BIFURCATION THEORY IN GEOMETRIC ANALYSIS

OUTLINE

1. BIFURCATION
2. CONSTANT MEAN CURVATURE ⟵ ISOPERIMETRIC PROBLEM
3. YAMABE PROBLEM ⟵ YAMABE INVARIANT & EINSTEIN METRICS

1. BIFURCATION

- EULER: BUCKLING OF STRUCTURES (FAILURE UNDER COMPRESSIVE STRESS)
- TURBULENCE / VORTICES IN FLUID DYNAMICS
- OSCILLATIONS IN ELECTRIC CIRCUITS
- POINCARÉ (1885). "TOPOLOGICAL CHANGE IN THE STRUCTURE OF A DYNAMICAL SYSTEM WHEN A PARAMETER CROSSES A BIFURCATION VALUE."

"STATES OF A SYSTEM" "ENERGY OF EACH STATE"

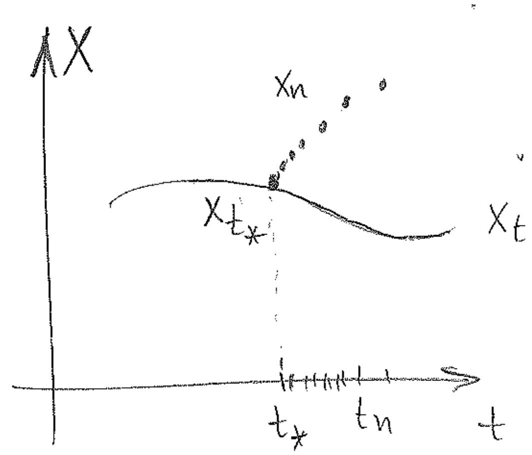
- $f_t : X \longrightarrow \mathbb{R}$ 1-PARAMETER FAMILY OF FUNCTIONALS
- $x_t \in X$ "TRIVIAL" BRANCH OF SOLUTIONS TO $df_t(x_t) = 0$

"GROUND STATE" EULER-LAGRANGE EQN

DEF: x_t SUFFERS BIFURCATION AT t_* IF $\exists\, t_n \to t_*$, $\exists\, x_n \to x_{t_*}$ SUCH THAT

- $df_{t_n}(x_n) = 0$
- $x_n \neq x_{t_n}$

I.E., THE IMPLICIT FUNCTION THEOREM FAILS AT x_{t_*}.



NECESSARY CONDITION: x_{t_*} IS A DEGENERATE CRITICAL POINT ($\mathrm{Hess}\, f_{t_*}(x_{t_*})$ NOT INVERTIBLE)

MORSE INDEX: $i_{Morse}(x) = \#\{\text{NEGATIVE EIGENVALUES OF } \mathrm{Hess}\, f_t(x)\}$

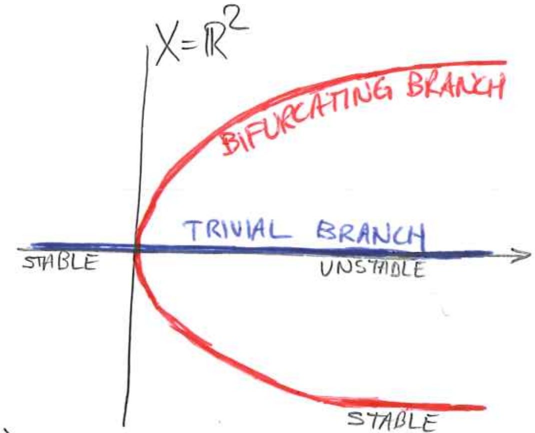
THM (KRASNOSEL'SKII). IF $\exists\, a<b$ S.T. $i_{Morse}(x_a) \neq i_{Morse}(x_b)$, THEN $\exists\, t_* \in (a,b)$ BIFURCATION INSTANT.

