NON-UNIQUENESS OF SOLUTIONS TO THE YAMABE PROBLEM ON COMPACT AND NONCOMPACT MANIFOLDS

MFO
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WORKSHOP ON ANALYSIS, GEOMETRY AND TOPOLOGY OF POSITIVE SCALAR CURVATURE METRICS
OBERWOLFACH, AUGUST 2017.

OUTLINE:

§1, YAMABE PROBLEM (AND MAIN RESULTS)

§2. 2 METHODS FOR NONUNIQUENESS

(§3) BONUS: Q-CURVATURE

§1. YAMABE PROBLEM

POSSIBLY NONCOMPACT AND INCOMPLETE

YAMABE PROBLEM: GIVEN (M^n, g_0), FIND COMPLETE CONFORMAL METRIC $g \in [g_0]$ WITH $scal_g = const.$

$\Updownarrow$

FIND $u : M \to \mathbb{R}$, $u > 0$, $u \nearrow +\infty$ FAST ENOUGH, S.T.

$$4\frac{n-1}{n-2}\Delta_{g_0} u + scal_{g_0} u = scal_g \cdot u^{\frac{n+2}{n-2}}$$

$$\left(g = u^{\frac{4}{n-2}} g_0\right)$$

EXISTENCE
- M COMPACT: YES [YAMABE, TRUDINGER, AUBIN, SCHOEN]
- M NONCOMPACT: NOT ALWAYS [JIN, MAZZEO-PACARD, GROSSE]

(☆) TODAYS TALK:

UNIQUENESS
- M COMPACT:
- M NONCOMPACT:

FAILS VERY OFTEN! ← EARLIER RESULTS BY [SCHOEN, POLLACK, MALCHIODI, ...]

CENTRAL BOARD:

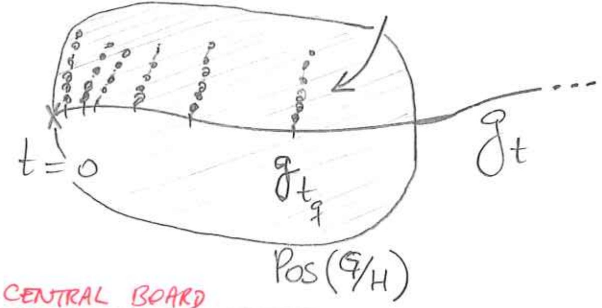
THM A. (B.-PICCIONE '13) LET $H < K < G$ BE COMPACT LIE GROUPS WITH $H \lhd K$ OR $K \lhd G$, AND $scal_{K/H} > 0$. LET g_t BE THE HOMOGENEOUS METRICS ON G/H OBTAINED RESCALING BY $t > 0$ THE FIBERS OF

$$K/H \to G/H \to G/K$$

THEN $\exists\, t_q \downarrow 0$ SEQUENCE OF BIFURCATION INSTANTS FOR g_t.

↑ RECENTLY GENERALIZED BY [OTOBA-PETEAN, '16]

EACH g_{t_q} IS A LIMIT OF OTHER METRICS WITH $scal = const.$



EXAMPLES: BERGER SPHERES

$$S^3 \longrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n$$

$$S^7 \longrightarrow S^{15} \longrightarrow S^8(1/2)$$

⚠ NORMALITY ASSUMPTIONS ARE VIOLATED...

CENTRAL BOARD

THM B (B.-PICCIONE-SANTORO '16). THERE EXIST INFINITELY MANY PAIRWISE NONHOMOTHETIC SOLUTIONS TO THE YAMABE PROBLEM ON $(S^n \setminus S^1, g_{round})$, $n \geq 5$, "BIFURCATING FROM g_{prod}"

$\hookrightarrow$ "SINGULAR YAMABE PROBLEM": $M = \overline{M} \setminus \Lambda$ [SCHOEN-YAU, MAZZEO-PACARD, ...]

$\overline{M}$ ↑ CLOSED MFLD

Λ ↑ CLOSED SUBSET

CENTRAL BOARD

THM C (B.-PICCIONE '16). LET (M,g) BE A CLOSED MANIFOLD WITH $scal_g = const > 0$ AND (N,h) BE A SIMPLY-CONNECTED SYMMETRIC SPACE OF NONCOMPACT OR EUCLIDEAN TYPE, S.T. $(M \times N, g \oplus h)$ HAS $scal > 0$. THEN THERE ARE INFINITELY MANY SOLUTIONS TO THE YAMABE PROBLEM ON $(M \times N, g \oplus h)$.

EXAMPLES: $S^m \times \mathbb{H}^d$, $2 \leq d \leq m$

$S^m \times \mathbb{R}^d$, $m \geq 2$, $d \geq 1$

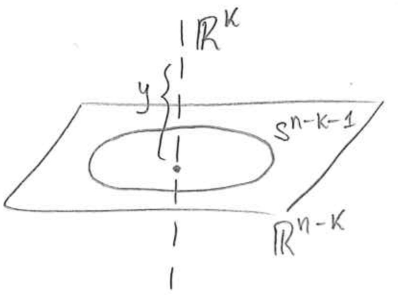
⚠ TRULY NONCOMPACT PHENOMENA: THERE ARE NO COMPACT QUOTIENTS TO WHICH ALL SOLUTIONS DESCEND.

