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NEW DEVELOPMENTS ON STRONGLY POSITIVE CURVATURE

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1. STRONGLY POSITIVE CURVATURE
2. EXAMPLES
3. BOCHNER TECHNIQUE

1. STRONGLY POSITIVE CURVATURE

- (M^n, g) CLOSED MFLD W/ $\sec > 0$ ARE RARE!
- EXAMPLES: S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$, $\mathbb{C}aP^2$ (CROSS)
  $(\pi_1 = \{1\})$
  OTHERS ONLY IN DIM 6, 7, 12, 13, 24.
- HOPF PROBLEM: DOES $S^2 \times S^2$ ADMIT A METRIC W/ $\sec > 0$?
- CONJECTURALLY, (M^4, g), $\pi_1(M) = \{1\}$, $\sec > 0 \Rightarrow M \cong S^4$ OR $\mathbb{C}P^2$.
- (M^n, g) RIEM. MFLD.
- $R: \Lambda^2 T_pM \to \Lambda^2 T_pM$ CURVATURE OPERATOR
  $\langle R(X \wedge Y), Z \wedge W \rangle = \langle R(X,Y)Z, W \rangle$
- $Gr_2 T_pM = \{ \sigma \in \Lambda^2 T_pM : \sigma \wedge \sigma = 0, \|\sigma\| = 1 \}$ GRASSMANNIAN
- $\sec: Gr_2 T_pM \to \mathbb{R}$, SECTIONAL CURVATURE
  $\sec(X \wedge Y) = \langle R(X \wedge Y), X \wedge Y \rangle$

- $\omega \in \Lambda^4 T_pM \rightsquigarrow \omega : \Lambda^2 T_pM \longrightarrow \Lambda^2 T_pM$

$$\langle \omega(X\wedge Y), Z\wedge W\rangle = \omega(X,Y,Z,W)$$

DEFINES $\Lambda^4 T_pM \hookrightarrow \mathrm{Sym}^2(\Lambda^2 T_pM)$.

- $b : \mathrm{Sym}^2(\Lambda^2 T_pM) \longrightarrow \Lambda^4 T_pM$ ORTHOGONAL PROJECTION IS THE "BIANCHI MAP"

$$b(S)(X,Y,Z,W) = \tfrac{1}{3}\Big(\langle S(X\wedge Y), Z\wedge W\rangle + \langle S(Y\wedge Z), X\wedge W\rangle + \langle S(Z\wedge X), Y\wedge W\rangle\Big)$$

- $R+\omega \in \mathrm{Sym}^2(\Lambda^2 T_pM)$ "MODIFIED CURVATURE OPERATOR"

$$\sec_{R+\omega}(X\wedge Y) = \langle (R+\omega)(X\wedge Y), X\wedge Y\rangle$$
$$= \langle R(X\wedge Y), X\wedge Y\rangle + \underbrace{\langle \omega(X\wedge Y), X\wedge Y\rangle}_{\substack{\| \\ \omega(X,Y,X,Y)}}$$
$$= \sec_R(X\wedge Y).$$

"THORPE'S TRICK"

DEF: (M,g) HAS STRONGLY POSITIVE CURVATURE (RESP. NONNEGATIVE) IF $\forall p\in M$, $\exists\, \omega_p \in \Lambda^4 T_pM$ SUCH THAT

$$(R_p+\omega_p) : \Lambda^2 T_pM \longrightarrow \Lambda^2 T_pM$$

IS POSITIVE-DEFINITE (RESP. POSITIVE-SEMIDEFINITE).

$$\boxed{R>0} \Longrightarrow \boxed{\text{STRONGLY POSITIVE CURVATURE}} \Longrightarrow \boxed{\sec>0}$$

"well-understood" ???

- IF $\dim \leq 3$, THEN ALL ARE EQUIVALENT
- IF $\dim = 4$, THEN STR. POS. CURV. $\Longleftrightarrow$ $\sec>0$.

