triangleleft G$ AND $\mathrm{scal}_{K/H} > 0$. THEN THE CANONICAL VARIATION $g_t = g|_{\mathrm{hor}} \oplus t\, g|_{\mathrm{ver}}$ ON $K/H \to G/H \to G/K$ HAS INFINITELY MANY BIFURCATION INSTANTS AS $t \searrow 0$.

NOTE: PREVIOUS EXAMPLE CORRESPONDS TO: $\underbrace{SO(n-1)}_{H} < \underbrace{SO(n)}_{K} \triangleleft \underbrace{SO(n) \times S^1}_{G}$

CF. DELAUNAY SURFACES IN COHOM 1 MFLDS!

SKETCH OF PROOF: USE $\mathcal{G}$ - EQUIVARIANT BIFURCATION:

$$\left.\begin{array}{l} H \triangleleft K \Rightarrow \mathcal{G} = K/H \\ K \triangleleft G \Rightarrow \mathcal{G} = K \end{array}\right\} \ \mathcal{G} \curvearrowright G/H \text{ WITH ORBITS EQUAL TO FIBERS OF } K/H \to G/H \to G/K$$

- $i^{\mathcal{G}}_{\mathrm{Morse}}(g_t) = \# \underbrace{\mathrm{Spec}(\Delta_{G/K})}_{\text{INDEPENDENT OF } t} \cap \left(-\infty, \frac{\mathrm{scal}_{g_t}}{n-1}\right) \nearrow +\infty$ AS $t \searrow 0$,

BECAUSE $\mathrm{scal}_{g_t} = \frac{1}{t} \mathrm{scal}_{K/H} + \mathrm{scal}_{G/K} - t|A|^2 \nearrow +\infty$ AS $t \searrow 0$. $\square$

THM (OTOBA - PETEAN, 2016). LET $F \to M \xrightarrow{\pi} B$ BE A HARMONIC RIEM. SUBMERSION WITH $\mathrm{scal}_F > 0$ AND ASSUME THAT THE CANONICAL VARIATION $g_t = g|_{\mathrm{hor}} \oplus t\, g|_{\mathrm{ver}}$ HAS $\mathrm{scal}_{g_t} \equiv \mathrm{const}$ FOR ALL $t > 0$. THEN g_t HAS INFINITELY MANY BIFURCATION INSTANTS AS $t \searrow 0$.

SKETCH OF PROOF. $F \to M \xrightarrow{\pi} B$ HARMONIC RIEM. SUBMERSION:

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\overline{u}"] & \\ B \arrow[r, "u"'] & \mathbb{R} \end{tikzcd} \qquad \overline{\Delta_B u} = \Delta_M \overline{u}$$

- DEFINE $L_t : C^\infty(B) \to C^\infty(B)$

$$L_t(u) = 4\frac{n-1}{n-2} \Delta_B u + \mathrm{scal}_{g_t} u$$

- THEN $L_{g_t}(\overline{u}) = \overline{L_t(u)}$ IS THE CONFORMAL LAPLACIAN ON (M, g_t)

- $g = \bar{u}^{\frac{4}{n-2}} g_t$ HAS $\mathrm{scal}_g \equiv c \iff L_t(u) = c \cdot u^{\frac{n+2}{n-2}}$

$\iff u \in C^\infty(B)$ IS A CRITICAL PT. OF $f : W^{1,2}(B) \to \mathbb{R}$, $f(u) = \int_B u\, L_t(u)$, SUBJECT TO CONSTRAINT $\|u\|_{L^{\frac{2n}{n-2}}(B)} = 1$.

OR: $f(u) = \int_B 2\frac{n-1}{n-2}|Du|^2 + \mathrm{scal}_{g_t}\left(\frac{u^2}{2} - \frac{u^p}{p}\right)$ $\left(p = \frac{2n}{n-2}\right)$ W/O CONSTRAINT

- JACOBI OPERATOR

$$J_t \psi = \Delta_B \psi - \frac{\mathrm{scal}_{g_t}}{n-1}\psi = L_t \psi - \frac{n+2}{n-2}\mathrm{scal}_{g_t} \cdot \psi$$

- $i_{Morse}(g_t) = \# \mathrm{Spec}(\Delta_B) \cap \left(-\infty, \frac{\mathrm{scal}_{g_t}}{n-1}\right) \nearrow +\infty$ AS $t \searrow 0$

BECAUSE $\mathrm{scal}_{g_t} = \frac{1}{t}\underbrace{\mathrm{scal}_F}_{>0} + \mathrm{scal}_B - t|A|^2 \nearrow +\infty$ AS $t \searrow 0$.

