# Convex Algebraic Geometry of Curvature Operators

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- First Bianchi identity:

$$\langle R(X\wedge Y), Z\wedge W\rangle + \langle R(Y\wedge Z), X\wedge W\rangle + \langle R(Z\wedge X), Y\wedge W\rangle = 0$$

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## Example

$(M^n, \mathrm{g})$ (pseudo-)Riemannian manifold, $p \in M$,

$$R_p\colon \Lambda^2 T_pM \to \Lambda^2 T_pM$$
$$R_p(X\wedge Y, Z\wedge W) = \mathrm{g}_p(R_p(X,Y)Z, W)$$

# Sectional curvature bounds

$R \in \mathsf{Sym}^2(\wedge^2\mathbb{R}^n)$

$\mathsf{Gr}_2(\mathbb{R}^n) = \left\{\sigma \in \wedge^2\mathbb{R}^n : \sigma \wedge \sigma = 0, \|\sigma\| = 1\right\}$

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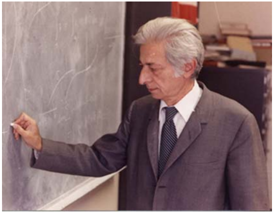
# Quantifier elimination

## Theorem (Tarski-Seidenberg)


A. Tarski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*Any finite list of* **quantified** *polynomial equalities and inequalities over the real numbers*

$$(\forall, \exists)\ t_1, t_2, t_3, \ldots$$
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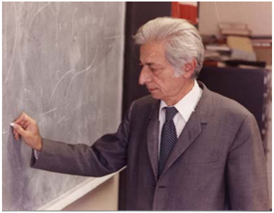

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*is equivalent to a list of* **quantifier-free** *polynomial equalities and inequalities*

$$\widetilde{F}_i(x_1, x_2, x_3, \ldots) = 0$$
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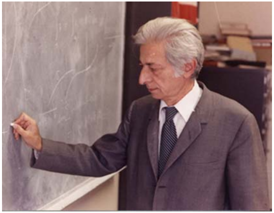

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# Quantifier elimination in practice

Quantified:

$$\exists t \in \mathbb{R} \quad \begin{aligned} & a t^2 + b t + c = 0 \\ & a \neq 0 \end{aligned}$$

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Quantifier-free:

$$F_i(R, k) \geq 0, \;\; 1 \leq i \leq N$$

# Questions

$$\boxed{\mathfrak{R}_{\mathrm{sec}\geq k}(n) := \left\{ R \in \mathrm{Sym}^2(\wedge^2\mathbb{R}^n) : \mathrm{sec}_R \geq k \right\}}$$

- **Q**: What is the structure of the set $\mathfrak{R}_{\mathrm{sec}\geq k}(n)$?
  **A**: Convex cone in $\mathrm{Sym}^2(\wedge^2\mathbb{R}^n)$; semialgebraic subset, i.e., described by finitely many polynomial inequalities on $R$
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  **A:** Maybe...?

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... need to do more Math!

# Crash course in Convex Algebraic Geometry

## Definition (Spectrahedron)

$$S = \left\{ x \in \mathbb{R}^d : A + \sum_{i=1}^{d} x_i B_i \succeq 0 \right\}, \quad \text{where } A, B_i \in \mathrm{Sym}^2(\mathbb{R}^m)$$

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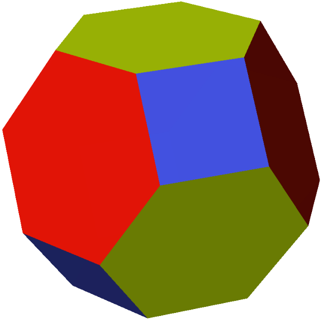
## Example

Polyhedron: $A, B_i$ *diagonal* matrices



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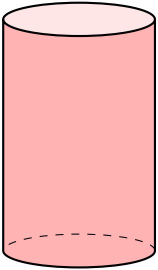
$$\begin{pmatrix} 1+x & y & 0 & 0 \\ y & 1-x & 0 & 0 \\ 0 & 0 & 1+z & 0 \\ 0 & 0 & 0 & 1-z \end{pmatrix} \succeq 0$$



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Elliptope:

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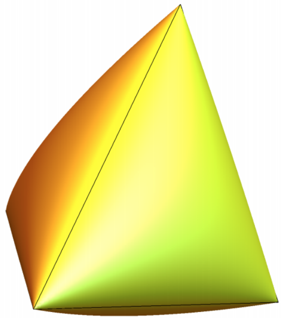

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## Definition (Spectrahedral shadow)

$$S = \left\{ x \in \mathbb{R}^d : \exists t \in \mathbb{R}^\ell,\ A + \sum_{i=1}^{d} x_i B_i + \sum_{j=1}^{\ell} t_j C_j \succeq 0 \right\},$$

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where $A, B_i, C_j \in \mathrm{Sym}^2(\mathbb{R}^m)$

- Linear programming: optimize linear functionals on polyhedra (solvable in polynomial time!)
- Semidefinite programming: optimize linear functionals on spectrahedra (also solvable in polynomial time!)

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*A convex semialgebraic cone $C = \overline{\mathrm{cone}(S)}$ is a spectrahedral shadow if and only if $\exists\, \phi\colon X \to \mathbb{A}^n$ morphism of affine $\mathbb{R}$-varieties and a finite-dimensional subspace $U \subset \mathbb{R}[X]$ s.t.:*

- $S \subset \phi(X(\mathbb{R}))$,
- $\forall f \in \mathbb{R}[x_1, \ldots, x_n]$ *homogeneous linear polynomial,* $f \geq 0$ *on* $S$, $\phi^*(f) \in \mathbb{R}[X]$ *is a sum of squares of elements in* $U$.

C. Scheiderer

Building on Scheiderer [SIAM 2018]; Blekherman, Smith, Velasco [JAMS 2016]:

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Upshot: sec $\geq k$ is <u>algebraically much harder</u> to verify if $n \geq 5$

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$\mathfrak{R}_{\sec\geq k}(4)$ *is the closure of a union of connected components of the semialgebraic set* $\big\{R : \mathrm{Disc}_x\big(\det(R - k\,\mathrm{Id} + x*)\big) \neq 0\big\}$.

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Some hope: outer approximation by *Weitzenböck spectrahedra*

## Theorem (B.-Mendes, 2017)

$$\mathfrak{R}_{\mathrm{sec}\geq k}(n) = \bigcap_{p\geq 2} \big\{R \in \mathrm{Sym}^2(\wedge^2\mathbb{R}^n) : \mathcal{K}\big(R - k\,\mathrm{Id}, \mathrm{Sym}^p_0\,\mathbb{R}^n\big) \succeq 0\big\}$$

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Thank you for your attention!