NONNEGATIVE SECTIONAL CURVATURE AND WEITZENBÖCK FORMULAE

WORKSHOP ON CURVATURE AND GLOBAL SHAPE

MÜNSTER, JULY 2017
[JOINT W/ R. MENDES]

OUTLINE

1. WEITZENBÖCK FORMULAE
2. ALGEBRAIC CHARACTERIZATION OF $\sec \geq 0$
3. KULKARNI–NOMIZU ALGEBRAS
4. TOWARDS A DECISION ALGORITHM FOR $\sec_R \geq 0$

1. WEITZENBÖCK FORMULAE

(M^n, g) RIEMANNIAN MANIFOLD

$\rho : O(n) \longrightarrow O(E)$ ORTHOGONAL REPRESENTATION

$$\begin{array}{ccc} O(n) \longrightarrow & \mathrm{Fr}(M) \\ & \downarrow \\ & M \end{array} \quad \text{FRAME BUNDLE} \quad \rightsquigarrow \quad \begin{array}{ccc} E \longrightarrow & \mathcal{E}_\rho \\ & \downarrow \\ & M \end{array} \quad \text{ASSOCIATED BUNDLE}$$

EXAMPLES:

$\rho : O(n) \curvearrowright \mathbb{R}^n \Rightarrow \mathcal{E}_\rho = TM$ VECTOR FIELDS

$\rho : O(n) \curvearrowright \mathrm{Sym}^p \mathbb{R}^n \Rightarrow \mathcal{E}_\rho = \mathrm{Sym}^p TM$ SYMMETRIC p-TENSORS

$\rho : O(n) \curvearrowright \Lambda^p \mathbb{R}^n \Rightarrow \mathcal{E}_\rho = \Lambda^p M$ p-FORMS.

WEITZENBÖCK FORMULA:

$$\Delta = \nabla^* \nabla + t\, K(R, \rho)$$

$\nabla^*\nabla = -\sum_i \nabla_{E_i} E_i$ CONNECTION LAPLACIAN

$t \in \mathbb{R}$

CURVATURE TERM

$\{X_a\}$ O.N. BASIS OF $\Lambda^2 \mathbb{R}^n \cong \mathfrak{so}(n)$

$$R = \sum_{a,b} R_{ab}\, X_a \otimes X_b \in \mathrm{Sym}^2(\Lambda^2 \mathbb{R}^n)$$

$$\boxed{K(R, \rho) = -\sum_{a,b} R_{ab}\, d\rho(X_a) \circ d\rho(X_b)}$$

$\rho : O(n) \to O(E)$; $d\rho : \mathfrak{o}(n) \cong \Lambda^2 \mathbb{R}^n \to \mathfrak{o}(E)$.

|* DO EXAMPLES FIRST!|

PROPERTIES:

- $\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \ni R \longmapsto K(R,\rho) \in \mathrm{Sym}^2(E)$ LINEAR AND $O(n)$-EQUIVARIANT
- $K(R,\rho_1 \oplus \rho_2) = K(R,\rho_1) \oplus K(R,\rho_2)$
- $K(R,\rho) = 0$ IF ρ IS TRIVIAL REPRESENTATION
- $K(R,\rho) > 0$ IF $R > 0$ AND ρ HAS NO TRIVIAL FACTORS.

EXAMPLES.

$\rho = \Lambda^1\mathbb{R}^n$: $\mathcal{E}_\rho = \Omega^1(M)$, $K(R,\rho) = Ric$, $t = 2$

$\rho = \mathbb{R}^n$: $\mathcal{E}_\rho = TM$, $K(R,\rho) = Ric$, $t = -2$

VANISHING THEOREM

(M,g) CLOSED, $Ric > 0$
$\Rightarrow b_1(M,\mathbb{R}) = 0$

(M,g) CLOSED, $Ric < 0$
$\Rightarrow |\mathrm{Iso}(M,g)| < \infty$.

THM (HITCHIN). $R \geq 0 \iff K(R,\rho) \geq 0$, $\forall \rho : O(n) \to O(E)$.

PROOF: $(\Rightarrow)$ CHOOSE DIAGONAL BASIS $\{X_a\}$ OF $\Lambda^2\mathbb{R}^n$

$$R X_a = \lambda_a X_a, \quad \lambda_a \geq 0,$$

SKEWSYMMETRIC

$$\langle K(R,\rho)\omega,\omega\rangle = -\sum_a R_{aa} \langle d\rho(X_a)\, d\rho(X_a)\omega, \omega\rangle$$
$$= \sum_a \lambda_a \|d\rho(X_a)\omega\|^2 \geq 0.$$

$(\Leftarrow)$ CONSIDER $X_a \in \mathfrak{X}(SO(n))$ AS LEFT-INVARIANT FIELDS,

$$L = -\sum_{a,b} R_{ab} X_a X_b : L^2(SO(n)) \longrightarrow L^2(SO(n))$$

SELF-ADJOINT 2^{ND} ORDER DIFFERENTIAL OPERATOR, EQUIVARIANT.