- CONCLUSION FOLLOWS FROM KRASNOSEL'SKII THM. $\square$

COROLLARIES:

(1) IF (F, g_F), (B, g_B) ARE CLOSED MANIFOLDS WITH POSITIVE CONSTANT SCALAR CURVATURE, THEN $(B \times F, [g_B \oplus t\, g_F])$ HAS INFINITELY MANY BIFURCATION INSTANTS FOR THE YAMABE PROBLEM AS $t \searrow 0$ AND $t \nearrow +\infty$. CF. [LIMA-PICCIONE-ZEDDA, 2012]

REVERSING ROLES OF F & B.

(2) [B.-PICCIONE, 2013]. THERE ARE INFINITELY MANY BIFURCATION INSTANTS AS $t \searrow 0$ FOR THE BERGER METRICS:

$$\begin{array}{ccccc} tS^3 \to (S^{4n+3}, g_t) & & tS^7 \to (S^{15}, g_t) & & t\,\mathbb{CP}^1 \to (\mathbb{CP}^{2n+1}, g_t) \\ \downarrow & , & \downarrow & , & \downarrow \\ \mathbb{HP}^n & & S^8(1/2) & & \mathbb{HP}^n \end{array}$$

(BUT NOT FOR $\boxed{tS^1} \to (S^{2n+1}, g_t) \to \mathbb{CP}^n$!) $\mathrm{scal} = 0$.

⚠ NORMALITY ASSUMPTIONS NOT SATISFIED.

RMK: IF ONLY DEGENERACIES ARISE FROM "JACOBI FIELDS" WHICH ARE CONSTANT ALONG THE FIBERS, THEN LOCALLY ALL SOLUTIONS ARE ALSO CONSTANT ALONG THE FIBERS.

THM (OTOBA - PETEAN, 2016). LET $F \to M^n \xrightarrow{\pi} B$ BE A HARMONIC RIEM. SUBMERSION WITH $scal_g$ CONSTANT ALONG FIBERS F. ASSUME THAT

$$\Delta_g u - \frac{scal_g}{n-1} = 0$$

ONLY HOLDS FOR $u \in C^\infty(M)$ WHICH ARE CONSTANT ALONG THE FIBERS THEN ALL SOLUTIONS TO THE YAMABE PROBLEM SUFF. CLOSE TO g ARE GIVEN BY CONFORMAL FACTORS CONSTANT ALONG FIBERS.

Pf: "DOUBLE" LYAPUNOV-SCHMIDT REDUCTION.

RMK: • IF $\dim M \leq 24$, $(M^n, g) \not\cong (S^n, g_{round})$, THEN SET OF SOLUTIONS TO YAMABE PROBLEM IN $[g]$ IS COMPACT [KHURI-MARQUES-SCHOEN '09]

• IF $\dim M \geq 25$, THERE ARE COUNTER-EXAMPLES [BRENDLE, BRENDLE-MARQUES '08]

LECTURE 4: BIFURCATING SOLUTIONS TO GENERALIZATIONS OF YAMABE PROBLEM

(GENERALIZED) YAMABE PROBLEM: GIVEN A RIEMANNIAN MANIFOLD (M^n, g_0), FIND A COMPLETE CONFORMAL METRIC $g \in [g_0]$ WITH $\mathrm{scal}_g \equiv \text{const}$.

OTHER GENERALIZATIONS STUDIED BY AKUTAGAWA, BOTVINNIK, ETC., BUT HERE WE SHALL FOCUS ON:

SINGULAR YAMABE PROBLEM: $M = M_0 \setminus \Lambda$, (M_0, g_0) CLOSED MANIFOLD, $\Lambda \subset M_0$ CLOSED SUBSET.

