RICCI FLOW AND NONNEGATIVE SECTIONAL CURVATURE

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OUTLINE

1. EVOLUTION OF CURVATURE UNDER RICCI FLOW & MAIN RESULT
2. COHOMOGENEITY ONE MANIFOLDS & EXAMPLES
3. DIAGONAL ISSUES & SKETCH OF PROOF

1. EVOLUTION OF CURVATURE UNDER RICCI FLOW

(M^n, g_t) CLOSED n-DIMENSIONAL RIEM. MFLD., $n \geq 3$.

$$\frac{\partial}{\partial t} g_t = -2\,\mathrm{Ric}(g_t) \qquad \text{RICCI FLOW}$$

[UNIFORMIZING PROPERTIES ANALOGOUS TO HEAT FLOW OF TEMPERATURE DISTR.]

AS IN OTHER GEOMETRIC FLOWS, KEY IS TO FIND "SPECIAL" GEOMETRIC PROPERTIES THAT ARE PRESERVED, SUCH AS CURVATURE CONDITIONS.

- HAMILTON '80s: ☺ $\mathrm{scal} \geq 0$ IS PRESERVED $\forall n$

  [$\mathrm{Ric} \geq 0$ IS PRESERVED FOR $n=3$

  $\sec \geq 0$ IS PRESERVED FOR $n=3$]

  $R \geq 0$ IS PRESERVED $\forall n$

(IN BOTH CASES, SHOW (R_t) EVOLVES TO $\sec \equiv \text{const}$.)

INGREDIENT IN THE PROOF THAT IF (M^3, g) HAS $\mathrm{Ric} > 0$, THEN $M \cong S^3/\Gamma$

INGREDIENT IN THE PROOF THAT IF (M^n, g) HAS $R > 0$, THEN $M \cong S^n/\Gamma$

$n=4$: HAMILTON

$n \geq 4$: BÖHM-WILKING

$$\frac{d}{dt}\mathrm{scal} = \Delta \mathrm{scal} + 2|\mathrm{Ric}|^2$$
$$\frac{d}{dt} R = \Delta R + 2(R^2 + R^\#)$$
+ MAXIMUM PRINCIPLE

HOMOGENEOUS EXAMPLES

- BÖHM-WILKING '07 ☹ $\sec \geq 0$ IS NOT PRESERVED FOR $n \geq 6$
- MÁXIMO '11, '14: $\mathrm{Ric} \geq 0$, $\mathrm{Ric} > 0$, NOT PRESERVED FOR $n \geq 4$ (KÄHLER)
- (L. Ni '04: $\sec \geq 0$ IS NOT PRESERVED FOR $n \geq 4$ IF M^n IS NONCOMPACT. EX: $M = TS^n$, $n \geq 2$)

THM (B.-KRISHNAN, '16). THERE ARE METRICS WITH $\sec \geq 0$ ON S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, AND $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ THAT IMMEDIATELY LOOSE $\sec \geq 0$ WHEN EVOLVED VIA RICCI FLOW.

COR: $\sec \geq 0$ IS NOT PRESERVED FOR $n \geq 4$. (TAKE PRODUCTS WITH SPHERES)

SHARP RESULT

$\rightsquigarrow$ LIMITATIONS OF RICCI FLOW TO:

- "GEOMETRIZATION" OF 4-MANIFOLDS
- STUDY MANIFOLDS WITH $\sec \geq 0$, $\sec > 0$.

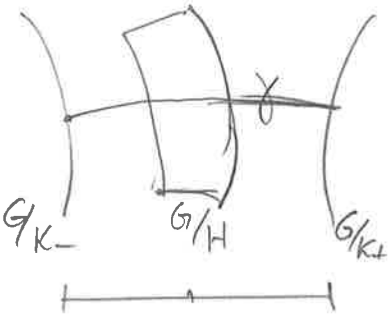
(2). COHOMOGENEITY ONE MANIFOLDS

- "NEXT STEP" AFTER HOMOGENEOUS;
- IMPLICITLY USED BY ANGENENT, ISENBERG, KNOPF, ...

$G \curvearrowright (M, g)$ ISOMETRIC GROUP ACTION, $\dim M/G = 1$

$$M/G = \begin{cases} [-1,1] \\ S^1 \\ [0,+\infty) \\ \mathbb{R} \end{cases}$$

$[-1,1]$, S^1: COMPACT. $[0,+\infty)$, $\mathbb{R}$: NON COMPACT.



