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MULTIPLICITY OF SOLUTIONS TO THE NONCOMPACT YAMABE PROBLEM

YAMABE PROBLEM: GIVEN A RIEM. MFLD (M^n, g_0), FIND A COMPLETE METRIC $g \in [g_0]$ WITH CONSTANT SCALAR CURVATURE.

$$\Delta_{g_0} u - \frac{n-2}{4(n-1)} \mathrm{scal}_{g_0} \cdot u = \frac{n-2}{4(n-1)} \mathrm{scal}_g \cdot u^{\frac{n+2}{n-2}} ; \quad g = u^{\frac{4}{n-2}} \cdot g_0$$

$u > 0$, $u \nearrow \infty$ "FAST ENOUGH" SO g IS COMPLETE.

"Critical Sobolev exponent; elliptic theory breaks down"

M COMPACT: EXISTENCE ✓

[YAMABE '60, TRUDINGER '68, AUBIN '76, SCHOEN '84]

UNIQUENESS X (FAILS IN GENERAL)

YES: IF $\mathrm{scal}_{g_0} \leq 0$, OR $(M, g_0) \neq (S^n, g_{\mathrm{round}})$ EINSTEIN, AND "GENERICALLY" AMONG $\mathrm{scal}_{g_0} > 0$, [ANDERSON POLLACK, ...]

NO: MULTIPLICITY OF SOLUTIONS VIA BIFURCATION, ETC.

M NONCOMPACT: EXISTENCE X

NO: EXAMPLES BY JIN '88

(M, g) closed with finitely many punctures

cf. Schoen: $S^n \setminus \{p_1, \dots, p_k\}$

YES: IF $M(\infty)$ IS TAME ENOUGH (MAZZEO-PACARD, SCHOEN, ...)

UNIQUENESS X

$\boxed{\text{TODAY: MULTIPLICITY OF SOLUTIONS}}$

OUTLINE:

1. MULTIPLICITY VIA COVERINGS
2. MULTIPLICITY VIA BIFURCATION
3. APPLICATION TO THE SINGULAR YAMABE PROBLEM ON $S^n \setminus S^k$.

④ MULTIPLICITY VIA COVERINGS

THM 1 (B. – PICCIONE, 2016). LET (M, g) BE A CLOSED MFLD WITH $\mathrm{scal}_g = \mathrm{const} > 0$, AND (N, h) BE A 1-CONNECTED SYMMETRIC SPACE OF NONCOMPACT OR EUCLIDEAN TYPE, SO THAT $(M \times N, g \oplus h)$ HAS $\mathrm{scal} > 0$. THEN THERE ARE INFINITELY MANY PERIODIC SOLUTIONS TO THE YAMABE PROBLEM ON $(M \times N, g \oplus h)$.

EXAMPLES: $S^m \times \mathbb{H}^d$, $2 \leq d < m$

$S^m \times \mathbb{R}^d$, $m \geq 2$, $d \geq 1$

Note: By Caffarelli-Gidas-Spruck these solutions do not depend on the S^m-variable...

MORE GENERALLY: • (M, g) AS IN THM ABOVE,

• (Σ, h) CLOSED MFLD, S.T. $\pi_1(\Sigma)$ HAS "INFINITE PROFINITE COMPLETION"

THEN THERE ARE INFINITELY MANY SOLUTIONS ON $(M \times \tilde{\Sigma}, g \oplus \tilde{h})$

FACTS:

1. BOREL '63: ANY SYMMETRIC SPACE (N, h) OF NONCOMPACT TYPE HAS COMPACT QUOTIENTS $\Sigma = N/\Gamma$.
2. IF (Σ, h) IS A LOCALLY SYMMETRIC SPACE, THEN $\pi_1(\Sigma)$ HAS INFINITE PROFINITE COMPLETION.

PROFINITE COMPLETION OF A (FIN. GEN.) GROUP G:

$$\hat{G} = \varprojlim G/\Gamma \qquad \Gamma \triangleleft G, \ [G : \Gamma] < \infty.$$

TOPOLOGY: $\pi_1(\Sigma)$ HAS INFINITE PROFINITE COMPLETION

$$\Updownarrow$$

$\exists$ $\tilde{\Sigma} \to \cdots \to \Sigma_k \to \cdots \to \Sigma_2 \to \Sigma_1 = \Sigma$ INFINITE CHAIN OF FINITE-SHEETED REGULAR COVERINGS OF Σ.

