Sectional Curvature and Weitzenböck Formulae

SECRET SEMINAR
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(JOINT W/ R. MENDES)

CONJECTURALLY, S^4 AND $\mathbb{CP}^2$ ARE THE ONLY 4-MFLDS W/ $\sec > 0$ AND $\pi_1 = \{1\}$.

THM A. LET (M^4, g) BE A CLOSED RIEM. MFLD, $\pi_1(M) = \{1\}$, WITH INDEFINITE INTERSECTION FORM AND $\sec > 0$. THEN THE SET $M \setminus \{p \in M : R_p > 0\}$ HAS AT LEAST 2 CONNECTED COMPONENTS.

↳ COUNTER-EXAMPLES TO HOPF CONJECTURE ON $S^2 \times S^2$ HAVE TO SATISFY THIS! USE IT TO RULE OUT?

COR: THERE DOES NOT EXIST CURVATURE-HOMOGENEOUS METRICS WITH $\sec > 0$ ON SUCH MANIFOLDS M.

THM B. LET (M^4, g) BE A CLOSED RIEM. MFLD, $\pi_1(M) = \{1\}$, WITH INDEFINITE INTERSECTION FORM AND $\sec \geq 0$. THEN EITHER

(i) $M \setminus \{p \in M : R_p \geq 0\}$ HAS AT LEAST 2 CONNECTED COMPONENTS;

(ii) $(M, g) \underset{\text{isom}}{=} (S^2, g_1) \times (S^2, g_2)$, WHERE (S^2, g_i) HAVE $\sec \geq 0$.

COR: ONLY CURVATURE-HOMOGENEOUS METRICS WITH $\sec \geq 0$ ON SUCH 4-MANIFOLDS ARE PRODUCTS $S^2(r_1) \times S^2(r_2)$.

THM C. LET $R : \Lambda^2 \mathbb{R}^n \to \Lambda^2 \mathbb{R}^n$ BE AN ALGEBRAIC CURVATURE OPERATOR. THEN $\sec_R \geq 0$ IF AND ONLY IF THE CURVATURE TERMS $K(R, \mathrm{Sym}^p_0(\mathbb{R}^n))$ IN THE WEITZENBÖCK FORMULAE FOR TRACELESS SYMMETRIC p-TENSORS ARE POSITIVE-SEMIDEFINITE $\forall p \geq 2$.

↳ CF. THM (HITCHIN). $R > 0 \Leftrightarrow K(R, \rho) > 0 \;\; \forall \rho$ IRREDUCIBLE.

RECALL:
$$K(R, \rho) = -\sum_{a,b} R_{ab}\, d\rho(X_a)\, d\rho(X_b), \quad \{X_a\} \text{ O.N. - BASIS OF } \Lambda^2\mathbb{R}^n \cong \mathfrak{so}(n).$$

WHERE $R = \sum_{a,b} R_{ab} X_a \otimes X_b$, AND $\Delta = \nabla^* \nabla + t\, K(R, \rho)$.

THM D. LET $R : \Lambda^2\mathbb{R}^n \longrightarrow \Lambda^2\mathbb{R}^n$ BE A MODIFIED CURVATURE OPERATOR. THEN

$$K(R, \Lambda^p\mathbb{R}^n) = \left(\frac{2(n-p)}{p-1} R_{\mathcal{U}} + \frac{n-2p}{p-1} R_{\mathcal{L}} - 2R_W + 4R_{\Lambda^4}\right) \owedge \frac{g^{\owedge(p-2)}}{(p-2)!}$$

$$K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) = \left(\frac{n+p-2}{n(p-1)} K(R_{\mathcal{U}}, \mathrm{Sym}^2_0(\mathbb{R}^n)) + \frac{n+2p-4}{n(p-1)} K(R_{\mathcal{L}}, \mathrm{Sym}^2_0(\mathbb{R}^n)) + K(R_W, \mathrm{Sym}^2_0(\mathbb{R}^n))\right) \varovee \frac{g^{\varovee(p-2)}}{(p-2)!}$$

WHERE $R = R_{\mathcal{U}} + R_{\mathcal{L}} + R_W + R_{\Lambda^4}$ IS THE DECOMPOSITION INTO $O(n)$-IRREDUCIBLE COMPONENTS, AND $\owedge$, $\varovee$ ARE THE SKEW-SYMMETRIC AND SYMMETRIC KULKARNI-NOMIZU PRODUCTS.

