KOELN, JUNE 2019

CONVEX ALGEBRAIC GEOMETRY OF CURVATURE OPERATORS

JOINT WORK WITH M. KUMMER AND R. MENDES

OUTLINE

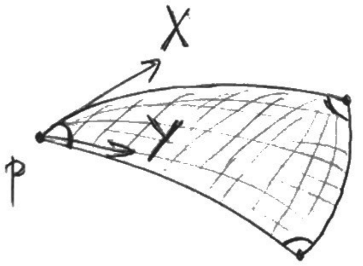
1. SECTIONAL CURVATURE & CURVATURE OPERATORS
2. SECTIONAL CURVATURE AS A FUNCTION & Q: WHEN IS $\sec \geq 0$?
3. INTERMEZZO: S.O.S. v. NONNEGATIVITY (HILBERT'S 17th PROBLEM)
4. MAIN RESULTS
5. SOME IDEAS OF PROOFS

1. SECTIONAL CURVATURE AND CURVATURE OPERATORS

(M^n, g) RIEMANNIAN MANIFOLD, DIMENSION $n \geq 2$.

ALLOWS TO MEASURE DISTANCES, ANGLES, AREAS, ...

$X, Y \in T_pM \rightsquigarrow$ GEODESIC TRIANGLES T WITH VERTEX $p \in M$ AND SIDES TANGENT TO X, Y.



ANGLE DEFECT: $\mathrm{def}(T) = \sum_{T} \measuredangle - \pi$

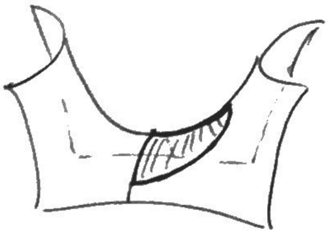
SECTIONAL CURVATURE: $\sec(X \wedge Y) = \lim_{|T| \to 0} \dfrac{\mathrm{def}(T)}{\mathrm{area}(T)}$

"FUNDAMENTAL LOCAL INVARIANT OF RIEMANNIAN GEOMETRY"

THM (KULKARNI '70; YAU '74). IF $f : (M_1^n, g_1) \longrightarrow (M_2^n, g_2)$, $n \geq 4$ IS A DIFFEOMORPHISM SUCH THAT $\sec_{g_2}(df(\sigma)) = \sec_{g_1}(\sigma)$ FOR ALL $\sigma \in Gr_2(TM_1)$, THEN f IS AN ISOMETRY "(M_1, g_1) & (M_2, g_2) ARE INDISTINGUISHABLE FROM GEOMETRIC VIEWPOINT"

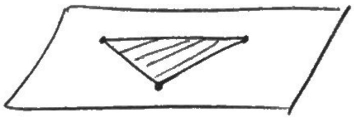
⚠ NEED (M_i, g_i) TO BE ANALYTIC AND $\sec \not\equiv K$. ALSO, FAILS IF $n \leq 3$.

$\sec < 0$



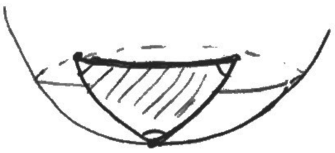
$\tilde{M} \cong \mathbb{R}^n$ (DIFFEO)

$\sec = 0$



$\tilde{M} \cong \mathbb{R}^n$ (ISOMETRIC)

$\sec > 0$



EXAMPLES: S^n, $\mathbb{CP}^n$, $\mathbb{HP}^n$, … ↑ BUT NOT SO MANY!

$|\pi_1(M)| < \infty$

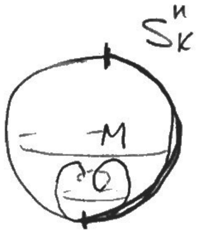
SECTIONAL CURVATURE BOUNDS AND GLOBAL CONSEQUENCES:

- CARTAN – HADAMARD: $\sec \leq 0 \implies \tilde{M} \cong \mathbb{R}^n$

- BONNET – MYERS: $\sec \geqslant K > 0 \implies M$ IS COMPACT

  $\operatorname{diam} M^n \leq \operatorname{diam} S^n_K = \frac{\pi}{\sqrt{K}}$

  $= \iff M \underset{isom.}{\cong} S^n_K$



- SPHERE THEOREMS:
  - GROVE – SHIOHAMA: $\sec \geqslant K > 0$, $\operatorname{diam} M^n > \frac{1}{2} \operatorname{diam} S^n_K = \frac{\pi}{2\sqrt{K}} \implies M^n \underset{homeo}{\cong} S^n$
  - BERGER – KLINGENBERG BRENDLE – SCHOEN: $1 < \sec \leq 4 \implies M^n \underset{diffeo}{\cong} S^n$