PETER-WEYL THM: $L^2(SO(n)) = \overline{\bigoplus_\rho E_\rho}$, ρ IRREDUCIBLE.

$$L(\varphi) = \underbrace{\langle K(R,\rho)\omega_\rho, \omega_\rho\rangle}_{\geq 0} \cdot \varphi, \quad \forall \varphi \in E_\rho \text{ FOR SOME } \omega_\rho \in \mathcal{E}_\rho.$$

HENCE THE SYMBOL OF L IS $R = \sigma(L) \geq 0$. $\square$

2. ALGEBRAIC CHARACTERIZATION OF $\sec \geq 0$

⚠ • $\sec_R \geq 0$ MUCH LESS UNDERSTOOD THAN $R \geq 0$!
• SYLVESTER CRITERION GIVES EASY ALG. CHAR. OF $R \geq 0$.

THM (B.-MENDES). $\sec_R \geq 0 \iff K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0, \ \forall p \geq 2$

NOTE: SIMILARLY, $\sec_R \geq \kappa \iff K(R - \kappa \mathrm{Id}, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0$
$(\leq)$ $(\leq)$

PROOF: $\mathrm{Sym}^p(\mathbb{R}^n) \cong \{ \varphi : \mathbb{R}^n \to \mathbb{R} \text{ HOMOG. POLY. OF DEGREE } p \}$

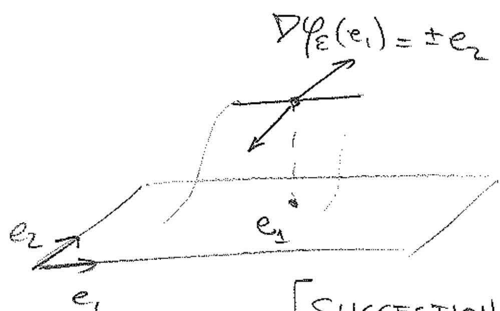
$\mathrm{Sym}^p_0(\mathbb{R}^n) \cong \{ \varphi \in \mathrm{Sym}^p(\mathbb{R}^n) : \Delta \varphi = 0 \}$.

DIRECT COMPUTATION [HEIL-MOROIANU-SEMMELMANN, 2015]:

$$\langle K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \varphi, \varphi \rangle = c_{p,n} \cdot \int_{S^{n-1}} R(X, \nabla\varphi, X, \nabla\varphi)\, dX$$

THUS $\sec_R \geq 0 \Rightarrow K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0, \ \forall p \geq 2$.

CONVERSELY, ARGUE BY CONTRADICTION: SUPPOSE $\exists\, \sigma = e_1 \wedge e_2 \in Gr_2 \mathbb{R}^n$ WITH $\sec_R(\sigma) < 0$.



LET $\varphi_\varepsilon : \mathbb{R}^n \to \mathbb{R}$, $X = (x_1, \dots, x_n)$

$$\varphi_\varepsilon(X) = \max\{0, \varepsilon^2 - |x_2| - \|X - e_1\|^2\}$$

[SUGGESTION BY MARCO RADESCHI]

SO $\nabla\varphi_\varepsilon(X) = \pm e_2 - 2(X - e_1)$

" $\nabla\varphi_\varepsilon(e_1) = \pm e_2$ "

$\mathrm{supp}\, \varphi_\varepsilon \subset B_\varepsilon(e_1)$, $\sec_R(e_1 \wedge e_2) = R(e_1, e_2, e_1, e_2) < 0 \Rightarrow \int_{S^{n-1}} R(X, \nabla\varphi_\varepsilon, X, \nabla\varphi_\varepsilon)\, dX < 0$

APPROXIMATING $\varphi_\varepsilon|_{S^{n-1}} \in W^{1,2}(\mathbb{R}^n)$ WITH (HOMOGENEOUS) POLYNOMIALS, AND SENDING $\varepsilon \searrow 0$, GET A CONTRADICTION WITH $K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0$. $\square$

3. KULKARNI-NOMIZU ALGEBRAS

GOAL: BETTER UNDERSTAND $K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0, \ \forall p \geq 2.$

KULKARNI: $\mathscr{C} = \bigoplus_{p=0}^{n} \mathrm{Sym}^2(\Lambda^p \mathbb{R}^n)$ COMMUTATIVE GRADED ALGEBRA