SKETCH OF PROOF OF THM D:

$\boxed{\text{LITTLEWOOD-RICHARDSON RULE FOR } GL(n,\mathbb{C})}$

$+$

$\boxed{\text{LITTLEWOOD RESTRICTION RULE}}$

$\Downarrow$

SCHUR FUNCTOR

LITTLEWOOD-RICHARDSON NUMBERS (DEFINED COMBINATORIALLY FROM PARTITIONS λ, μ, ν)

$$\mathbb{S}_\lambda \otimes \mathbb{S}_\mu = \bigoplus_\nu N_{\lambda\mu\nu} \mathbb{S}_\nu$$

$$\mathrm{Res}^{GL(n,\mathbb{C})}_{O(n,\mathbb{C})} \mathbb{S}_\nu(\mathbb{C}^n) = \bigoplus_{\bar\lambda} N_{\nu\bar\lambda} \mathbb{S}_{[\bar\lambda]}(\mathbb{C}^n)$$

$$N_{\nu\bar\lambda} = \sum_\delta N_{\delta\bar\lambda\nu}$$

- $\forall n \geq 4$, $\forall 2 \leq p \leq n-2$, $\mathrm{Sym}^2(\Lambda^p\mathbb{R}^n)$ CONTAINS <u>EXACTLY ONE</u> COPY OF EACH $O(n)$-IRREDUCIBLE $\mathcal{U} = \mathbb{R}$, $\mathcal{L} = \mathrm{Sym}^2_0(\mathbb{R}^n)$, W, $\Lambda^4\mathbb{R}^n$. ← ALL $O(n)$-IRREDUCIBLES IN $\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n)$
- $\forall n \geq 4$, $\forall p \geq 2$, $\mathrm{Sym}^2(\mathrm{Sym}^p_0(\mathbb{R}^n))$ CONTAINS <u>EXACTLY ONE</u> COPY OF EACH $O(n)$-IRREDUCIBLE $\mathcal{U} = \mathbb{R}$, $\mathcal{L} = \mathrm{Sym}^2_0(\mathbb{R}^n)$, AND W. <u>NO COPIES</u> OF $\Lambda^4\mathbb{R}^n$!

- $\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \ni R \longmapsto K(R,\rho) \in \mathrm{Sym}^2(\rho)$ IS $O(n)$-EQUIVARIANT
- $K(R,\rho_1\oplus\rho_2) = K(R,\rho_1)\oplus K(R,\rho_2)$
- $K(R,\rho)=0$ IF ρ IS THE TRIVIAL REPRESENTATION
- $K(R,\rho)$ IS POSITIVE-DEFINITE IF R IS POSITIVE-DEFINITE AND ρ DOESN'T CONTAIN TRIVIAL FACTORS.

- USING THE ABOVE OBSERVATIONS, IT FOLLOWS THAT THE FORMULAS FOR $K(R,\rho)$ CAN BE PROVED VIA SCHUR'S LEMMA BY COMPUTING BOTH SIDES IN A REPRESENTATIVE $R_{\mathcal{U}}, R_{\mathcal{L}}, R_{\mathcal{W}}, R_{\Lambda^4}$ OF EACH IRREDUCIBLE. ← MESSY, LENGTHY COMPUTATIONS! FOR $\Lambda^p\mathbb{R}^n$, SEE LABBI TOO

NOVELTY: SYMMETRIC KULKARNI-NOMIZU PRODUCT $\circledvee$

ALLOWS TO USE $K(R,\mathrm{Sym}^2)$ AS "BASIS" FOR COMPARISON

$$\mathcal{A} = \bigoplus_{p=0}^{+\infty} \mathrm{Sym}^2(\mathrm{Sym}^p(\mathbb{R}^n))$$

MAIN DIFFERENCE FROM CLASSIC KULKARNI-NOMIZU ALGEBRA, $\bigoplus_{p=0}^{\infty} \mathrm{Sym}^2(\Lambda^p\mathbb{R}^n)$ →

$$\mathcal{A}_0 = \bigoplus_{p=0}^{+\infty} \mathrm{Sym}^2(\mathrm{Sym}^p_0(\mathbb{R}^n))$$