- IMPORTANT (OPEN) CONJECTURES: SUPPOSE M IS CLOSED

HOPF + … : M^4, $\pi_1 M = \{1\}$, $\sec > 0 \implies M^4 \underset{diffeo}{\cong} S^4, \mathbb{CP}^2$

M^4, $\pi_1 M = \{1\}$, $\sec \geqslant 0 \implies M^4 \underset{diffeo}{\cong} S^4, \mathbb{CP}^2, S^2 \times S^2,$
$\mathbb{CP}^2 \# \mathbb{CP}^2, \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}.$

BOTT – GROVE – HALPERIN: M^n, $\pi_1 M = \{1\}$, $\sec \geqslant 0 \implies M$ IS RATIONALLY ELLIPTIC, I.E.

$$\sum_{i=1}^{+\infty} \dim_{\mathbb{Q}} \pi_i(M) \otimes \mathbb{Q} < \infty$$

2. SECTIONAL CURVATURE AS A FUNCTION:

- $Gr_2(TM) = \{\sigma \subset TM \text{ 2-PLANE}\}$

  $= \{\sigma = X \wedge Y \in \Lambda^2 TM : |X \wedge Y| = 1\}$ GRASSMANNIAN OF 2-PLANES

- $\sec : Gr_2(TM) \longrightarrow \mathbb{R}$

  $\sec(X \wedge Y) = \langle R(X \wedge Y), X \wedge Y \rangle$

  WHERE $R : \Lambda^2 TM \to \Lambda^2 TM$ IS THE CURVATURE OPERATOR:

  $$\langle R(X \wedge Y), Z \wedge W \rangle = \langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, W \rangle$$

- I.E., $\sec = q|_{Gr_2(TM)}$ WHERE $q(\alpha) = \langle R\alpha, \alpha \rangle$ IS THE QUADRATIC FORM ASSOCIATED TO $R \in Sym^2(\Lambda^2 TM)$.

Q: WHEN IS $\sec \geqslant 0$? (OR $\sec \geqslant K$, $\sec \leq K$, ...)

Q': HOW TO "CERTIFY" THAT $\sec \geqslant 0$?

A0: E.G., $R \succeq 0 \implies q(\alpha) = \langle R\alpha, \alpha \rangle \geqslant 0, \ \forall \alpha \in \Lambda^2 TM$

$\implies \sec = q|_{Gr_2(TM)} \geqslant 0.$

BUT: $R \succeq 0$ IS MUCH STRONGER THAN $\sec \geqslant 0$.

MANIFOLDS WITH $R \geqslant 0$ ARE CLASSIFIED (W/ RICCI FLOW)
(RECALL: $R > 0 \implies \widetilde{M} = S^n$.)

$\hookrightarrow$ EACH FACTOR IN DERHAM DECOMPOSITION OF $\widetilde{M}$ IS: $(\mathbb{R}^n, g_{flat})$, $(S^n, g_{R \geqslant 0})$, COMPACT (IRRED.) SYMM. SP., OR COMPACT KÄHLER MFLD WITH $R|_{\Lambda^{1,1}_{\mathbb{R}}} \geq 0$.

NOTE: $R \in Sym^2(\Lambda^2 TM) \implies R$ DIAGONALIZABLE

$R = \text{diag}(\lambda_1, \ldots, \lambda_{\binom{n}{2}})$

$R \geqslant 0 \implies q(\alpha) = \sum_{i=1}^{\binom{n}{2}} \lambda_i \alpha_i^2, \quad \lambda_i \geqslant 0$

IS A SUM OF SQUARES (S.O.S.)

3. INTERMEZZO: S.O.S. V. NONNEGATIVITY

Q: $p \in \mathbb{R}[x_1, \dots, x_n]$ HOMOGENEOUS POLYNOMIAL OF $\deg = 2d$

$$p(x) \geqslant 0, \ \forall x \in \mathbb{R}^n \iff p(x) = \sum_{i=1}^{N} q_i(x)^2 \quad ?$$

$\nwarrow$ ($q_i \in \mathbb{R}[x_1, \dots, x_n]_d$ HOMOGENEOUS)

A: (HILBERT, 1893) TRUE IF AND ONLY IF:

(i) $2d = 2$, $\forall n$ – QUADRATIC POLYNOMIALS
(SPECTRAL THM: SYMM. $\Rightarrow$ DIAG.)