EXISTENCE:
- FAILS IN GENERAL; E.G. $M_0 \setminus \{p_1, \dots, p_k\}$ [JIN, 1988]
- DIMENSION RESTRICTIONS ($n = \dim M_0$):

$\exists$ SOLUTION WITH $\mathrm{scal} < 0$ $\Longrightarrow$ $\dim_H \Lambda > \frac{n-2}{2}$

(IN THIS CASE, UNIQUENESS HOLDS BY ASYMPTOTIC MAX. PRINC.) $\Longleftarrow$ (IF $\Lambda \subset M$ IS A SUBMANIFOLD [LOEWNER-NIRENBERG '74] $M_0 = S^n$ [AVILES-McOWEN, 1988] ANY M_0)

$\exists$ SOLUTION WITH $\mathrm{scal} \geq 0$ AND $M_0 = S^n$ $\Longrightarrow$ $\dim_H \Lambda \leq \frac{n-2}{2}$ [SCHOEN-YAU, '88]

$\Longleftarrow$ (IF $\Lambda = \{p_1, \dots, p_k\}$ [SCHOEN, 1988] OR Λ = DISJOINT UNION OF SUBMANIFOLDS WITH $0 \leq \dim \leq \frac{n-2}{2}$ [MAZZEO-PACARD, 1999])

BIFURCATIONS, NON-UNIQUENESS, AND MODULI SPACE OF SOLUTIONS:

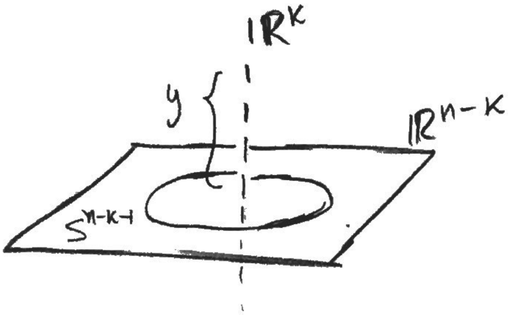
THM (SCHOEN, 1988; MAZZEO-POLLACK-UHLENBECK, 1994). THE SOLUTIONS TO THE SING. YAMABE PROBLEM ON $S^n \setminus \{p_1, \dots, p_k\}$ FORM A k-DIM MANIFOLD.

k "DELAUNAY NECK PARAMETERS"

THM (B.-PICCIONE-SANTORO, 2016). THERE EXIST UNCOUNTABLY MANY BRANCHES OF SOLUTIONS TO THE SINGULAR YAMABE PROBLEM ON $S^n \setminus S^1$, $n \geq 5$, WITH (CONSTANT) SCALAR CURVATURE $\approx (n-4)(n-1)$.

A USEFUL CONFORMAL EQUIVALENCE:

$$\left(S^n \setminus S^k, g_{round}\right) \xrightarrow[\text{projection}]{\text{Stereographic}} \left(\mathbb{R}^n \setminus \mathbb{R}^k, g_{flat}\right) \xrightarrow{\cong} \left(S^{n-k-1} \times \mathbb{H}^{k+1}, g_{round} \oplus g_{hyp}\right)$$



$$g_{flat} = dr^2 + r^2 d\theta^2 + dy^2$$

$$\xrightarrow{\cdot \frac{1}{r^2}} d\theta^2 + \frac{dr^2 + dy^2}{r^2} = g_{round} \oplus g_{hyp}$$

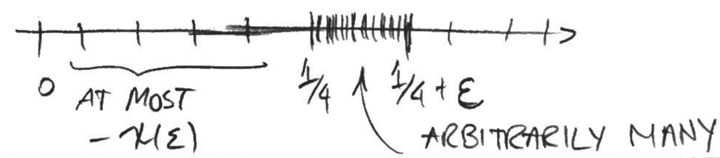
THIS PRODUCES A (TRIVIAL) SOLUTION WITH CONSTANT SCALAR CURVATURE

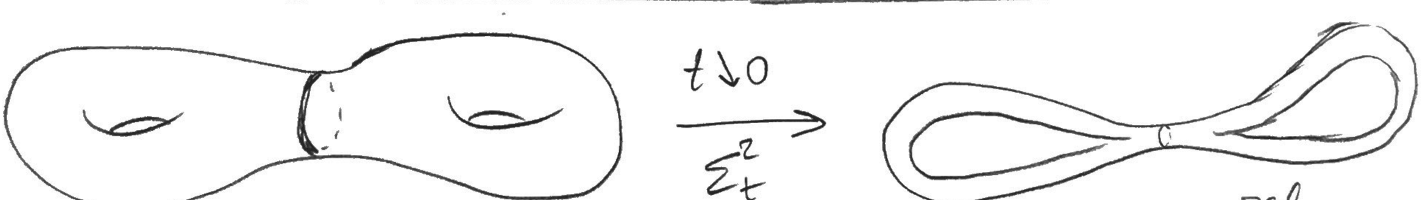
$$scal_{k,n} = (n-2k-2)(n-1) > 0 \iff k < \frac{n-2}{2} \quad \text{(C.F. [SCHOEN-YAU])}$$