DEFINITION OF $\circledvee$ →

$$\underbrace{(\underbrace{x_1\vee\dots\vee x_p}_{\in\mathrm{Sym}^p(\mathbb{R}^n)})\otimes(y_1\vee\dots\vee y_p)}_{\in\mathrm{Sym}^2(\mathrm{Sym}^p(\mathbb{R}^n))} \circledvee \underbrace{(v_1\vee\dots\vee v_q)\otimes(w_1\vee\dots\vee w_q)}_{\in\mathrm{Sym}^2(\mathrm{Sym}^q(\mathbb{R}^n))}$$

$$:= (x_1\vee\dots\vee x_p\vee v_1\vee\dots\vee v_q)\otimes(y_1\vee\dots\vee y_p\vee w_1\vee\dots\vee w_q) \in \mathrm{Sym}^2(\mathrm{Sym}^{p+q}(\mathbb{R}^n)),$$

PROPERTIES:
- $\circledvee$ IS BILINEAR AND $O(n)$-EQUIVARIANT
- $A\geq 0 \Rightarrow (A\circledvee g)\geq 0$
- $g^{\circledvee p} = p!\,\mathrm{Id}_{\mathrm{Sym}^p(\mathbb{R}^n)}$

$$\underbrace{\mathcal{A}_0}_{\text{SUBALGEBRA}} \oplus \underbrace{\bigoplus_{p=2}^{\infty}\left[\mathrm{Sym}^2(\mathrm{Sym}^{p-2}(\mathbb{R}^n)) \oplus \mathrm{Sym}^p_0(\mathbb{R}^n)\otimes\mathrm{Sym}^{p-2}_0(\mathbb{R}^n)\right]}_{\text{THIS IS AN IDEAL!}} = \mathcal{A}$$

METRIC → $g\in\mathrm{Sym}^2(\mathbb{R}^n)\subset\mathcal{A}$, AND $g^{\circledvee k} = k!\cdot\mathrm{Id}_{\mathrm{Sym}^k(\mathbb{R}^n)}$

PROOF OF THEOREM A:

THAT IS, $*$ (HODGE STAR)

- (M^4, g) $\sec > 0 \overset{\text{THORPE}}{\iff} \exists f: M \to \mathbb{R}$ SUCH THAT $(R + f\,\mathrm{vol}) > 0$.

- SINCE $R_p > 0$ $\forall p \in f^{-1}(0)$, <u>IT SUFFICES TO SHOW</u> $\exists p_\pm \in M$ S.T. $f(p_-) < 0 < f(p_+)$ AND $R_{p_\pm}$ NOT POSITIVE-DEFINITE.

- IF $\nexists p_+$ AS ABOVE, $R_p > 0$ WHENEVER $f(p) > 0$. THEN, CAN <u>REPLACE f WITH $f_+ = \max\{f, 0\} \geq 0$,</u> THAT IS, $R + f_+ \mathrm{vol} > 0$.

- As M HAS <u>INDEFINITE INTERSECTION FORM,</u> BY <u>HODGE THEORY,</u>
$\hookrightarrow b_+(M) > 0$ & $b_-(M) > 0$
$\exists \alpha_\pm \in \Omega^2_\pm(M) \setminus \{0\}$ NONTRIVIAL SELF-DUAL/ ANTI-SELF-DUAL HARMONIC 2-FORMS, THAT IS, $\Delta \alpha_\pm = 0$, $* \alpha_\pm = \pm \alpha$, $\alpha_\pm \neq 0$.

- <u>BOCHNER TECHNIQUE:</u> INTEGRATE THE WEITZENBÖCK FORMULA FOR $\rho = \Lambda^2 \mathbb{R}^4$

$$0 = \langle \Delta\alpha, \alpha \rangle = \|\nabla\alpha\|^2 + \langle K(R, \Lambda^2\mathbb{R}^4)\alpha, \alpha \rangle$$

$$= \|\nabla\alpha\|^2 + \underbrace{\langle K(R + f_+ \mathrm{vol}, \Lambda^2\mathbb{R}^4)\alpha, \alpha \rangle}_{> 0 \text{ b/c } R + f_- \mathrm{vol} > 0,\ \alpha \neq 0} - \underbrace{\langle f_+ K(\mathrm{vol}, \Lambda^2\mathbb{R}^4)\alpha, \alpha \rangle}$$