(ii) $n \leq 2$, $\forall d$ – UNIVARIATE (INHOMOGENEOUS) POLYNOMIALS
RMK: 2 SQUARES SUFFICE!

(iii) $n = 3$, $2d = 4$ – TERNARY QUARTICS

SO, IN MOST CASES, IT IS FALSE!

EXAMPLE (MOTZKIN): $p(x, y, z) = z^6 + x^4 y^2 + x^2 y^4 - 3x^2 y^2 z^2$

- $p(x,y,z) \geqslant 0$ BY ARITHMETIC – GEOMETRIC INEQUALITY
- $p(x,y,z)$ IS NOT A S.O.S. BY INSPECTION, OR NEWTON POLYTOPE

HILBERT'S 17th PROBLEM: $p(x) \geqslant 0, \forall x \in \mathbb{R}^n \iff p(x) = \sum_{i=1}^{N} \left(\dfrac{q_i(x)}{r_i(x)} \right)^2 ?$

RATIONAL FUNCTIONS $\nearrow$

THM (ARTIN, 1927). YES!

SO, TO CERTIFY $p(x) \geqslant 0$, FIRST MULTIPLY BY $r(x) \geqslant 0$ TO GET S.O.S.

BACK TO OUR PROBLEM...

A1: JUST NEED $q(\alpha) = \langle R\alpha, \alpha \rangle$ TO BE S.O.S / NONNEGATIVE ON $Gr_2 TM \subset \Lambda^2 TM$

$\swarrow$ (HOMOG. QUADRATIC POLY)

FINSLER'S LEMMA (1936): IF $V = \{ \phi(x) = 0 \} \subset \mathbb{R}^n$ IS A QUADRIC, AND $p \in \mathbb{R}[x]_2$ IS A HOMOGENEOUS QUADRATIC POLYNOMIAL, THEN

$$p|_V \geqslant 0 \iff \exists\, a \in \mathbb{R}, \quad p(x) = \sum_{i=1}^{N} q_i(x)^2 + a\, \phi(x)$$

$\uparrow$ LINEAR FUNCTIONS

$V = Gr_2 \mathbb{R}^n$ IS A (SINGLE) QUADRIC $\iff n = 4$.

PLÜCKER RELATION: $\phi(\alpha) = \langle *\alpha, \alpha \rangle$, $*: \Lambda^2\mathbb{R}^4 \longrightarrow \Lambda^2\mathbb{R}^4$ HODGE STAR

COROLLARY ("THORPE'S TRICK", 1972): ON A 4-MANIFOLD (M^4, g),

$$\sec \geqslant 0 \iff \exists\, a : M^4 \to \mathbb{R}, \quad R + a* \succeq 0$$

THIS GIVES A STRUCTURAL RESULT ABOUT THE CONVEX CONE

$$\mathcal{R}_{\sec\geqslant 0}(n) = \left\{ R \in \mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) : \sec_R \geqslant 0 \right\} \quad (\text{IF } n=4)$$

COROLLARY: $\mathcal{R}_{\sec\geqslant 0}(4)$ IS A SPECTRAHEDRAL SHADOW.

DEF: A SPECTRAHEDRON IS A SET $S \subset \mathbb{R}^n$ OF THE FORM

$$S = \left\{ x \in \mathbb{R}^n : A + \sum_{i=1}^{n} x_i B_i \succeq 0 \right\}, \quad A, B_i \in \mathrm{Sym}^2(\mathbb{R}^d)$$

I.E., THE INTERSECTION OF $\mathrm{Sym}^2_{\succeq 0}(\mathbb{R}^d)$ WITH AN AFFINE SUBSPACE

POSITIVE-SEMIDEFINITE $d \times d$ SYMM. MATRICES

EXAMPLE: POLYHEDRA $\iff$ A, B_i DIAGONAL

DEF: A SPECTRAHEDRAL SHADOW IS A SET $S \subset \mathbb{R}^n$ OF THE FORM

$$S = \left\{ x \in \mathbb{R}^n : \exists\, y \in \mathbb{R}^m, \quad A + \sum_{i=1}^{n} x_i B_i + \sum_{j=1}^{m} y_j C_j \succeq 0 \right\}$$

WHERE $A, B_i, C_j \in \mathrm{Sym}^2(\mathbb{R}^d)$, I.E., A LINEAR PROJECTION OF A SPECTRAHEDRON.

NOTE: SPECTRAHEDRAL SHADOWS ARE FEASIBLE REGIONS OF SEMIDEFINITE PROGRAMS.

