A WEITZENBÖCK VIEWPOINT ON SECTIONAL CURVATURE

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OUTLINE

1. WEITZENBÖCK FORMULAE
2. ALGEBRAIC CHARACTERIZATION OF $\sec \geq 0$
3. APPLICATIONS IN DIMENSION 4

1. WEITZENBÖCK FORMULAE

(M^n, g) RIEMANNIAN MANIFOLD

$\rho : O(n) \longrightarrow O(E)$ ORTHOGONAL REPRESENTATION:

$$\begin{tikzcd} O(n) \arrow[r] & Fr(M) \arrow[d] \\ & M \end{tikzcd} \quad \text{FRAME BUNDLE} \quad \rightsquigarrow \quad \begin{tikzcd} E \arrow[r] & \mathcal{E}_\rho \arrow[d] \\ & M \end{tikzcd} \quad \text{ASSOCIATED BUNDLE}$$

EXAMPLES:

$\rho : O(n) \curvearrowright \mathbb{R}^n \Rightarrow \mathcal{E}_\rho = TM$ VECTOR FIELDS

$\rho : O(n) \curvearrowright Sym^p \mathbb{R}^n \Rightarrow \mathcal{E}_\rho = Sym^p TM$ SYMMETRIC p-TENSORS

$\rho : O(n) \curvearrowright \Lambda^p \mathbb{R}^n \Rightarrow \mathcal{E}_\rho = \Lambda^p TM$ p-FORMS

WEITZENBÖCK FORMULA: $\Delta = \nabla^* \nabla + t \, K(R, \rho)$

CONNECTION LAPLACIAN

$t \in \mathbb{R}$

CURVATURE TERM

$\boxed{\nabla^* \nabla = -\sum_i \nabla_{E_i} E_i}$

$\{X_a\}$ O.N. BASIS OF $\Lambda^2 \mathbb{R}^n \cong \mathfrak{so}(n)$

$$R = \sum_{a,b} R_{ab} X_a \otimes X_b \in Sym^2(\Lambda^2 \mathbb{R}^n) \qquad \boxed{K(R, \rho) = -\sum_{a,b} R_{ab} \, d\rho(X_a) \circ d\rho(X_b)}$$

$\rho : O(n) \to O(E), \; d\rho(X) : \mathfrak{so}(n) \cong \Lambda^2 \mathbb{R}^n \to \mathfrak{o}(E)$

EXAMPLES:

$\rho = \Lambda^1 \mathbb{R}^n$, $\mathcal{E}_\rho = \Lambda^1 TM$, $K(R,\rho) = Ric_R$, $t = 2$

$\rho = \mathbb{R}^n$, $\mathcal{E}_\rho = TM$, $K(R,\rho) = Ric_R$, $t = -2$

VANISHING THEOREMS

(M,g) CLOSED, $Ric > 0$
$\Rightarrow b_1(M,\mathbb{R}) = 0$

(M,g) CLOSED, $Ric < 0$
$\Rightarrow |Iso(M,g)| < \infty$.

PROPERTIES:

- $Sym^2(\Lambda^2\mathbb{R}^n) \ni R \longmapsto K(R,\rho) \in Sym^2(E)$ LINEAR AND $O(n)$-EQUIVARIANT
- $K(R, \rho_1 \oplus \rho_2) = K(R,\rho_1) \oplus K(R,\rho_2)$
- $K(R,\rho) = 0$ IF ρ IS THE TRIVIAL REPRESENTATION

$$\boxed{\underline{\text{THM}}\ (\text{HITCHIN}).\ R \geq 0 \iff K(R,\rho) \geq 0,\ \forall \rho : O(n) \to O(E)}$$

PF. $(\Rightarrow)$ CHOOSE DIAGONAL BASIS $\{X_a\}$ OF $\Lambda^2\mathbb{R}^n$

$RX_a = \lambda_a X_a$, $\lambda_a \geq 0$

SKEW SYMMETRIC

$$\langle K(R,\rho)\omega,\omega\rangle = -\sum_a R_{aa} \langle d\rho(X_a)\, d\rho(X_a)\omega,\omega\rangle$$

$$= \sum_a \lambda_a \| d\rho(X_a)\omega\|^2 \geq 0$$

$(\Leftarrow)$ CONSIDER $X_a \in \mathfrak{X}(SO(n))$ AS LEFT-INVARIANT FIELDS,

$$L = -\sum_{a,b} R_{ab} X_a X_b : L^2(SO(n)) \longrightarrow L^2(SO(n))$$

SELF-ADJOINT 2^{nd} ORDER DIFFERENTIAL OPERATOR, EQUIVARIANT.