"GROUP DIAGRAM"

$$\begin{tikzcd} & G \arrow[dl, dash] \arrow[dr, dash] & \\ K_- \arrow[dr, dash] & & K_+ \arrow[dl, dash] \\ & H & \end{tikzcd}$$

FIX HORIZONTAL GEODESIC $\gamma(r)$, THEN:

$$g = dr^2 + g_r, \quad g_r \text{ HOMOGENEOUS METRICS ON } G/H.$$

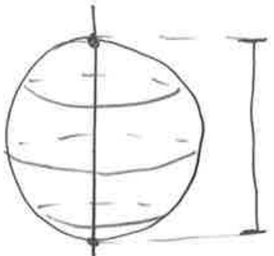
$$g_r = \sum_{i=1}^{n-1} f_i(r)^2 \, dx_i^2 \quad \text{"DIAGONAL"}$$

dx_i: KILLING FIELDS

EXAMPLE: $SO(n) \curvearrowright S^n$ ROTATIONS

$$g_{round} = dr^2 + \sin^2 r \, \underbrace{d\theta^2}$$

ROUND METRIC ON $S^{n-1} = G/H = \frac{SO(n)}{SO(n-1)}$



$$\begin{tikzcd} & SO(n) \arrow[dl, dash] \arrow[dr, dash] & \\ SO(n) \arrow[dr, dash] & & SO(n) \arrow[dl, dash] \\ & SO(n-1) & \end{tikzcd}$$

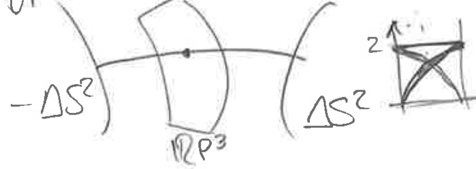
$$S^n / SO(n) = [0, \pi]$$

EXAMPLE: $SO(3) \curvearrowright S^2 \times S^2 \subset \mathbb{R}^3 \oplus \mathbb{R}^3$ DIAGONAL ACTION

$$A \cdot (x, y) = (Ax, Ay), \quad A \in SO(3), \quad S^2 \times S^2 / SO(3) = [0, \pi/2]$$

$$g_{prod} = dr^2 + 2\sin^2 r \, dx_1^2 + 2\cos^2 r \, dx_2^2 + 2\, dx_3^2$$

$$\begin{tikzcd} & SO(3) \arrow[dl, dash] \arrow[dr, dash] & \\ SO(2) \arrow[dr, dash] & & SO(2) \arrow[dl, dash] \\ & \{1\} & \end{tikzcd}$$

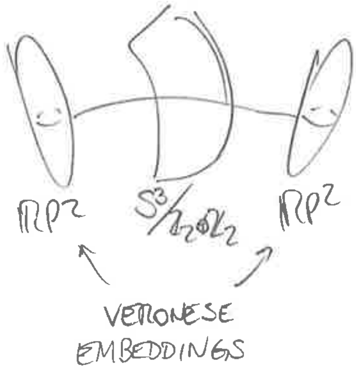


$x = \pm y \rightsquigarrow$ SINGULAR ORBIT $K_\pm = SO(2)$

$x \neq \pm y \rightsquigarrow$ PRINCIPAL ORBIT $H = \{1\}$.

EXAMPLE: $SO(3) \curvearrowright V^5 = \{$ SYMM. TRACELESS 3×3 MATRICES $\}$
CONJUGATION

$SO(3) \curvearrowright S^4$ $\qquad$ $S^4/SO(3) = [0, \pi/3]$



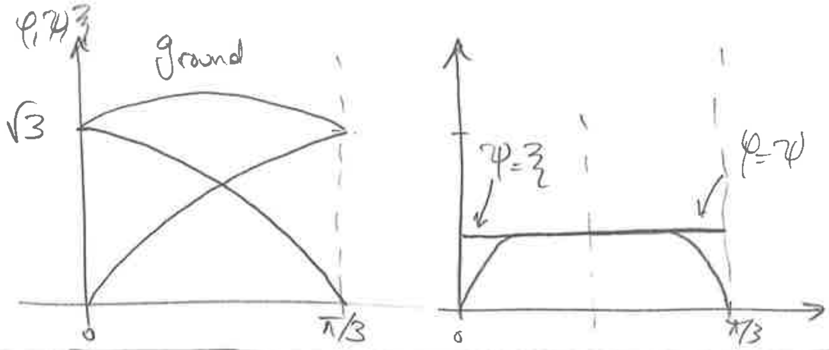
$$\begin{tikzcd}[column sep=small] & SO(3) \arrow[dl, dash] \arrow[dr, dash] & \\ S(O(2)O(1)) \arrow[dr, dash] & & S(O(1)O(2)) \arrow[dl, dash] \\ & S(O(1)O(1)O(1)) \cong \mathbb{Z}_2\oplus\mathbb{Z}_2 & \end{tikzcd}$$