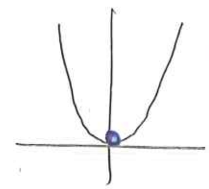
EXAMPLE (TOY MODEL):

$f_t : \mathbb{R}^2 \longrightarrow \mathbb{R}$

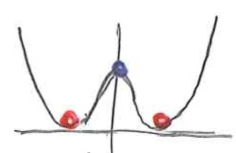
$f_t(x,y) = \frac{1}{2}\left(x^2 + y^4 - t y^2\right)$

$df_t(x,y) = \left(x,\, 2y^3 - ty\right)$





$t<0$



$t<0$

TRIVIAL BRANCH: $(x_t, y_t) = (0,0)$,

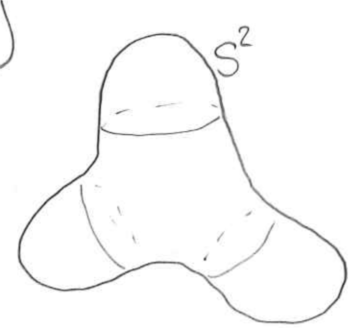
$$i_{Morse}(0,0) = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases} \implies \text{BIFURCATION AT } t_* = 0. \qquad (x_t, y_t) = \left(0, \pm\sqrt{t/2}\right)$$

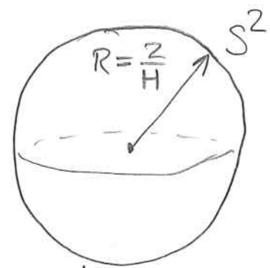
$$\mathrm{Hess}\, f = \begin{bmatrix} 1 & 0 \\ 0 & -t \end{bmatrix}$$

2. CONSTANT MEAN CURVATURE

MOTIVATION (ISOPERIMETRIC PROBLEM): WHAT SURFACE $\Sigma^n \subset M^{n+1}$ HAS LEAST AREA AMONG THOSE ENCLOSING A FIXED VOLUME?

$X = \mathrm{Emb}(\Sigma^n, M^{n+1})$ E.G., $\mathrm{Emb}(S^2, \mathbb{R}^3)$





$df_H(x)=0$

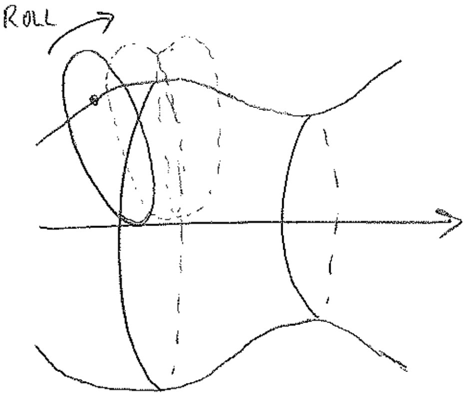
$f_H : X \longrightarrow \mathbb{R}$

$f_H(x) = \mathrm{Area}(x) + H \cdot \mathrm{Vol}(x)$

$\uparrow$ LAGRANGE MULTIPLIER

$$\boxed{df_H(x) = 0 \iff x : \Sigma \hookrightarrow M \text{ HAS CONSTANT MEAN CURVATURE (CMC)}}$$

DELAUNAY (1841): CMC SURFACES OF REVOLUTION IN $\mathbb{R}^3$



UNDULOID

Q: CONSTRUCT CMC SURFACES IN OTHER SPACES?

(SHED LIGHT ON ISOPERIMETRIC PROBLEM)

HARD (GLOBAL) PROBLEM! OPEN IN MOST CASES, E.G. $\mathbb{C}P^n$.

THM (B. - PICCIONE, 2016). THERE ARE INFINITELY MANY BRANCHES OF CMC EMBEDDINGS $\Sigma^{n-1} \hookrightarrow M^n$ THAT BIFURCATE FROM PRINCIPAL ORBITS Σ_t ON COHOMOGENEITY ONE MANIFOLDS.

"TRIVIAL FAMILY" "DELAUNAY-TYPE"

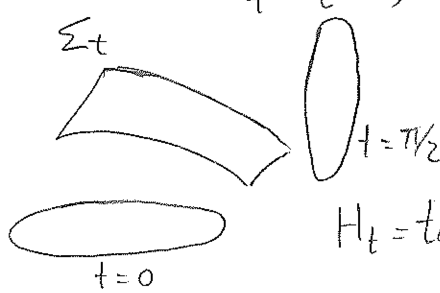
M^n	Σ^{n-1}
S^n	$S^1 \times S^{n-2}$
$\mathbb{C}P^n$	S^{2n-1}, $T_1 \mathbb{R}P^n$
$\mathbb{H}P^n$	S^{4n-1}
KERVAIRE SPHERE M^{2n-1}	$S^1 \times T_1 S^{n-1}$

ISOPERIMETRIC PROBLEM IS OPEN HERE!