PETER-WEYL THM: $L^2(SO(n)) = \overline{\bigoplus_\rho E_\rho}$, ρ IRREDUCIBLE

$$L(\varphi) = \underbrace{\langle K(R,\rho)\omega_\rho,\omega_\rho\rangle}_{\geq 0} \cdot \varphi,\ \forall \varphi \in E_\rho, \text{ FOR SOME } \omega_\rho \in \mathcal{E}_\rho.$$

HENCE THE SYMBOL OF L IS $R = \sigma(L) \geq 0$. $\square$

2, ALGEBRAIC CHARACTERIZATION OF $\sec \geq 0$

⚠ • $\sec \geq 0$ MUCH LESS UNDERSTOOD THAN $R \geq 0$
• SYLVESTER CRITERION GIVES "EASY" ALGEBRAIC CHAR. OF $R \geq 0$.

$$\boxed{\underline{\text{THM}}\ (\text{B.- MENDES}).\quad \sec_R \geq 0 \iff K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0,\ \forall p \geq 2.}$$

<u>NOTE</u>: MORE GENERALLY, $\underset{(\leq)}{\sec_R \geq K} \iff K(R - K.\mathrm{Id}, \mathrm{Sym}^p_0(\mathbb{R}^n)) \underset{(\leq)}{\geq 0}$, $\forall p \geq 2$

<u>PF.</u> $\mathrm{Sym}^p(\mathbb{R}^n) \cong \{\varphi : \mathbb{R}^n \to \mathbb{R}$ HOMOGENEOUS POLY. OF DEGREE $p\}$.

$\mathrm{Sym}^p_0(\mathbb{R}^n) \cong \{\varphi \in \mathrm{Sym}^p(\mathbb{R}^n) : \Delta\varphi = 0\}$.

DIRECT COMPUTATION [HEIL-MOROIANU-SEMMELMANN, 2015]:

$$\boxed{\langle K(R, \mathrm{Sym}^p_0(\mathbb{R}^n))\varphi, \varphi\rangle = C_{p,n} \cdot \int_{S^{n-1}} R(X, \nabla\varphi, X, \nabla\varphi)\, dX}$$

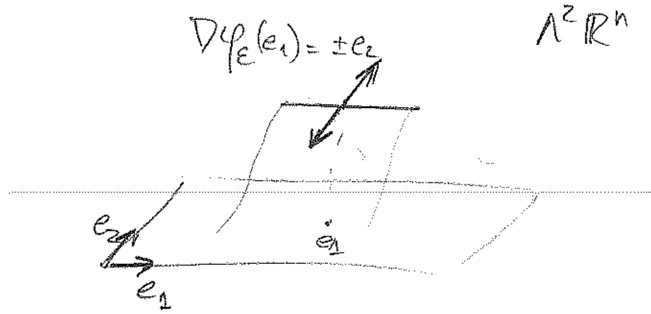
THUS $\sec_R \geq 0 \implies K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0$, $\forall p \geq 2$.