4. MAIN RESULTS

THM A (B.-KUMMER-MENDES, 2019)

(i) $\mathcal{R}_{\sec\geqslant 0}(n)$, $n \geqslant 5$, IS NOT A SPECTRAHEDRAL SHADOW

(ii) $\mathcal{R}_{\sec\geqslant 0}(4)$ IS NOT A SPECTRAHEDRON (BUT IS A SPECTR. SH.)

(iii) $\mathcal{R}_{\sec\geqslant 0}(2)$ AND $\mathcal{R}_{\sec\geqslant 0}(3)$ ARE SPECTRAHEDRA (TRIVIAL)

BUT NOT ALL IS LOST WHEN $n \geq 5$...

PROP: $\mathcal{R}_{\sec\geq 0}(n)$ IS A SEMIALGEBRAIC SET, THAT IS,

$$\mathcal{R}_{\sec\geq 0}(n) = \left(\bigcup_{j=1}^{N}\right)\left\{ R \in \mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) : \underset{\uparrow \text{ POLYNOMIALS}}{P_{i,(j)}}(R) \geq 0,\ 1 \leq i \leq N_{(j)} \right\}$$

PF: TARSKI – SEIDENBERG THM / QUANTIFIER ELIMINATION

(QUANTIFIED) $\quad \exists x \in \mathbb{R},\ ax^2+bx+c=0,\ a\neq 0 \quad \Big| \quad \forall \sigma \in Gr_2\mathbb{R}^n,\ \sec_R(\sigma)\geq 0$

$\Updownarrow$

(QUANTIFIER-FREE) $\quad b^2-4ac \geq 0 \quad \Big| \quad P_{i,(j)}(R) \geq 0,\ 1 \leq i \leq N_{(j)}$ (FOR SOME j)

- FINDING $P_{i(j)}(R)$ EXPLICITLY IS HARD / HOPELESS / USELESS, IN GENERAL.
- HOWEVER, IF $n=4$, SOMETHING CAN BE SAID:

THM B (B.–KUMMER–MENDES, 2019). $\mathcal{R}_{\sec\geq 0}(4) = \overline{\mathcal{C}}$, WHERE $\mathcal{C}$ IS THE CONNECTED COMPONENT OF THE SET WHERE $P(R) = \mathrm{disc}_x\big(\det(R + x*)\big) > 0$ WITH $\mathrm{Id} \in \overline{\mathcal{C}}$. "ALGEBRAIC INTERIOR w/ MINIMAL POLY. P"

NOTE: ALSO MIGHT FOLLOW FROM INDEPENDENT WORK BY DAN FODOR

5. SOME IDEAS OF PROOFS:

THM A. • [BLEKHERMAN – SMITH – VELASCO, JAMS 2016]

$V = \{\phi_1(x)=0, \ldots, \phi_m(x)=0\} \subset \mathbb{R}^n$ PROJECTIVE VARIETY

HOMOGENEOUS POLYNOMIALS

HAS MINIMAL DEGREE $\deg(V) = 1 + \mathrm{codim}(V)$

IF AND ONLY IF $\forall p \in \mathbb{R}[x]_2$ HOMOG. QUADRATIC POLY.

$$p|_V \geq 0 \iff \exists a \in \mathbb{R}^m,\ p(x) = \sum_{i=1}^{N} q_i(x)^2 + \sum_{j=1}^{m} a_j \phi_j(x)$$

- $Gr_2\mathbb{R}^n \xhookrightarrow[\text{PLÜCKER}]{} \mathbb{RP}^{\binom{n}{2}-1}$ HAS:

$$\operatorname{Codim}\left(Gr_2\mathbb{R}^n\right) = \frac{(n-2)(n-3)}{2}$$

$$\deg\left(Gr_2\mathbb{R}^n\right) = \frac{(2(n-2))!}{(n-2)!\,(n-1)!}$$

HENCE HAS MINIMAL DEGREE IFF $n \leq 4$.

- THUS, $\forall n \geqslant 5$, $\exists\, p_* \in \mathbb{R}[Gr_2\mathbb{R}^n]$ NONNEGATIVE WHICH IS NOT A S.O.S. ON $Gr_2\mathbb{R}^n$
- FEED p_* INTO A CRITERION OF [SCHEIDERER, SIAM 2018] THAT CHARACTERIZES SPECTRAHEDRAL SHADOWS TO CONCLUDE. □

MORE ABOUT SCHEIDERER'S CRITERION:

[NEMIROVSKI, ICM 2006], [HELTON-NIE, SIAM 2009]. IS EVERY CONVEX SEMIALGEBRAIC SET IN $\mathbb{R}^n$ A SPECTRAHEDRAL SHADOW?