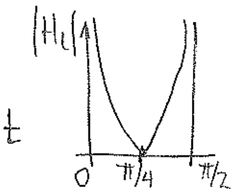
EXAMPLE: $S^3 = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$

$\Sigma_t = \{(z,w) \in S^3 : |z| = \cos t, |w| = \sin t\}$.

$t \in [0, \pi/2]$



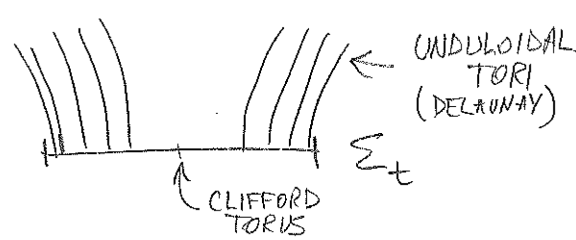
$H_t = \tan t - \cot t$

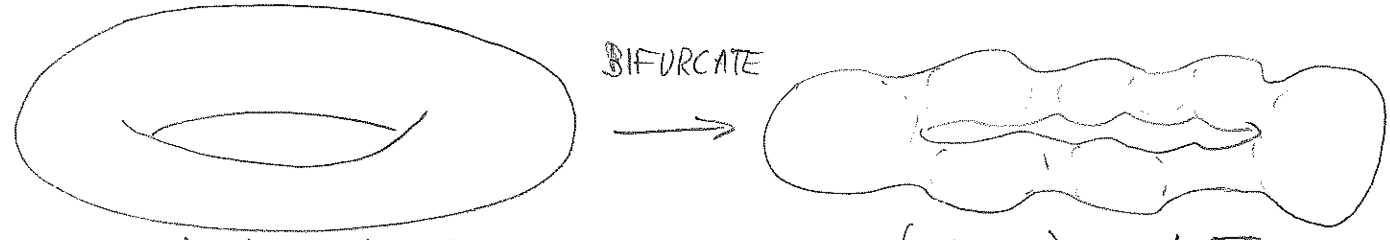


$$i_{Morse}(\Sigma_t) = \#\{\underbrace{\text{EIGENVALUES OF } \Delta_{\Sigma_t}}_{\text{(some stay bounded)}} \text{ THAT ARE } < \mathrm{Ric}(\vec{n}) + \underbrace{|A_{\Sigma_t}|^2}_{(\to\infty)}\} \nearrow +\infty$$

$\Longrightarrow$ SEQUENCE OF BIFURCATIONS

$\Longrightarrow$ NEW CMC SURFACES. (PARTIAL SYMMETRY BREAK) $\square$



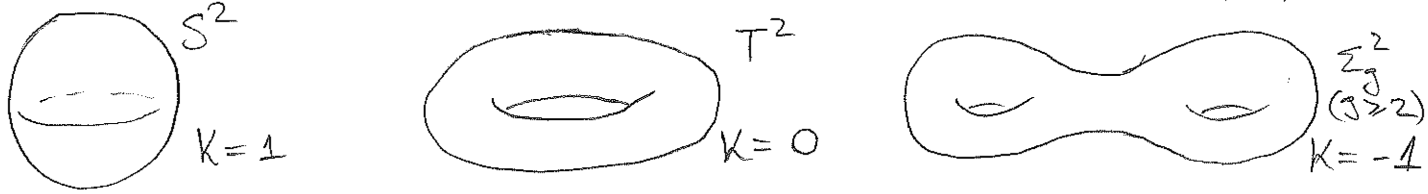


$\mathrm{Iso}(\Sigma_t) = S^1 \times S^1$

$\mathrm{Iso}(\Sigma_{Delaunay}) = S^1 \times \mathbb{Z}_m$

3. YAMABE PROBLEM

UNIFORMIZATION THEOREM: (M^2, g_0) SURFACE $\xrightarrow[\text{CHANGE}]{\text{CONF.}}$ CONSTANT CURVATURE $K = -1, 0, 1$



HIGHER DIMENSIONS:

$\boxed{\text{}}$ THM (YAMABE, TRUDINGER, AUBIN, SCHOEN). (M^n, g_0) CLOSED MANIFOLD ADMITS A CONFORMAL METRIC WITH CONSTANT SCALAR CURVATURE.