$\alpha, \beta \in \Lambda^p \mathbb{R}^n, \ \gamma, \delta \in \Lambda^q \mathbb{R}^n$

$$\underbrace{(\alpha \otimes \beta)}_{\in (\Lambda^p \mathbb{R}^n)^{\otimes 2}} \owedge \underbrace{(\gamma \otimes \delta)}_{\in (\Lambda^q \mathbb{R}^n)^{\otimes 2}} := (\alpha \wedge \gamma) \otimes (\beta \wedge \delta) \in \left(\Lambda^{p+q} \mathbb{R}^n\right)^{\otimes 2}$$

EXTEND $\owedge$ BY LINEARITY TO $\bigoplus_{p=0}^{n} (\Lambda^p \mathbb{R}^n)^{\otimes 2}$ AND NOTE $\mathscr{C}$ IS INVARIANT.

SIMILARLY, $\mathcal{A} = \bigoplus_{p=0}^{+\infty} \mathrm{Sym}^2(\mathrm{Sym}^p(\mathbb{R}^n))$ COMMUTATIVE GRADED ALGEBRA

$\alpha, \beta \in \mathrm{Sym}^p \mathbb{R}^n, \ \gamma, \delta \in \mathrm{Sym}^q \mathbb{R}^n$

$$\underbrace{(\alpha \otimes \beta)}_{\in (\mathrm{Sym}^p(\mathbb{R}^n))^{\otimes 2}} \ovee \underbrace{(\gamma \otimes \delta)}_{\in (\mathrm{Sym}^q \mathbb{R}^n)^{\otimes 2}} := (\alpha \vee \gamma) \otimes (\beta \vee \delta) \in \left(\mathrm{Sym}^{p+q}(\mathbb{R}^n)\right)^{\otimes 2}$$

EXTEND $\ovee$ BY LINEARITY TO $\bigoplus_{p=0}^{+\infty} (\mathrm{Sym}^p(\mathbb{R}^n))^{\otimes 2}$ AND NOTE $\mathcal{A}$ IS INVARIANT

PROP: $\left(\mathcal{A}_0 = \bigoplus_{p=0}^{+\infty} \mathrm{Sym}^2(\mathrm{Sym}^p_0(\mathbb{R}^n)), \ovee\right)$ IS A (QUOTIENT) COMMUTATIVE GRADED ALGEBRA

EXAMPLE:

- $g^{\owedge k} = k! \, \mathrm{Id}_{\Lambda^k \mathbb{R}^n} \in \mathrm{Sym}^2(\Lambda^k \mathbb{R}^n) \subset \mathscr{C}$
- $g^{\ovee k} = k! \, \mathrm{Id}_{\mathrm{Sym}^k(\mathbb{R}^n)} \in \mathrm{Sym}^2(\mathrm{Sym}^k \mathbb{R}^n) \subset \mathcal{A}$

PROPERTIES:

- $\owedge$, $\ovee$ ARE BILINEAR AND $O(n)$ - EQUIVARIANT
- $(\cdot) \owedge g$, $(\cdot) \ovee g$ ARE $O(n)$ - EQUIVARIANT AND PRESERVE $A \geq 0$ $(A \geq 0 \Rightarrow A \owedge g \geq 0 \ \& \ A \ovee g \geq 0)$

$$\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) = (\mathbb{R}\, g \owedge g) \oplus (\mathrm{Sym}^2_0(\mathbb{R}^n)\owedge g) \oplus \mathcal{W} \oplus \Lambda^4\mathbb{R}^n$$

$$R = R_{\mathcal{U}} + R_{\mathcal{L}} + R_{\mathcal{W}} + R_{\Lambda^4}$$

IRREDUCIBLE $O(n)$ - COMPONENTS OF CURVATURE OPERATORS

$R_{\Lambda^4} = 0$ IFF R SATISFIES FIRST BIANCHI

$$R_{\mathcal{U}} = \frac{\mathrm{scal}}{2n(n-1)}\, g \owedge g$$

$$R_{\mathcal{L}} = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{\mathrm{scal}}{n} g\right) \owedge g$$

THM (B. - MENDES) DENOTING BY $\pi = \mathrm{Sym}^2_0(\mathbb{R}^n)$,

$$\boxed{K(R, \mathrm{Sym}^p_0(\mathbb{R}^n)) = \left(\frac{n+p-2}{n(p-1)} K(R_{\mathcal{U}},\pi) + \frac{n+2p-4}{n(p-1)} K(R_{\mathcal{L}},\pi) + K(R_{\mathcal{W}},\pi)\right) \owedge \frac{g^{\owedge(p-2)}}{(p-2)!}}$$

cf. $$K(R,\Lambda^p\mathbb{R}^n) = \left(\frac{2(n-p)}{p-1} R_{\mathcal{U}} + \frac{n-2p}{p-1} R_{\mathcal{L}} - 2R_{\mathcal{W}} + 4R_{\Lambda^4}\right) \owedge \frac{g^{\owedge(p-2)}}{(p-2)!}$$ [LABBI]