CONVERSELY, ARGUE BY CONTRADICTION: SUPPOSE $\exists\, \sigma = e_1 \wedge e_2 \in \underset{\Lambda^2\mathbb{R}^n}{\underset{\cap}{Gr_2 \mathbb{R}^n}}$

WITH $\sec_R(\sigma) = -1$ (UP TO NORMALIZATION)

LET $\varphi_\varepsilon : \mathbb{R}^n \to \mathbb{R}$, $X = (x_1, \dots, x_n)$

$\varphi_\varepsilon(X) = \max\{0, \varepsilon^2 - |x_2| - \|X - e_1\|^2\}$



SO $\nabla\varphi_\varepsilon(X) = \pm e_2 - 2(X - e_1)$

"$\nabla\varphi_\varepsilon(e_1) = \pm e_2$"

$\mathrm{supp}\,\varphi_\varepsilon \subset B_\varepsilon(e_1)$, $\sec_R(e_1 \wedge e_2) = R(e_1, e_2, e_1, e_2) = -1$

$$\implies \int_{S^{n-1}} R(X, \nabla\varphi_\varepsilon, X, \nabla\varphi_\varepsilon)\, dX < 0$$

APPROXIMATING $\varphi_\varepsilon|_{S^{n-1}}$ WITH (HOMOGENEOUS) POLYNOMIALS IN $W^{1,2}(S^{n-1})$

AND SENDING $\varepsilon \searrow 0$ GET A CONTRADICTION WITH $K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0, \forall p \geq 2$

$\square$

TO BETTER UNDERSTAND $K(R, \mathrm{Sym}^p_0(\mathbb{R}^n))$:

$$\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) = (\mathbb{R}\, g \owedge g) \oplus (\mathrm{Sym}^2_0(\mathbb{R}^n)\owedge g) \oplus W \oplus \Lambda^4\mathbb{R}^n$$

IRREDUCIBLE $O(n)$-FACTORS OF CURVATURE OPERATORS

$$R = R_{\mathcal{U}} + R_{\mathcal{L}} + R_W + R_{\Lambda^4}$$

$R_{\Lambda^4} = 0$ IFF R SATISFIES FIRST BIANCHI.

$$R_{\mathcal{U}} = \frac{\mathrm{scal}}{2n(n-1)}\, g \owedge g$$

$$R_{\mathcal{L}} = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{scal}}{n} g\right)\owedge g$$

"SYMMETRIC VERSION OF KULKARNI-NOMIZU PRODUCT" $\bigoplus_{p=0}^{\infty} \mathrm{Sym}^2(\mathrm{Sym}^p\mathbb{R}^n)$

THM. (B.-MENDES). DENOTING BY $\pi = \mathrm{Sym}^2_0(\mathbb{R}^n)$,

$$\boxed{K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) = \left(\frac{n+p-2}{n(p-1)} K(R_{\mathcal{U}},\pi) + \frac{n+2p-4}{n(p-1)} K(R_{\mathcal{L}},\pi) + K(R_W,\pi)\right) \owedge \frac{g^{\owedge p-2}}{(p-2)!}}$$

c.f. $$K(R, \Lambda^p\mathbb{R}^n) = \left(\frac{2(n-p)}{p-1} R_{\mathcal{U}} + \frac{n-2p}{p-1} R_{\mathcal{L}} - 2R_W + 4R_{\Lambda^4}\right) \owedge \frac{g^{\owedge p-2}}{(p-2)!}$$ [LABBI]

PF: $K(\cdot, \mathrm{Sym}^p_0(\mathbb{R}^n)) : \mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \longrightarrow \mathrm{Sym}^2(\mathrm{Sym}^p_0(\mathbb{R}^n))$ IS $O(n)$-EQUIVARIANT

$\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) = \mathcal{U}\oplus\mathcal{L}\oplus W\oplus\Lambda^4$

$\mathrm{Sym}^2(\mathrm{Sym}^p_0(\mathbb{R}^n)) = \underbrace{\mathcal{U}\oplus\mathcal{L}\oplus W}_{\text{EXACTLY ONE COPY OF EACH!}} \oplus \dots$ ← ONLY OTHER INEQUIVALENT IRREP'S

LITTLEWOOD-RICHARDSON RULE FOR $GL(n,\mathbb{C})$ & LITTLEWOOD RESTRICTION RULE & WEYL'S CONSTRUCTION

SCHUR'S LEMMA: SUFFICES TO COMPUTE IN ONE VECTOR. □

3. APPLICATIONS IN DIMENSION 4

CONJECTURALLY, S^4 AND $\mathbb{CP}^2$ ARE THE ONLY 4-MFLDS W/ $\sec>0$ AND $\pi_1=\{1\}$

HOPF PROBLEM: DOES $S^2\times S^2$ ADMIT $\sec>0$?