COROLLARY. THERE ARE INFINITELY MANY SOLUTIONS TO THE (SINGULAR) YAMABE PROBLEM ON $S^n \setminus S^k$, FOR ALL $0 \leq k < \frac{n-2}{2}$.

MAXIMAL RANGE FOR K WHERE NONUNIQUENESS IS POSSIBLE

PROOF (THM C $\Rightarrow$ COR.):

$$(S^n \setminus S^k, g_{round}) \xrightarrow[\text{stereog. proj}]{\cong} (\mathbb{R}^n \setminus \mathbb{R}^k, g_{flat}) \xrightarrow[\cdot \frac{1}{r^2}]{\cong} (S^{n-k-1} \times \mathbb{H}^{k+1}, g_{prod})$$



$$g_{flat} = dr^2 + r^2 d\theta^2 + dy^2$$

$$\cdot \frac{1}{r^2} \downarrow$$

$$d\theta^2 + \frac{dr^2 + dy^2}{r^2} = g_{S^{n-k-1}} \oplus g_{\mathbb{H}^{k+1}} =: g_{prod}.$$

NOTE: $scal(g_{prod}) = (n-2k-2)(n-1) > 0 \iff k < \frac{n-2}{2}$

§2. 2 METHODS FOR NON UNIQUENESS

BUT FIRST: CRASH COURSE ON YAMABE PROBLEM:

- $\mathcal{A} : [g_0]_1 \longrightarrow \mathbb{R}, \quad \mathcal{A}(g) = \int_M \mathrm{scal}_g \cdot \mathrm{vol}_g$
- $d\mathcal{A}(g) = 0 \iff g \in [g_0]_1$ HAS $\mathrm{scal}_g = \mathrm{const.}$
- YAMABE INVARIANT: $Y(M,[g_0]) = \inf\limits_{g \in [g_0]} \mathcal{A}(g)$

THM. $\exists\, g_Y \in [g_0]$ YAMABE METRIC, S.T. $Y(M,[g_0]) = \mathcal{A}(g_Y)$, AND ("AUBIN INEQUALITY") $Y(M,[g_0]) \leq Y(S^n,[g_{\mathrm{round}}])$, WITH EQUALITY IF AND ONLY IF (M,g_0) IS CONFORMALLY EQUIV. TO $(S^n, g_{\mathrm{round}})$.

- BIFURCATION METHOD:

(M,g_t) COLLAPSING FAMILY OF METRICS WITH $\mathrm{scal} = \mathrm{const} > 0$ $\rightsquigarrow$ $i_{\mathrm{Morse}}(g_t) \nearrow +\infty$ AS $t \searrow 0$ (COLLAPSE) $\rightsquigarrow$ $\exists\, t_q \searrow 0$ SEQUENCE OF BIFURCATION INSTANTS

SPECTRAL ANALYSIS

KRASNOSELS'KII THM

PROOF OF THM A:

$K/H \to (G/H, g_t) \to G/K$ TOT. GEOD. FIBERS

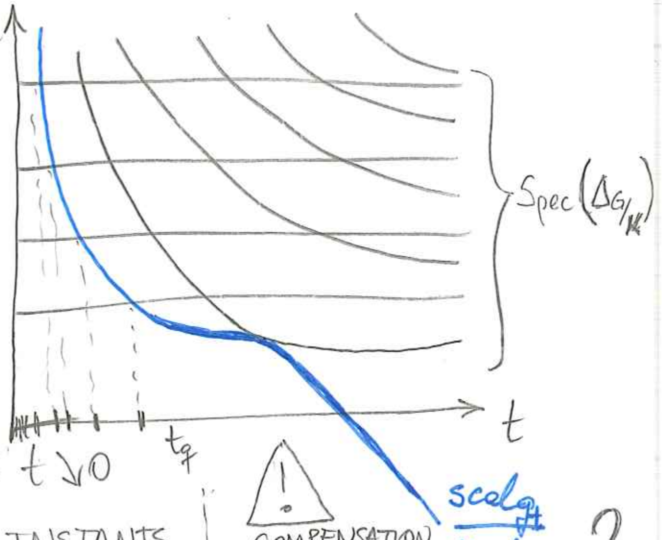
$\Rightarrow \mathrm{Spec}(\Delta_{G/K}) \subset \mathrm{Spec}(\Delta_{g_t}), \ \forall t > 0$

$\mathrm{scal}_{K/H} > 0 \Rightarrow \mathrm{scal}_{g_t} \nearrow +\infty$ AS $t \searrow 0$.

THUS, SINCE $d^2\mathcal{A}(g)(\psi,\psi) = \int_M \left(\Delta_g \psi - \frac{\mathrm{scal}_g}{n-1}\right)\psi$,

$i_{\mathrm{Morse}}(g_t) = \#\, \mathrm{Spec}(\Delta_{g_t}) \cap \left(-\infty, \frac{\mathrm{scal}_{g_t}}{n-1}\right) \nearrow +\infty$ AS $t \searrow 0$

$\Rightarrow \exists\, t_q \searrow 0$ SEQUENCE OF BIFURCATION INSTANTS.