PROP: IF $\pi:(\overline{M},\overline{g})\to(M,g)$ IS A RIEM. SUBMERSION AND $(\overline{M},\overline{g})$ HAS STRONGLY POSITIVE CURVATURE, THEN SO DOES (M,g).

PF: LET $\alpha = A^*A$ I.E. $\langle \alpha(X\wedge Y), Z\wedge W\rangle = \langle A_X Y, A_Z W\rangle$

REARRANGING GRAY-O'NEILL FORMULA: $\left(\underline{\text{RECALL}}: A_X Y = \frac{1}{2}[\overline{X},\overline{Y}]^v\right)$

$$\langle R(X\wedge Y), Z\wedge W\rangle = \langle \overline{R}(\overline{X}\wedge\overline{Y}), \overline{Z}\wedge\overline{W}\rangle + 3\langle \alpha(\overline{X}\wedge\overline{Y}), \overline{Z}\wedge\overline{W}\rangle - 3b(\alpha)(\overline{X},\overline{Y},\overline{Z},\overline{W})$$

IF $\overline{R}+\overline{\omega}>0$, THEN $\omega := \left(\overline{\omega}+3b(\alpha)\right)\big|_{\Lambda^4 T_p M}$ IS S.T. $R+\omega>0$. $\square$

2. EXAMPLES:

CROSS: S^n $R>0$ ✓

$\mathbb{CP}^n$ $R+\frac{1}{2}(\omega_{FS}\wedge\omega_{FS})>0$ ✓ [NOTE: $R\geq 0$ HAS KERNEL! NON-DECOMP. ELEMENTS]

$\mathbb{HP}^n$ ... ✓

PROP: $\mathbb{C}a\mathbb{P}^2$ DOES NOT HAVE STRONGLY POSITIVE CURVATURE

PF: IF $R+\omega>0$, AVERAGE ω TO GET $\overline{\omega}\in\Omega^4(\mathbb{C}a\mathbb{P}^2)^{F_4}$ S.T. $R+\overline{\omega}>0$, AND $\Delta\overline{\omega}=0$. THEN $0\neq[\overline{\omega}]\in H^4(\mathbb{C}a\mathbb{P}^2;\mathbb{R})=\{0\}$. $\square$

THM (B.-MENDES). ALL SIMPLY-CONNECTED HOMOGENEOUS SPACES WITH $\sec>0$ ADMIT A HOMOGENEOUS METRIC W/ STRONGLY POSITIVE CURVATURE, EXCEPT $\mathbb{C}a\mathbb{P}^2$.

THM (B.-MENDES). "ALL KNOWN EXAMPLES" OF MANIFOLDS WITH $\sec\geq 0$ ADMIT METRICS WITH STRONGLY NONNEGATIVE CURVATURE.

3. BOCHNER TECHNIQUE

- (M,g) CLOSED RIEM. MFLD., ORIENTED, $\dim M = n$.
- $\rho : SO(n) \to E$ REPRESENTATION OF $SO(n)$, $Spin(n)$, ...
- $E \to M$ VECTOR BUNDLE ASSOC. TO PRINCIPAL FRAME BUNDLE BY ρ.
- WEITZENBÖCK FORMULA: $\Delta = D^* D + K^\rho(R)$

$$R = \sum_{a,b} R_{ab} X_a \otimes X_b \rightsquigarrow K^\rho(R) = -\sum_{a,b} R_{ab}\, d\rho(X_a) \circ d\rho(X_b)$$

$\left(\{X_a\} \text{ BASIS OF } \Lambda^2 T_pM \cong \mathfrak{so}(n) \right)$

- IF ρ HAS NO TRIVIAL FACTORS, $R > 0 \Rightarrow K^\rho(R) > 0$.
- $K^\rho : \mathrm{Sym}^2(\Lambda^2 T_pM) \to \mathrm{Sym}^2(E)$ IS $SO(n)$ – EQUIVARIANT.