EXAMPLE: $\Sigma = T^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\pi_1(\Sigma) = \mathbb{Z}^2$ HAS INFINITE PROFINITE COMPLETION

$$\mathbb{R}^2 \to T^2 \to \dots \to T^2 \to T^2$$ E.G. DOUBLE COVERS.

$\hat{\Sigma} = \tilde{\Sigma}/\Gamma$

GEOMETRY: FOR SUCH (Σ, h), $\forall V > 0$, $\exists\, \hat{\Sigma} \to \Sigma$ FINITE-SHEETED REGULAR COVERING S.T. $Vol(\hat{\Sigma}, \hat{h}) > V$.

TO SOLVE CLOSED YAMABE PROBLEM, STUDY

$$Y(M, [g_0]) = \inf_{g \in [g_0]} \frac{\int_M scal_g \, vol_g}{Vol(M,g)^{\frac{n-2}{2}}} = A(g)$$

THM. THE ABOVE INF IS ATTAINED AT $g_Y \in [g_0]$ CALLED A YAMABE METRIC, WHICH HAS $scal_{g_Y} = const.$ MOREOVER, $Y(M^n, [g_0]) \le Y(S^n, [g_{round}])$.

w/ equality iff $(M, g) \underset{\text{conf. diff.}}{\cong} (S^n, g_{round})$.

PF OF THM 1: $(M \times \Sigma, g \oplus h)$ HAS $scal = const$

If (M,g) has $scal_g = const$, $A(g) = Vol(M,g)^{\frac{2}{n}} \cdot scal_g$

- IF $g \oplus h$ NOT YAMABE, $\exists\, g_Y \in [g \oplus h]$ NEW SOLUTION. SET $g_1 = g_Y$ (OR $g_1 = g \oplus h$).
- LET $\Sigma_2 \xrightarrow{p_2} \Sigma_1 = \Sigma$ BE A FINITE COVERING S.T. $A(p_2^* g_1) > Y(S^n, [g_{round}])$. THEN $\exists\, g_2 \in [p_2^* g_1]$ YAMABE SOLUTION
- ITERATE: $\Sigma_3 \xrightarrow{p_3} \Sigma_2 \xrightarrow{p_2} \Sigma_1$, $A(p_3^* g_2) > Y(S^n, [g_{round}])$ SO $\exists\, g_3 \in [p_3^* g_2]$ YAMABE SOLUTION ...

- EVENTUALLY, LIFTING ALL THESE METRICS g_K FROM $M \times \Sigma_K$ TO $M \times \tilde{\Sigma}$, GET INFINITELY MANY SOLUTIONS TO THE YAMABE PROBLEM ON $M \times \tilde{\Sigma}$. $\square$

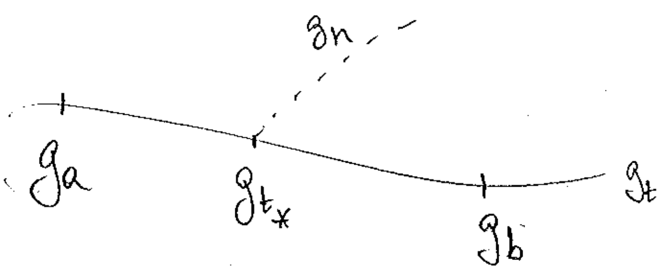
FACT: Lifts of conformal metrics are conformal.

2) MULTIPLICITY VIA BIFURCATION

CONSIDER $(\Sigma, h) = \begin{cases} \mathbb{H}^2/\Gamma & \text{CLOSED HYPERBOLIC SURFACE; OR} \\ \mathbb{R}^n/\pi & \text{CLOSED FLAT MANIFOLD} \end{cases}$

- $\mathcal{H}(\Sigma)$ = MODULI SPACE OF HYPERBOLIC/FLAT METRICS ON Σ.
- $h_t \in \mathcal{H}(\Sigma)$ PATH OF METRICS
- $g_t = g_M \oplus h_t$ PATH OF PRODUCT METRICS ON $M \times \Sigma$.