PROOF: $K(\cdot, \mathrm{Sym}^p_0(\mathbb{R}^n)) : \mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) \longrightarrow \mathrm{Sym}^2(\mathrm{Sym}^p_0(\mathbb{R}^n))$ IS $O(n)$-EQUIV.

$$\mathrm{Sym}^2(\Lambda^2\mathbb{R}^n) = \mathcal{U}\oplus\mathcal{L}\oplus\mathcal{W}\oplus\Lambda^4 \rightsquigarrow \mathrm{Sym}^2(\mathrm{Sym}^p_0(\mathbb{R}^n)) = \underbrace{\mathcal{U}\oplus\mathcal{L}\oplus\mathcal{W}}_{\text{EXACTLY ONE COPY OF EACH!}}\oplus \cdots$$

$\cdots$ ONLY OTHER INEQUIVALENT IRREPS

LITTLEWOOD-RICHARDSON RULE FOR $GL(n,\mathbb{C})$ & LITTLEWOOD RESTRICTION RULE,

WEYL'S CONSTRUCTION, YOUNG DIAGRAMS...

SCHUR'S LEMMA: SUFFICES TO COMPUTE IN ONE VECTOR. $\square$

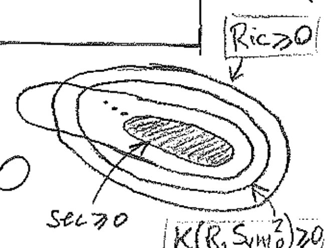
4. TOWARDS A DECISION ALGORITHM FOR $\sec_R \geq 0$

$\boxed{\text{TARSKI (QUANTIFIER ELIMINATION)}} \Longrightarrow \boxed{\exists \text{ FINITE \# OF POLYNOMIAL INEQUALITIES ON } R \text{ EQUIVALENT TO } \sec_R \geq 0.}$

"DECISION ALGORITHM"

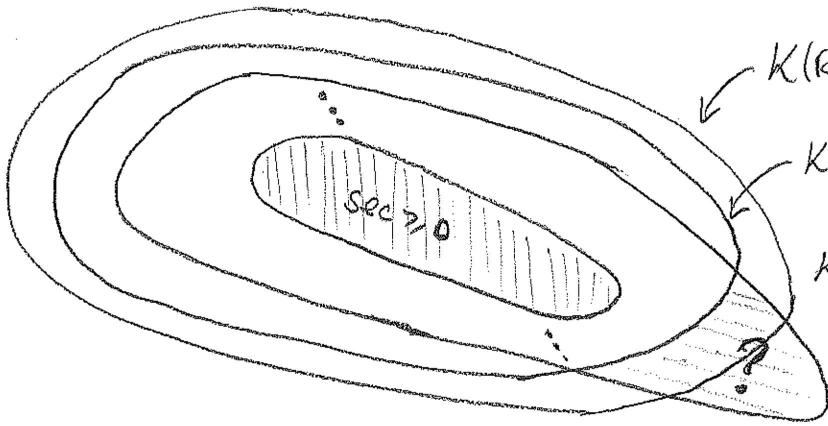
QUESTION: ARE THEY OF THE FORM $K(R,\mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0$?

ALL HAVE EXPLICIT ALG. PARAMETRIZATION $\rightsquigarrow$ DOES THE LIMIT HAVE IT TOO?



BERGER: $K(R,\mathrm{Sym}^2_0(\mathbb{R}^n)) \geq 0 \Longrightarrow K(R,\underbrace{\mathrm{Sym}^1(\mathbb{R}^n)}_{\mathbb{R}^n}) = \mathrm{Ric}_R \geq 0$

IF $K(R,\mathrm{Sym}^{p+1}_0(\mathbb{R}^n)) \geq 0 \overset{(*)}{\Longrightarrow} K(R,\mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0$, THEN WOULD GET $\sec_R \geq 0 \iff$ "$\lim_{p\to\infty} K(R,\mathrm{Sym}^p_0(\mathbb{R}^n)) \geq 0$" ... BUT $(*)$ REMAINS UNCLEAR!



$$\left.\begin{array}{c} K(R, \mathrm{Sym}^1(\mathbb{R}^n)) = \mathrm{Ric}_R \geq 0 \\ \Uparrow \\ K(R, \mathrm{Sym}^2_0(\mathbb{R}^n)) \geq 0 \\ \Uparrow ? \\ K(R, \mathrm{Sym}^3_0(\mathbb{R}^n)) \geq 0 \end{array}\right\}$$

ALL OF THESE HAVE DESCRIPTION