X_1, X_2, X_3 KILLING FIELDS

$\{\dot\gamma, X_1, X_2, X_3\}$ O.N.-FRAME, $\dot\gamma = \frac{\partial}{\partial r}$

$$g_{round} = dr^2 + \varphi(r)^2 dx_1^2 + \psi(r)^2 dx_2^2 + \xi(r)^2 dx_3^2$$ "DIAGONAL METRIC"

$\varphi(r) = 2\sin r$
$\psi(r) = \sqrt{3}\cos r + \sin r$
$\xi(r) = \sqrt{3}\cos r - \sin r$



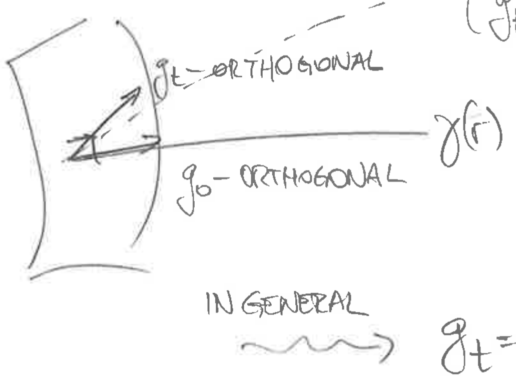
THM (GROVE-ZILLER). IF M^n IS A COHOMOGENEITY ONE MANIFOLD WITH SINGULAR ORBITS OF CODIMENSION 2, THEN $\exists\, g_{GZ}$ INVARIANT METRIC WITH $\sec \geq 0$. WILL BE USED AS INITIAL METRICS!

- FEATURES:
  - ISOMETRIC TO A PRODUCT NEAR "MIDDLE"
  - LOTS OF FLAT PLANES $\dot\gamma \wedge X_i$ AT ALL POINTS
  - NOT C^ω

METRICS WITH $\sec \geq 0$ ON MANY EXOTIC SPHERES.

- UNIQUENESS FOR RICCI FLOW $\Longrightarrow$ ISOMETRIES ARE PRESERVED
- $G \curvearrowright (M, g_0)$ ISOM., COHOM 1 $\Longrightarrow$ $G \curvearrowright (M, g_t)$ $\forall t \geq 0$, ISOM., COHOM 1

PROBLEM: • FIXED HORIZONTAL GEODESIC CAN "MOVE" (g_t-ORTHOGONALITY CHANGES WITH $t > 0$).



- DIAGONAL METRICS MAY NO LONGER STAY DIAGONAL!

$$g_0 = dr^2 + f_1(r)^2 dx_1^2 + f_2(r)^2 dx_2^2 + \cdots$$

BAD TERMS

IN GENERAL $\rightsquigarrow$ $g_t = \underline{\rho(t,r)}\, dr^2 + \underline{f_{01}(t,r)^2}\, dr\, dx_1 + f_{11}(t,r)^2 dx_1^2 + \underline{f_{12}(t,r)^2}\, dx_1 dx_2\, 2$

IF WE KNEW THAT DIAGONAL METRICS STAY DIAGONAL

$$(*) \qquad g_t = \rho(t,r)^2 dr^2 + \varphi(t,r)^2 dx_1^2 + \psi(t,r)^2 dx_2^2 + \xi(t,r)^2 dx_3^2$$

THEN RICCI FLOW REDUCES TO A SYSTEM OF PDE'S IN (t,r):

$$\begin{cases} \rho_t = -\left(\dfrac{\varphi_r}{\varphi} + \dfrac{\psi_r}{\psi} + \dfrac{\xi_r}{\xi}\right)\dfrac{\rho_r}{\rho^2} + \left(\dfrac{\varphi_{rr}}{\varphi} + \dfrac{\psi_{rr}}{\psi} + \dfrac{\xi_{rr}}{\xi}\right)\dfrac{1}{\rho} \\ \varphi_t = \dfrac{1}{\rho^2}\varphi_{rr} + \dfrac{1}{\rho\psi\xi}\left(\dfrac{\psi\xi}{\rho}\right)_r \varphi_r - \dfrac{2}{\psi^2\xi^2}\varphi^3 + \dfrac{2(\psi^2-\xi^2)^2}{\psi^2\xi^2}\dfrac{1}{\varphi} \\ \psi_t = \dfrac{1}{\rho^2}\psi_{rr} + \cdots \\ \xi_t = \dfrac{1}{\rho^2}\xi_{rr} + \cdots \end{cases}$$

DEGENERATE PARABOLIC; CAN BE MADE (STRICTLY) PARABOLIC BY FIXING A GAUGE, PARAMETRIZING $\gamma(r)$ BY g_t-ARCLENGTH SO $\rho(t,r) \equiv 1$.

AND OVERDETERMINED BOUNDARY CONDITIONS, SO THAT (*) IS C^∞ AT $r=0,L$.