NOTE: $\mathcal{H}(\Sigma) \cong \mathbb{R}^{6\gamma-6}$ IF $\text{genus}(\Sigma^2) = \gamma$.



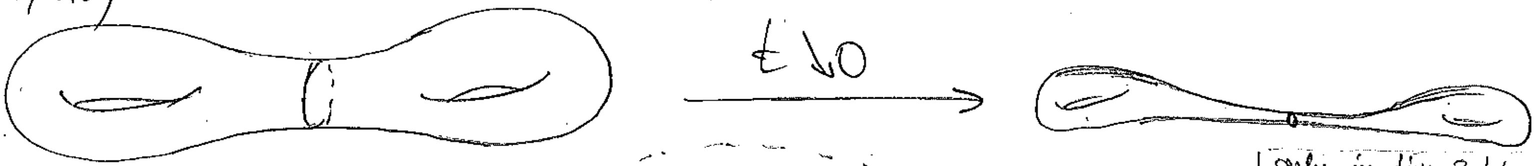
DEF: t_* IS A BIFURCATION VALUE IF $\exists t_n \to t_*$, $\exists g_n \to g_{t_*}$, S.T. $g_n \in [g_{t_n}]$ HAS CONST. SCAL. CURV, $\mathrm{Vol}(g_n) = \mathrm{Vol}(g_{t_n})$, AND $g_n \neq g_{t_n}$.

THM (KRASNOSEL'SKII). $i_{\mathrm{Morse}}(g_a) \neq i_{\mathrm{Morse}}(g_b) \implies \exists t_* \in (a,b)$ BIFURCATION INSTANT

Actually, need also g_a & g_b to be nondegenerate critical points

$$i_{Morse}(g) = \left| \mathrm{Spec}(\Delta_g) \cap \left(-\infty, \frac{scal_g}{n-1}\right) \right|$$ MORSE INDEX OF g AS A CRITICAL PT. OF $\mathcal{A}:[g_0]_1 \to \mathbb{R}$.

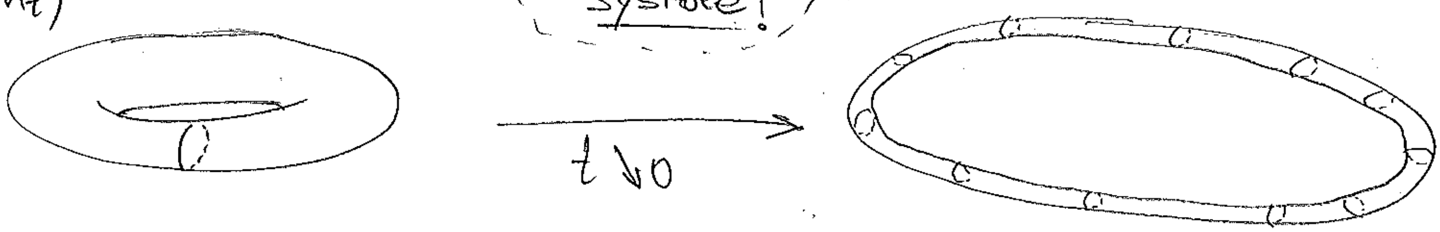
(Σ^2, h_t)



only in dim=2 b/c of Mostow Rigidity!

$(\mathbb{R}^n/\pi, h_t)$

Shrink the systole!



$\rightsquigarrow$ $\mathrm{Spec}(\Sigma, h_t)$ HAS (INFINITELY MANY) EIGENVALUES $\searrow 0$

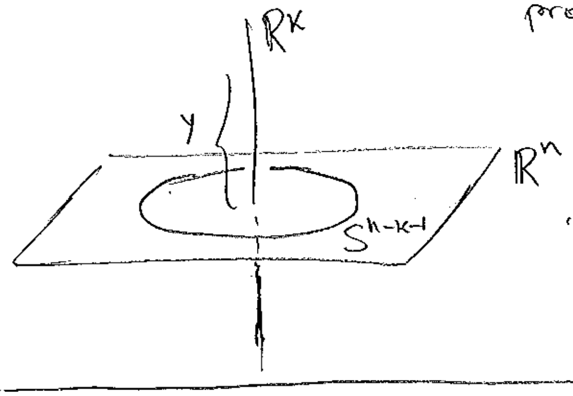
$\Sigma = \mathbb{H}^2/\Gamma$ Buser
$\Sigma = \mathbb{R}^n/\pi$ explicit