THM (SCHEIDERER, 2018). YES, IF $n=2$; NO, IF $n \geqslant 3$.

↳ NECESSARY & SUFFICIENT CONDITION FOR A CONVEX SEMIALGEBRAIC SET TO BE A SPECTRAHEDRAL SHADOW:

A CONVEX SEMIALGEBRAIC CONE $C = \overline{\operatorname{cone}(S)}$ IS A SPECTRAHEDRAL SHADOW IF AND ONLY IF $\exists\, \Phi: X \to \mathbb{A}^n$ MORPHISM OF AFFINE $\mathbb{R}$-VARIETIES AND A FINITE-DIMENSIONAL SUBSPACE $U \subset \mathbb{R}[X]$ SUCH THAT:

(1) $S \subset \Phi(X(\mathbb{R}))$

(2) IF $f \in \mathbb{R}[x_1, \ldots, x_n]$ IS A HOMOG. LINEAR POLY. WITH $f|_S \geqslant 0$, THEN $\Phi^*(f) \in \mathbb{R}[X]$ IS A S.O.S. OF ELEMENTS OF U.

THM B. $P : \mathrm{Sym}^2(\Lambda^2 \mathbb{R}^4) \longrightarrow \mathbb{R}$

$$P(R) = \mathrm{disc}_x\big(\det(R + x\,*)\big)$$

CLAIM 1: $P|_{\partial \mathcal{R}_{\sec \geq 0}(4)} = 0$

PF. • $R \in \partial \mathcal{R}_{\sec\geq 0}(4) \iff \exists\, a_0 \in \mathbb{R},\ R + a_0 * \succeq 0,\ \ \mathrm{Ker}(R + a_0 *) \neq \{0\}$.

(SO $\det(R + a_0 *) = 0$)

- NOTE: $\det(R + x*) = -\det(TRT + x\,\mathrm{Id}),\quad T = \begin{array}{cc} \Lambda^2_+ & \Lambda^2_- \\ \left(\begin{array}{c:c} \mathrm{Id}_{3\times3} & \\ \hdashline & \sqrt{-1}\,\mathrm{Id}_{3\times 3}\end{array}\right) \end{array}$

SO $T^2 = *$.

$\implies \mathrm{disc}_x(\det(R + x*)) = \mathrm{disc}(TRT)$

DISCRIMINANT OF A MATRIX IS THE DISCRIMINANT OF ITS CHARACTERISTIC POLY.

- IF $a_0 \in \mathbb{R}$ IS A SIMPLE ROOT, THEN:

$-a_0$ IS AN EIGENVALUE OF TRT WITH MULTIPLICITY 1

$\implies \dim \mathrm{Ker}(R + a_0 *) = \dim \mathrm{Ker}\big(T^{-1}(TRT + a_0 \mathrm{Id})T\big) = 1$

$\implies R + a_0 * \succeq 0$ MUST HAVE 5 POSITIVE EIGENVALUES

$\implies$ FOR $x \approx a_0$, $\quad R + x* \succ 0 \iff \det(R + x*) > 0$

$\implies \exists\, x \approx a_0$ WITH $R + x* \succ 0$, A <u>CONTRADICTION!</u>

- THUS $a_0 \in \mathbb{R}$ IS A ROOT W/ MULTIPLICITY ≥ 2 HENCE $P(R) = 0$.

CLAIM 2: $\mathcal{R}_{\sec\geq 0}(4) \setminus \{P = 0\}$ IS CONNECTED.

PF: SHOW $\{P = 0\}$ HAS CODIMENSION ≥ 2 INSIDE $\mathcal{R}_{\sec\geq 0}(4)$ ☐

ABOUT $\mathcal{R}_{\sec\geq 0}(4)$ <u>NOT BEING A SPECTRAHEDRON</u> (IN THM A):

- $P(R) = \mathrm{disc}_x(\det(R + x*))$ IS IRREDUCIBLE, BUT $P(\mathrm{Id}) = 0$
- [HELTON–VINNIKOV, CPAM 2007]. THE MINIMAL DEFINING POLYNOMIAL OF A SPECTRAHEDRON (AS AN ALGEBRAIC INTERIOR) <u>DOES NOT VANISH</u> IN THE INTERIOR, CONTRADICTING $P(\mathrm{Id}) = 0$. ☐