$$\overset{K(\mathrm{vol}, \Lambda^2\mathbb{R}^4) = 4\,\mathrm{vol}}{=} 4\langle f_+ \mathrm{vol}(\alpha), \alpha \rangle$$

$$= 4\langle f_+ \mathrm{vol}, \alpha \wedge \alpha \rangle$$

$$= 4\int_M f_+ \cdot \underbrace{\alpha \wedge \alpha}_{\in \Omega^4(M)}$$

$$\overset{\alpha = \alpha_- \in \Omega^2_-(M)}{=} -4\int_M f_+ \cdot \alpha \wedge *\alpha$$

$$= -4\int_M f_+ |\alpha|^2 \leq 0$$

- THIS GIVES A <u>CONTRADICTION,</u> HENCE $\exists p_+ \in M$ AS ABOVE.

- ANALOGOUSLY, USE $b_+(M) > 0$ (I.E. $\alpha_+ \in \Omega^2_+(M) \setminus \{0\}$, $\Delta\alpha_+ = 0$) TO SHOW $\exists p_- \in M$ AS ABOVE.

$\square$

PROOF OF THM B:

PROCEED AS IN THM A, BUT NOTE:

- IF $\exists p_\pm \in M$, $f(p_-) < 0 < f(p_+)$, $R_{p_\pm}$ NOT POSITIVE-SEMIDEFINITE, THEN $M \setminus \{p \in M : R_p \geq 0\}$ HAS AT LEAST 2 COMPONENTS, I.E., (i) HOLDS
- IF $\nexists p_\pm \in M$, THEN $f \equiv 0$ HENCE $R \geq 0$. BY THE CLASSIFICATION OF MANIFOLDS WITH $R \geq 0$, IT FOLLOWS THAT $(M,g) \underset{\text{isom.}}{=} (S^2 \times S^2, g_1 \oplus g_2)$ WITH (S^2, g_i) OF $\sec \geq 0$. $\square$

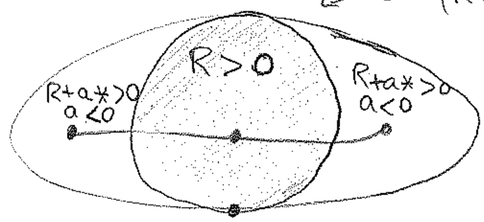
THM: (M^n, g) $R \geq 0$, THEN $\widetilde{M}$ IS A PRODUCT OF FACTORS
- $(\mathbb{R}^n, g)$ WITH $R \geq 0$
- (S^n, g) WITH $R \geq 0$
- COMPACT IRRED. SYMMETRIC SPACE
- COMPACT KÄHLER MFLD BIHOLOM. TO $\mathbb{CP}^n$ WITH $R|_{\Lambda^{1,1}} \geq 0$.

REMARKS:

(1) CANNOT USE THIS TO PROVE THE "LOCAL HOPF CONJECTURE"; IN FACT IT DOES NOT RULE OUT THE EXISTENCE OF $g_n \in \mathrm{Met}(S^2 \times S^2)$ WITH $\sec_{g_n} > 0$ AND $g_n \to g_{\text{product}}$.

IDEA WOULD BE TO FURTHER EXPLORE THAT $R : S^2 \times S^2 \to \mathrm{Sym}^2(\Lambda^2 \mathbb{R}^4) / O(4)$ MUST SATISFY:

$\{R : \sec_R > 0\}$



$\subset \mathrm{Sym}^2(\Lambda^2 \mathbb{R}^4) / O(4)$

MAYBE $GL(6,\mathbb{R})$?

COMMON ACCUMULATION POINTS OF ABOVE 3 SETS ARE PRECISELY PRODUCT METRICS ON $S^2 \times S^2$.