EXAMPLE: $\rho = \Lambda^2 T_pM$, $K^\rho(\omega) = 4\omega$, $\forall \omega \in \Lambda^4 T_pM$

THM: (M^n, g) CLOSED, W/ STRONGLY POSITIVE CURVATURE. IF $\alpha \in \Omega^2(M)$ SATISFIES $\Delta\alpha = 0$ AND $\alpha \wedge \alpha \equiv 0$, THEN $\alpha \equiv 0$

PF:
$$0 = \langle \Delta\alpha, \alpha \rangle = \langle D^* D\alpha + K^\rho(R+\omega)\alpha - K^\rho(\omega)\alpha, \alpha \rangle$$
$$= \underbrace{|\nabla\alpha|^2}_{\geq 0} + \underbrace{\langle K^\rho(R+\omega)\alpha, \alpha \rangle}_{> 0} - 4 \langle \omega, \underbrace{\alpha \wedge \alpha}_{= 0} \rangle. \qquad \square$$

COR: (M^n, g) CLOSED, WITH 2-DIM FOLIATION $\mathcal{F}$ S.T. $\mathcal{F}$ AND $\mathcal{F}^\perp$ HAVE MINIMAL LEAVES. THEN (M^n, g) DOES NOT HAVE STR. POS. CURV.

PF: $\chi_\mathcal{F}$ CHARACTERISTIC 2-FORM OF $\mathcal{F}$. $\Delta \chi_\mathcal{F} = 0 \iff \mathcal{F}$ AND $\mathcal{F}^\perp$ MINIMAL.

$\boxed{\text{THM (B.-MENDES). } (M^4,g) \text{ CLOSED, } \pi_1 M=\{1\},\ sec>0, \text{ S.T. } \forall p\in M, R_p : \Lambda^2 T_pM \to \Lambda^2 T_pM \text{ IS \underline{NOT} POSITIVE-DEFINITE. THEN } M \cong \#^k \mathbb{CP}^2}$

COR: IF $S^2\times S^2$ (OR $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, OR ANY INDEFINITE 4-MFLD) HAS A METRIC g WITH $sec_g>0$, THEN $\exists N$ 2-SIDED HYPERSURFACE SUCH THAT $R|_N$ IS POSITIVE-DEFINITE.

PF: M^4, $sec>0 \implies \exists f: M\to\mathbb{R}$ S.T. $R+f\cdot vol>0$

BY HYPOTHESIS, $f(p)\neq 0$, $\forall p\in M$; SO ASSUME $f>0$.

IF $\alpha\in\Omega^2(M)$ SATISFIES $\Delta\alpha=0$, $*\alpha=-\alpha$, THEN

$$0=\langle\Delta\alpha,\alpha\rangle=\underbrace{|D\alpha|^2}_{\geq 0}+\underbrace{\langle K^P(R+f vol)\alpha,\alpha\rangle}_{>0}-\langle K^P(f vol)\alpha,\alpha\rangle$$

AND $\langle K^P(f vol)\alpha,\alpha\rangle=\langle 4f\, vol,\alpha\wedge\alpha\rangle=-4\int_M f|\alpha|^2$

$\int_M \alpha\wedge *\alpha=\int_M|\alpha|^2 vol$

HENCE $\alpha=0$, AND $b_2^-(M)=0$.

BY DONALDSON-FREEDMAN, $M\cong\#^k\mathbb{CP}^2$. $(k=b_2^+(M))$ □

PF (COR). LET $N=f^{-1}(0)$.

WORK IN PROGRESS:

THM: (M^n,g) CLOSED, W/ STRONGLY POSITIVE CURVATURE. IF $h\in Sym^k(TM)$ SATISFIES $\Delta h=0$, THEN

(i) $h=0$ IF k ODD

(ii) $h=\lambda(\underbrace{g\vee\dots\vee g}_{k/2})$ IF k EVEN

SPANS THE TRIVIAL ISOTYPIC COMPONENT.