"Non-product"! $\rightsquigarrow$ $i_{Morse}(g_t) \nearrow +\infty$ (B/C $scal_{g_t} = \text{const}$).
$\rightsquigarrow$ NEW (BIFURCATING) SOLUTIONS

Cf: In (1), solutions on $M \times \tilde{\Sigma}$ come from infinitely many distinct compact quotients $M \times \Sigma_k$. In (2), solutions come from a single quotient. However, in both cases, get infinitely many periods being realized.

(3) APPLICATION TO SINGULAR YAMABE PROBLEM ON $S^n \setminus S^k$

$$(S^n \setminus S^k, g_{round}) \xrightarrow[\text{stereog. proj.}]{\cong} (\mathbb{R}^n \setminus \mathbb{R}^k, g_{flat}) \xrightarrow{\cong} (S^{n-k-1} \times \mathbb{H}^{k+1}, g_{round} \oplus h)$$



$$dr^2 + r^2 d\theta^2 + dy^2 = g_{flat}$$

$\frac{1}{r^2}$

$$d\theta^2 + \frac{dr^2 + dy^2}{r^2} = g_{round} \oplus h \quad \text{"TRIVIAL SOLUTION"}$$

SING. YAMABE PROBLEM: GIVEN (M, g_0) CLOSED RIEM. MFLD, $\Lambda \subset M$ CLOSED SUBSET, FIND COMPLETE METRIC g ON $M \setminus \Lambda$, CONFORMAL TO g_0, WITH CONSTANT SCALAR CURVATURE.

$\mathrm{scal}_{g_{\mathrm{round}} \oplus h} = (n-2k-2)(n-1) > 0 \iff k < \frac{n-2}{2}$.

To our Knowledge, the first nontrivial solutions to be found

THM 2: THERE ARE INFINITELY MANY PERIODIC SOLUTIONS TO THE SINGULAR YAMABE PROBLEM ON $S^n \setminus S^k$ FOR ALL $0 \leq k < \frac{n-2}{2}$.

$\underline{k=0}$: $S^n \setminus \{\pm p\} \cong \mathbb{R}^n \setminus \{0\} \cong S^{n-1} \times \mathbb{R}$ ← APPLY THM 1

$\underline{1 \leq k < \frac{n-2}{2}}$: $S^n \setminus S^k \cong S^{n-k-1} \times \mathbb{H}^{k+1}$ ← APPLY THM 1

SPECIAL CASE $k=1$: $S^n \setminus S^1 \cong S^{n-2} \times \mathbb{H}^2$, ALTERNATIVELY CAN APPLY BIFURCATION RESULT [B.-PICCIONE-SANTORO, 2014]

$\underline{k \geq \frac{n-2}{2}}$: $S^n \setminus S^k$ HAS $\mathrm{scal} \leq 0$, SO TRIVIAL SOLUTION IS THE UNIQUE PERIODIC SOLUTION, BY THE ASYMPTOTIC MAXIMUM PRINCIPLE.

(SO THE ABOVE RANGE $0 \leq k < \frac{n-2}{2}$ IS THE MAXIMUM POSSIBLE)

FUTURE DIRECTIONS:

1. USE ABOVE TECHNIQUES TO SEARCH FOR MULTIPLICITY RESULTS ON OTHER CONFORMALLY INVARIANT PROBLEMS (Q-CURVATURE, Q_s-CURVATURE / "FRACTIONAL" YAMABE PROB. ...)
[W/ P. PICCIONE AND Y. SIRE]

2. EXTEND ABOVE RESULT FROM $S^n \setminus S^k$ TO OTHER CASES THAT SHARE SAME STRUCTURE NEAR "SINGULAR" LOCUS S^k.

3. CONICAL & STRATIFIED YAMABE PROBLEMS...