(2) (M^4, g) EINSTEIN, $\sec > 0$, INDEFINITE INTERSECTION FORM

$$\implies |W^\pm|^2 < 6\left(\frac{scal}{12} \pm f\right)^2 \implies \chi(M) < \frac{3}{8\pi^2}\int_M \left(\frac{scal^2}{24} + 4f^2\right)$$

$$\overset{\text{BISHOP}}{\implies} \underset{2 + b_2(M)}{\underset{\parallel}{\chi(M)}} < 6 + \frac{3}{2\pi^2}\int_M f^2$$

cf. LEBRUN-GURSKY

$$\frac{15}{4}|\tau| < \chi < 9$$

SO BETTER ESTIMATE IF $\frac{1}{2\pi^2}\int_M f^2 < 1$

PROOF OF THEOREM C:

$$\mathrm{Sym}^p(\mathbb{R}^n) \cong \{\varphi : \mathbb{R}^n \to \mathbb{R} \text{ HOMOGENEOUS POLYNOMIALS OF DEGREE } p\}$$

$$\mathrm{Sym}^p_0(\mathbb{R}^n) \cong \{\varphi \in \mathrm{Sym}^p(\mathbb{R}^n),\ \Delta\varphi = 0\}.$$

IRREDUCIBLE (AS $O(n)$-REPRESENTATION)

EXAMPLE: IN $\mathbb{R}^2$, $\{e_1, e_2\}$

$\mathrm{Sym}^2(\mathbb{R}^2) = \mathrm{span}\{x_1^2, x_2^2, x_1x_2\}$

$\mathrm{Sym}^2_0(\mathbb{R}^2) = \mathrm{span}\{x_1^2 - x_2^2\}$

DIRECT COMPUTATION GIVES:

$$(*) \quad \left\langle K\left(R, \mathrm{Sym}^p_0(\mathbb{R}^n)\right)\varphi, \psi\right\rangle = c_p \int_{S^{n-1}} R(X, \nabla\varphi, X, \nabla\psi)\, dX$$

THUS $\quad \sec_R \geq 0 \implies K\left(R, \mathrm{Sym}^p_0(\mathbb{R}^n)\right) \geq 0, \ \forall p \geq 2.$

FOR THE CONVERSE, ASSUME $K\left(R, \mathrm{Sym}^p_0(\mathbb{R}^n)\right) \geq 0, \ \forall p \geq 2$

DECOMPOSE $\underbrace{\varphi}_{\text{ANY POLYNOMIAL IN } \mathbb{R}^n} = \varphi_{even} + \varphi_{odd}$. REPLACING φ WITH A POLY WITH SAME RESTRICTION TO S^{n-1}, MAY ASSUME $\varphi_{even}, \varphi_{odd}$ ARE HOMOGENEOUS. SINCE $R(X, \nabla\varphi_{even}, X, \nabla\varphi_{odd})$ IS ODD,

$$\int_{S^{n-1}} R(X, \nabla\varphi, X, \nabla\varphi)\, dX = \int_{S^{n-1}} R(X, \nabla\varphi_{even}, X, \nabla\varphi_{even})\, dX + \int_{S^{n-1}} R(X, \nabla\varphi_{odd}, X, \nabla\varphi_{odd})\, dX$$

$$\mathrm{Sym}^k(\mathbb{R}^n) = \mathrm{Sym}^k_0(\mathbb{R}^n) \oplus \mathrm{Sym}^{k-2}_0(\mathbb{R}^n) \oplus \dots \oplus \underbrace{\mathrm{Sym}^0_0(\mathbb{R}^n)}_{\cong \mathbb{R}} \quad O(n)\text{-IRRED.}$$

(k EVEN)

$$\varphi_{even} = \varphi_k + \varphi_{k-2} + \dots + \varphi_0$$

$$(*) \Rightarrow \int_{S^{n-1}} R(X, \nabla\varphi_{even}, X, \nabla\varphi_{even})\, dX = \sum_{\ell=0}^{k/2} \frac{1}{c_{2\ell}} \underbrace{\left\langle K\left(R, \mathrm{Sym}^{2\ell}_0(\mathbb{R}^n)\right)\varphi_{2\ell}, \varphi_{2\ell}\right\rangle \geq 0}_{\text{BY ASSUMPTION}}$$

SIMILARLY FOR φ_{odd}, HENCE $\int_{S^{n-1}} R(X, \nabla\varphi, X, \nabla\varphi)\, dX \geq 0, \forall \varphi \in C^\infty(\mathbb{R}^n)$