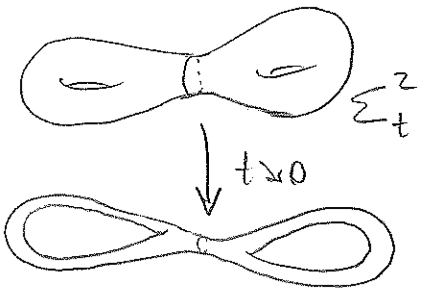
⚠ COMPENSATION ISSUE...

SOL: EQUIV. BIF., OR [OTOBA-PETEAN]

PROOF OF THM B: $S^n \setminus S^1 \cong S^{n-2} \times \mathbb{H}^2$

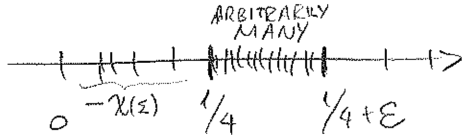
$$\downarrow$$

$S^{n-2} \times \Sigma^2_t$ $\qquad \Sigma^2_t = \mathbb{H}^2/\Gamma_t$ HYPERBOLIC SURFACES



HYPERBOLIC METRICS

THM (BUSER '77). $\forall \varepsilon > 0,\ j \in \mathbb{N},\ \exists h \in \mathcal{H}(\Sigma)$ WITH $\lambda_j(\Sigma, h) < \frac{1}{4} + \varepsilon$.



$$\Rightarrow i_{Morse}(g_{round} \oplus h_t) = \#\Big\{\lambda_i(S^{n-2}, g_{round}) + \underbrace{\lambda_j(\Sigma^2, h_t)}_{\substack{\text{ONLY PART}\\ \text{THAT CHANGES}\\ \text{WITH } t>0}} < \underset{\frac{scal}{n-1}}{\underset{\shortparallel}{n-4}}\Big\} \nearrow +\infty.$$

PULL-BACK BIF. METRICS TO $S^n \setminus S^1$. □

NOTE: THIS IDEA FAILS FOR $k \geq 2$, BY MOSTOW RIGIDITY!

- COVERINGS METHOD (SUGGESTED BY C. LEBRUN)

(M, g) WITH $scal_g = const > 0$ AND $\pi_1(M)$ HAS INFINITE PROFINITE COMPLETION $\rightsquigarrow$ FOR $k \gg 1$, PULLBACK METRIC g_k $(M_k, g_k) \to (M, g)$ IS NOT THE YAMABE METRIC, OTHERWISE VIOLATES AUBIN INEQUALITY $\rightsquigarrow$ ITERATING, GET INFINITELY MANY SOLUTIONS ON $\tilde{M}$

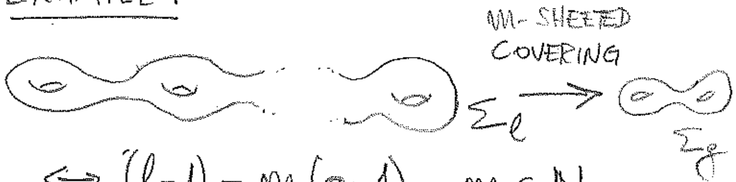
$\Updownarrow$ $\exists$ INFINITE CHAIN OF PROPER FINITE COVERINGS:

$\tilde{M} \to \cdots \to M_k \to \cdots \to M_1 \to M$

(DEF. $\hat{\pi} = \varprojlim \pi/\Gamma,\quad \Gamma \triangleleft \pi,\ [\pi:\Gamma] < \infty$

E.G.: π RESIDUALLY FINITE $\Rightarrow \pi \hookrightarrow \hat{\pi}$)

EXAMPLE:



$$\Leftrightarrow \underset{\frac{1}{2}\chi(\Sigma_\ell)}{\underset{\shortparallel}{(\ell-1)}} = m\,\underset{\frac{1}{2}\chi(\Sigma_g)}{\underset{\shortparallel}{(g-1)}},\quad m \in \mathbb{N}$$

PROOF OF THM C:

- BOREL: N SYMMETRIC SPACE OF NONCOMPACT TYPE $\Rightarrow \exists\, N/\Gamma$ COMPACT QUOTIENT
- SELBERG-MALCEV LEMMA $\Rightarrow \Gamma \subset Iso(N) \subset GL(\ldots, \mathbb{C})$ IS RESIDUALLY FINITE
- THUS, $\pi_1(M \times N/\Gamma) \cong \Gamma$ HAS INFINITE PROFINITE COMPLETION.

BY ABOVE METHOD, GET INFINITELY MANY SOLUTIONS ON $M \times N$. □

§3. Q-CURVATURE [B.-PICCIONE-SIRE], IN PROGRESS...

ALMOST EXACT SAME SITUATIONS PROVIDE NONUNIQUENESS OF CONSTANT Q-CURVATURE METRICS, BUT ALSO NEW PHENOMENA: BIFURCATION WITH $Q < 0$ & $scal < 0$!