SETTING $k=1$:

$$S^n \setminus S^1 \cong S^{n-2} \times \mathbb{H}^2 \longrightarrow S^{n-2} \times \Sigma^2_t, \quad \Sigma^2_t = \mathbb{H}^2/\Gamma_t \text{ HYPERBOLIC SURFACES.}$$

THM (BUSER '77). $\forall \varepsilon > 0, j \in \mathbb{N}, \exists h$ HYPERBOLIC METRIC ON Σ, WITH $\lambda_j(\Sigma, h) < \frac{1}{4} + \varepsilon$.





$$\Rightarrow i_{Morse}(g_{round} \oplus h_t) = \# \operatorname{Spec}(S^{n-2} \times \Sigma^2, g_{round} \oplus h_t) \cap (-\infty, \overbrace{n-4}^{\frac{scal}{n-1}})$$

$$= \# \Big\{ \underbrace{\lambda_i(S^{n-2}, g_{round})}_{\text{INDEPENDENT OF } t} + \underbrace{\lambda_j(\Sigma, h_t)}_{\text{ARBITRARILY CLOSE TO } 1/4} < n-4 \Big\} \nearrow +\infty$$

- UP TO SMALL PERTURBATIONS, $g_{round} \oplus h_t$ ARE NONDEGENERATE [WOLPERT, 92]
- CONCLUSION FOLLOWS FROM KRASNOSEL'SKII'S THM APPLIED TO $S^{n-2} \times \Sigma^2$ AND THEN LIFTING SOLUTIONS TO $S^{n-2} \times \mathbb{H}^2 \cong S^n \setminus S^1$. (ALL ARE CONFORMAL TO $g_{round} \oplus g_{hyp}$)

RMK: IF $K \geq 2$, THIS METHOD FAILS DUE TO MOSTOW RIGIDITY.

HOWEVER, WITH OTHER TECHNIQUES (THAT DO NOT INVOLVE BIFURCATION):

> **THM** (B.–PICCIONE, 2018) THERE ARE INFINITELY MANY (PERIODIC) SOLUTIONS TO THE SING. YAMABE PROBLEM ON $S^n \setminus S^k$, $0 \leq k < \frac{n-2}{2}$.

MAXIMUM RANGE WHERE NON UNIQUENESS CAN HOLD!

4TH ORDER COUSIN OF YAMABE PROBLEM:

(M^n, g) RIEM. MFLD, $n \geq 5$.

SCHOUTEN TENSOR: $\quad A_g = \frac{1}{n-2}\left(Ric_g - \frac{scal_g}{2(n-1)} g\right)$

RECALL: $R = A \owedge g + W$

Q-CURVATURE [BRANSON, 1985]

$$Q_g = \Delta_g \sigma_1(A_g) + 4\sigma_2(A_g) + \frac{n-4}{2}\sigma_1(A_g)^2$$

$$= \frac{1}{2(n-1)}\Delta_g scal_g - \frac{2}{(n-2)^2}\|Ric_g\|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2} scal_g^2$$

PANEITZ OPERATOR:

$$P_g u = \Delta_g^2 u + div_g\left[\left(4A_g - (n-2)\sigma_1(A_g)g\right)(\nabla u, \cdot)\right] + \frac{n-4}{2} Q_g u$$

$$= \Delta_g^2 u + \frac{4}{n-2} div_g\left(Ric_g(\nabla u, e_i)e_i\right) - \frac{n^2-4n+8}{2(n-1)(n-2)} div_g(scal_g \cdot \nabla u) + \frac{n-4}{2} Q_g u$$

CONFORMALLY COVARIANT:

$$P_{u^{\frac{4}{n-4}}g}(\phi) = u^{-\frac{n+4}{n-4}} P_g(u\phi), \qquad Q_g = \frac{2}{n-4} P_g(1)$$

CF:

$$L_{u^{\frac{4}{n-2}}g}(\phi) = u^{-\frac{n+2}{n-2}} L_g(u\phi), \quad L_g(u) = 4\frac{n-1}{n-2}\Delta_g u + scal_g \cdot u$$

CONFORMAL LAPLACIAN

CONSTANT Q-CURVATURE PROBLEM: GIVEN A RIEM. MFLD (M^n, g_0), $n \geq 5$
FIND A COMPLETE CONFORMAL METRIC $g \in [g_0]$ WITH $Q_g \equiv$ const.