THM (B.-MENDES). LET (M^4,g) BE A CLOSED RIEM. MFLD, $\pi_1 M=\{1\}$, WITH INDEFINITE INTERSECTION FORM AND $\sec_g>0$. THEN $M\setminus\{p\in M: R_p>0\}$ HAS AT LEAST 2 CONNECTED COMPONENTS W/ NONEMPTY INTERIOR.

RIGIDITY VERSION: $\sec\geq 0 \Longrightarrow \begin{cases} \bullet\ M\setminus\{p\in M: R_p>0\} \text{ HAS 2 CONNECTED COMPONENTS} \\ \bullet\ M \underset{\text{ISOM.}}{\cong} (S^2,g_1)\times(S^2,g_2),\ (S^2,g_i) \text{ HAS } \sec\geq 0. \end{cases}$

PF: (M^4, g) $\sec > 0 \overset{\text{THORPE}}{\Longleftrightarrow} \exists f : M \to \mathbb{R}$ SUCH THAT $R + f* > 0$

- CLAIM: f HAS A ZERO.
- $b_\pm(M) > 0 \xrightarrow[\text{THEORY}]{\text{HODGE}} \exists\, \alpha_\pm \in \Lambda^2_\pm TM \setminus \{0\}, \quad \Delta\alpha_\pm = 0$

  I.E., $*\alpha_\pm = \pm\alpha_\pm$
- $\langle K(*, \Lambda^2\mathbb{R}^4)\alpha_\pm, \alpha_\pm\rangle = 4\langle *\alpha_\pm, \alpha_\pm\rangle = \pm 4\|\alpha_\pm\|^2$
- BOCHNER TECHNIQUE: $\alpha = \alpha_\pm \in \Lambda^2_\pm TM$,

$$0 = \int_M \langle \Delta\alpha, \alpha\rangle \,\mathrm{vol} = \int_M \|\nabla\alpha\|^2 + 2\langle K(R, \Lambda^2\mathbb{R}^4)\alpha, \alpha\rangle \,\mathrm{vol}$$

$$= \int_M \|\nabla\alpha\|^2 + 2\langle K(R + f*, \Lambda^2\mathbb{R}^4)\alpha, \alpha\rangle - 2\langle K(f*, \Lambda^2\mathbb{R}^4)\alpha, \alpha\rangle \,\mathrm{vol}$$

$$= \int_M \|\nabla\alpha\|^2 + 2\langle K(R + f*, \Lambda^2\mathbb{R}^4)\alpha, \alpha\rangle \mp 8f\|\alpha\|^2 \,\mathrm{vol}$$

SO $\mp f > 0 \Rightarrow \alpha_\pm = 0$ (CONTRADICTION), SO f MUST HAVE A ZERO.

- NOTE: $R|_{f^{-1}(0)} > 0$
- SUFFICES TO SHOW $\exists\, p_\pm \in M$ S.T. $f(p_-) < 0 < f(p_+)$ AND $R_p : \Lambda^2 TM \to \Lambda^2 TM$ NOT POSITIVE-DEFINITE $\forall p$ NEAR $p_\pm \in M$.
- IF $\nexists\, p_+$, $R_p \geq 0$ $\forall p \in M$ S.T. $f(p) > 0$.

  THUS, CAN REPLACE f WITH $f_0 = \min\{0, f\} \leq 0$ AND $R + f_0 * \geq 0$
- f HAS A ZERO $\Rightarrow R + f_0 * > 0$ ON SOME OPEN SUBSET $\mathcal{U} \subset M$

  $\Rightarrow \alpha_+ \equiv 0$ ON $\mathcal{U}$ HENCE ON M (CONTRADICTION).
- ANALOGOUSLY IF $\nexists\, p_-$ (USING α_-).

$\square$