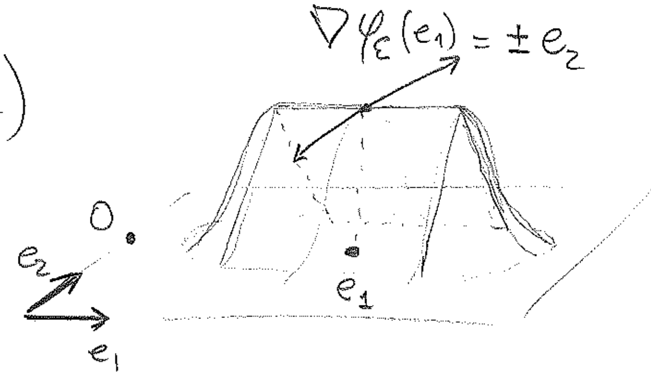
BY DENSITY OF POLY'S.

SUPPOSE BY CONTRADICTION $\exists\, \sigma \in Gr_2(\mathbb{R}^n)$ WITH $\sec_R(\sigma) < 0$.

WLOG, $\sigma = e_1 \wedge e_2$.

LET $\varphi_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}$, $X = (x_1, x_2, \dots, x_n)$

$$\varphi_\varepsilon(X) = \max\{0, \varepsilon^2 - |x_2| - \|X - e_1\|^2\}.$$



NOTE: $0 \leq \varphi_\varepsilon(X) \leq \varepsilon^2$, $\quad \operatorname{supp}(\varphi_\varepsilon) \subset B_\varepsilon(e_1)$, AND

$$\nabla\varphi_\varepsilon(X) = \pm e_2 - 2(X - e_1) \qquad \left(\text{SO } "\nabla\varphi_\varepsilon(e_1) = \pm e_2"\right)$$

SINCE $R(e_1, e_2, e_1, e_2) = \sec(e_1 \wedge e_2) < 0$, BY CONTINUITY, $\exists\, \varepsilon > 0$ S.T. $R(X, \nabla\varphi_\varepsilon, X, \nabla\varphi_\varepsilon) \leq 0$ ON S^{n-1} AND NOT IDENTICALLY ZERO, SO THAT

$$\int_{S^{n-1}} R(X, \nabla\varphi_\varepsilon, X, \nabla\varphi_\varepsilon)\, dX < 0$$

APPROXIMATING φ_ε WITH FUNCTIONS $\psi_j \in C^\infty(\mathbb{R}^n)$ WE OBTAIN THE DESIRED CONTRADICTION (B/C $\int_{S^{n-1}} R(X, \nabla\psi, X, \nabla\psi)\, dX \geq 0, \forall \psi \in C^\infty(\mathbb{R}^n)$) $\square$

NOTE: $\psi_n \in C^\infty(\mathbb{R}^n)$, $\psi_n \longrightarrow \varphi_\varepsilon$ IN $W^{1,2}(\mathbb{R}^n)$ SUFFICES,

AND $\varphi_\varepsilon \in \overline{C^\infty(\mathbb{R}^n)}^{W^{1,2}(\mathbb{R}^n)} =: W^{1,2}(\mathbb{R}^n)$.

$\psi_n \to \varphi_\varepsilon$ IN $L^2(\mathbb{R}^n)$ AND $\nabla\psi_n \to \nabla\varphi_\varepsilon$ IN $L^2(\mathbb{R}^n, \mathbb{R}^n)$.

REMARKS: $K(R, \operatorname{Sym}^2_0(\mathbb{R}^n)) \geq 0 \overset{\text{BERGER}}{\Longrightarrow} K(R, \underbrace{\operatorname{Sym}^1(\mathbb{R}^n)}_{\cong \mathbb{R}^n}) = \operatorname{Ric}_R \geq 0$.

- NO EVIDENCE THAT $K(R, \operatorname{Sym}^{p+1}_0(\mathbb{R}^n)) \geq 0 \Longrightarrow K(R, \operatorname{Sym}^p_0(\mathbb{R}^n)) \geq 0$ FOR $p \geq 2$, UNLESS $n = 3$ (IN WHICH CASE IT SEEMS TO HOLD)
- IF ABOVE WAS TRUE, $\sec_R \geq 0 \Longleftrightarrow$ "$\lim_{p\to\infty} K(R, \operatorname{Sym}^p_0(\mathbb{R}^n)) \geq 0$" ?