PDE SETUP: $P_{g_0} u = \lambda u^{\frac{n+4}{n-4}}$, $\lambda = \frac{n-4}{2} Q_g$, $g = u^{\frac{4}{n-4}} g_0$

(CF. YAMABE PDE SETUP: $L_{g_0} u = \lambda u^{\frac{n+2}{n-2}}$, $\lambda = \mathrm{scal}_g$, $g = u^{\frac{4}{n-2}} g_0$)

VARIATIONAL SETUP:

- $X = [g_0]_1 = \{ g \in [g_0] : \mathrm{Vol}(M, g) = 1 \} = \{ u : M \to \mathbb{R}_+ : \int_M u^{\frac{2n}{n-4}} = 1 \}$ ($g = u^{\frac{4}{n-4}} g_0$)
- $f : X \to \mathbb{R}$ TOTAL Q-CURVATURE

$$f(g) = \int_M Q_g \, \mathrm{vol}_g$$

- $df(g) = 0 \iff Q_g \equiv$ const.

EXISTENCE RESULTS: [A. CHANG, M. GURSKY, F. HANG, Y. LIN, P. YANG, ...]

NON-UNIQUENESS RESULTS: [G. LI, WEI-ZHAO, ...]

MAIN DIFFERENCES FROM YAMABE PROBLEM:
- 4^{TH} ORDER NONLINEAR ELLIPTIC PDE
- NO MAXIMUM PRINCIPLE
- CRITICAL SOBOLEV EXPONENT $W^{2,2}(M) \hookrightarrow L^{\frac{2n}{n-4}}(M)$ NONCPCT

ANALOGOUSLY TO YAMABE PROB.

SETTING FOR BIFURCATION RESULT:

$F \to M \xrightarrow{\pi} B$ RIEM. SUBMERSION

- HARMONIC (I.E., MINIMAL FIBERS): $\overline{\Delta_B u} = \Delta_M \overline{u}$

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "\overline{u}"] & \\ B \arrow[r, "u"] & \mathbb{R} \end{tikzcd}$$

- HORIZONTALLY EINSTEIN:

$$\mathrm{Ric}_g = \mathrm{Ric}_{\mathcal{H}} \oplus \mathrm{Ric}_{\mathcal{V}} ; \quad \mathrm{Ric}_{\mathcal{H}} = K \cdot \pi^*(g_B)$$

- $g_t = g|_{hor} \oplus t\, g|_{ver}$ CANONICAL VARIATION

- $\alpha_t := \dfrac{(n^2-4n+8)\,\mathrm{scal}_{g_t} - 8(n-1)K_t}{4(n-1)(n-2)}$; $\beta_t := -2Q_t$

- $P_t(u) := \Delta^2_{g_B} u + 2\alpha_t \Delta_{g_B} u - \dfrac{n-4}{4}\beta_t u$, $u : B \to \mathbb{R}$

(POLYNOMIAL ON Δ_{g_B})

$P_{g_t}(\overline{u}) = \overline{P_t(u)}$

- $g = \overline{u}^{\frac{4}{n-4}} g_t$ HAS $Q_g \equiv c \iff P_t u = \dfrac{n-4}{2} c\, u^{\frac{n+4}{n-4}}$

$\iff u \in C^\infty(B)$ IS A CRITICAL POINT OF $f : W^{2,2}(B) \to \mathbb{R}$,

$f(u) = \int_B u P_t u$, SUBJECT TO CONSTRAINT $\|u\|_{L^{\frac{2n}{n-4}}(B)} = 1$.

- JACOBI OPERATOR

$$J_t\psi = \frac{1}{2} P_t \psi - \frac{n+4}{4} Q_t \psi = \frac{1}{2}\Delta^2_{g_B}\psi + \alpha_t \Delta_{g_B}\psi + \beta_t\psi$$

$$\mathrm{Spec}(J_t) = \left\{ \tfrac{1}{2}\lambda^2 + \alpha_t\lambda + \beta_t : \lambda \in \mathrm{Spec}(\Delta_{g_B}) \right\}$$

- $i_{Morse}(u) = \#\,\mathrm{Spec}(J_t) \cap (-\infty, 0)$ JUMPS AT $t = t_*$ IF $\exists \lambda \in \mathrm{Spec}(\Delta_{g_B})$ S.T.

$$\begin{cases} \frac{1}{2}\lambda^2 + \alpha_{t_*}\lambda + \beta_{t_*} = 0 \\ \alpha'_{t_*}\lambda + \beta'_{t_*} \neq 0 \end{cases} \qquad (*)$$

HENCE $t = t_*$ IS A BIFURCATION INSTANT IF $(*)$ HOLDS.