HARD PDE PROBLEM (CRITICAL SOBOLEV EXPONENT)

TOOK 24 YEARS TO FIX YAMABE'S MISTAKE. !

- $X = [g_0]_1 = \{$METRICS g CONFORMAL TO g_0 WITH $Vol(M,g)=1\}$
- $f : X \longrightarrow \mathbb{R}$ HILBERT-EINSTEIN FUNCTIONAL
- $f(g) = \int_M scal_g \, vol_g$

PROOF OF THM ABOVE: FIND $\min_{g \in X} f(g)$

$df(g) = 0 \iff g$ HAS CONSTANT SCALAR CURVATURE

Q: IS IT UNIQUE?

THM. THERE ARE INFINITELY MANY BRANCHES OF SOLUTIONS TO THE YAMABE PROBLEM THAT BIFURCATE FROM "TRIVIAL" METRICS g_t ON:

(B. - PICCIONE, 2013). S^{4n+3}, g_t = (HOMOGENEOUS) BERGER METRICS

(B. - PICCIONE, 2013). G/H, g_t = (HOMOGENEOUS) BUNDLE METRICS $H < K < G \leadsto K/H \to G/H \to G/K$.

(B. - PICCIONE - SANTORO, 2016) $S^n \setminus S^1$, $g_t = g_{round} \oplus g_{hyperbolic}$

"SINGULAR YAMABE PROBLEM" (ALSO ON $S^n \setminus S^k$...)

KEY ELEMENT IN PROOFS: JUMP OF MORSE INDEX + ...

(B. - PICCIONE - SIRE, IN PREP.). BIFURCATION FOR CONSTANT Q-CURVATURE PROBLEM (4^{th} ORDER PDE)

"CHEAT SHEET"

CONSTANT MEAN CURVATURE:

$u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $\mathrm{graph}(u) \subset \mathbb{R}^{n+1}$ HAS CMC $= H$

$$\iff \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H$$

$$d^2 f(x)(\psi,\psi) = \int_\Sigma \underbrace{\left(\Delta_\Sigma \psi - (\mathrm{Ric}(\vec{n}) + |A_\Sigma|^2)\psi\right)}_{J_\Sigma \psi} \psi \, \mathrm{vol}_\Sigma$$

YAMABE PROBLEM:

(M^n, g) $\tilde{g} = \varphi^{\frac{4}{n-2}} g$ HAS $\mathrm{scal}_{\tilde{g}} = c \iff$

$$-4\frac{n-1}{n-2}\Delta_g \varphi + \mathrm{scal}_g \cdot \varphi = c \cdot \varphi^{\frac{n+2}{n-2}}, \quad f(\varphi) = \frac{\int_M 4\frac{n-1}{n-2}|\nabla\varphi|^2 + \mathrm{scal}_g \varphi^2}{\left(\int_M \varphi^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}$$

$$d^2 f(\tilde{g})(\psi,\psi) = \frac{(n-1)(n-2)}{2}\int_M \left(\Delta_{\tilde{g}}\psi - \frac{\mathrm{scal}_{\tilde{g}}}{n-1}\right)\psi \cdot \mathrm{vol}_{\tilde{g}}$$

KERVAIRE SPHERE:

$M_d^{2n-1} \subset \mathbb{C}^{n+1}$ $\quad \begin{cases} z_0^d + z_1^2 + \dots + z_n^2 = 0 \\ |z_0|^2 + |z_1|^2 + \dots + |z_n|^2 = 1 \end{cases}$ $\quad$ (COHOM. ONE ACTION $G = SO(2) \times SO(n)$)

n, d ODD $\Rightarrow M_d^{2n-1}$ HOMEOMORPHIC TO S^{2n-1}

$2n-1 \equiv 1 \bmod 8 \Rightarrow M_d^{2n-1}$ NOT DIFFEOMORPHIC TO S^{2n-1}. (EXOTIC)