THM (B.-PICCIONE-SIRE, 2018). SUPPOSE $F \to M \xrightarrow{\pi} B$ IS A RIEM. SUBMERSION WITH TOTALLY GEODESIC FIBERS AND

$$\mathrm{Ric}_{g_B} = \Lambda_B \cdot g_B, \quad \mathrm{Ric}_{g_F} = \Lambda_F \cdot g_F, \quad \ell = \dim F, \quad n = \dim M$$

$$(A_X, A_Y) = \sum_{i=1}^{n-\ell} g(A_X X_i, A_Y X_i) = \zeta\, g(X,Y)$$

$$(AU, AV) = \sum_{i=1}^{n-\ell} g(A_{X_i} U, A_{X_i} V) = \eta \cdot g(U,V).$$

THEN THE CANONICAL VARIATION $g_t = g|_{hor} \oplus t\, g|_{ver}$ HAS CONSTANT Q-CURVATURE, CONSTANT SCALAR CURVATURE, AND $\pi_t : (M, g_t) \to (B, g_B)$ IS HORIZONTALLY EINSTEIN WITH $K_t = \Lambda_B - 2\zeta t$.

(1) IF $\Lambda_F > 0$ AND EITHER $5 \leq n \leq 8$ AND $\ell \geq 3$ OR $n \geq 9$ AND $\ell \geq 2$, THEN g_t HAS INFINITELY MANY BIFURCATION INSTANTS AS $t \searrow 0$.

(2) IF $\zeta > 0$, $\eta > 0$, $\dfrac{\eta}{\zeta} > \dfrac{8(n-1)\sqrt{n-\ell}}{\sqrt{(n^3 - 4n^2 + 16n - 16)\ell^2 - 16(n-1)^2 \ell}}$

AND EITHER $5 \leq n \leq 8$ AND $\ell \geq 3$, OR $n \geq 9$ AND $\ell \geq 2$, OR $n \geq 21$ AND $\ell = 1$, THEN g_t HAS INFINITELY MANY BIFURCATION INSTANTS AS $t \nearrow +\infty$.

MOREOVER, FOR $t \simeq 0$ AND $t \simeq +\infty$, THE ABOVE BIFURCATING METRICS <u>DO NOT</u> HAVE CONSTANT SCALAR CURVATURE.

<u>METHOD OF PROOF</u>: ANALYZE ASYMPTOTICS OF α_t AND β_t AS $t \searrow 0$ AND $t \nearrow +\infty$.

NOTE: (1) APPLIES TO PRODUCT MANIFOLDS $M = F \times B$, BUT

(2) DOES NOT, SINCE IT REQUIRES $|A| > 0$, BUT CF. COROLLARY!

APPLYING THE ABOVE TO BERGER SPHERES:

THM (B.-PICCIONE-SIRE, 2018). THERE ARE INFINITELY MANY BIFURCATING BRANCHES OF METRICS WITH $Q \equiv \text{const.}$ WHEN:

$F \to M \to B$	$t \searrow 0$	$t \nearrow +\infty$
$S^1 \to S^{2q+1} \to \mathbb{C}P^q$	NO	$q \geq 6$
$S^3 \to S^{4q+3} \to \mathbb{H}P^q$	$q \geq 1$	$q \geq 2$
$\mathbb{C}P^1 \to \mathbb{C}P^{2q+1} \to \mathbb{H}P^q$	$q \geq 2$	$q \geq 3$
$S^7 \to S^{15} \to S^8(1/2)$	YES	YES.

NOTE: AS $t \nearrow +\infty$, $\mathrm{scal}_{g_t} \searrow -\infty$ AND $Q_t \searrow -\infty$ ON:

S^{2q+1}, $6 \leq q \leq 9$

S^{4q+3}, $q = 1$

$\mathbb{C}P^{2q+1}$, $q = 2$

APPLYING THE ABOVE, CASE (1), TO PRODUCT MANIFOLDS:

COROLLARY: IF F AND B ARE CLOSED EINSTEIN MANIFOLDS WITH POSITIVE EINSTEIN CONSTANT AND DIMENSION ≥ 3, THEN $g_t = g_B \oplus t\, g_F$ ON $M = B \times F$ HAS INFINITELY MANY BIFURCATION INSTANTS FOR THE CONSTANT Q-CURVATURE PROBLEM AS $t \searrow 0$ AND $t \nearrow +\infty$.