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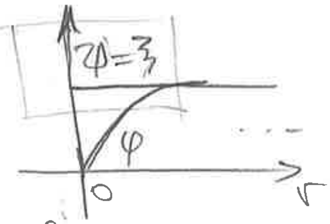
③ SKETCH OF PF (ASSUMING (*), WORKS SIMULTANEOUSLY ON $S^4, \mathbb{C}P^2, S^2\times S^2, \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$)

$$\sec(\dot\gamma \wedge X_2) = -\frac{\psi_{rr}}{\psi} \equiv 0 \quad \text{AT } t=0, \text{ NEAR } r=0.$$

COULD BE REPLACED BY ANY LINEAR COMBINATION OF X_2, X_3.



$$\frac{d}{dt}\sec_{g_t}(\dot\gamma\wedge X_2)\Big|_{t=0} \overset{\substack{\psi_r=0 \\ \text{near } r=0}}{=} -\frac{\psi_{rrt}}{\psi} = -\frac{4\left(\underbrace{\varphi_r^2}_{\neq 0 \text{ at } r=0 \text{ (smoothness)}} + \underbrace{\varphi_{rr}\varphi}_{=0 \text{ at } r=0}\right)}{\psi^4} < 0$$

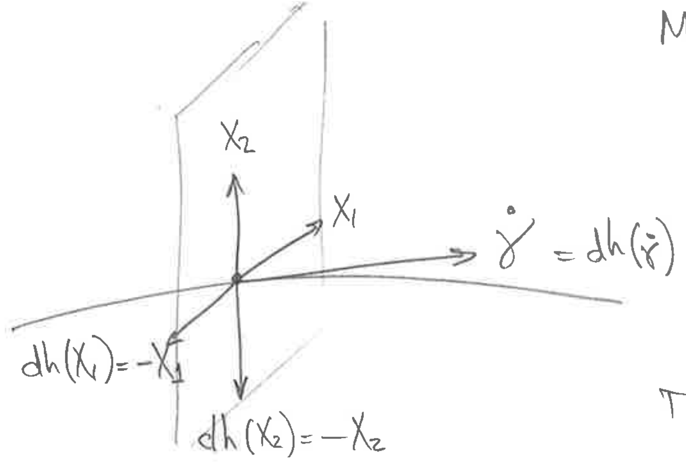
KEY COMPUTATIONAL SIMPLIFICATION

$\boxed{(RF) \;\&\; \xi = \psi \equiv \text{const near } r=0 \text{ if } t=0}$

SO $\sec_{g(t)}(\dot\gamma\wedge X_2) < 0$ FOR ALL SMALL $t>0$. ☐

HOW TO PROVE (*)?

ANSWER: MORE SYMMETRIES!
IN EACH OF THE COHOM 1 MFLDS $S^4, \mathbb{C}P^2, S^2\times S^2, \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ WITH THE METRIC g_{GZ}, THERE ARE OTHER ISOMETRIES THAT FORCE g_t TO STAY DIAGONAL (AD HOC SOLUTION).



MORE PRECISELY, CAN FIND ISOMETRIES h_i THAT;

- FIX $\gamma(r)$ POINTWISE
- REVERSE X_i BUT FIX ALL X_j, $j \neq i$.

THAT IS,

$$g(\dot\gamma, X_i) = g\big(dh(\dot\gamma), dh(X_i)\big) = -\, g(\dot\gamma, X_i)$$

$$g(X_i, X_j) = g\big(dh(X_i), dh(X_j)\big) = -\, g(X_i, X_j)$$

So THESE TERMS ARE FORCED TO BE ZERO, $\forall t \geq 0$.

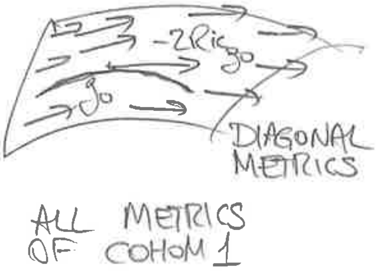
<u>EXAMPLE</u>: $SO(3) \curvearrowright S^4$, $h = \mathrm{diag}(\pm 1, \pm 1, \pm 1) \in H \subset SO(3)$, $H = S(O(1)O(1)O(1))$

SATISFY $dh(\gamma(t)) = \mathrm{diag}(1, \underbrace{\pm 1, \pm 1, \pm 1}_{h}) : T_{\gamma(r)} S^4 \longrightarrow T_{\gamma(r)} S^4$

WITH RESPECT TO THE FRAME $\{\dot\gamma, X_1, X_2, X_3\}$.

HOPE TO RESOLVE DIAGONAL ISSUE <u>IN GENERAL</u>:

- HEURISTICALLY, IF Ric_g IS DIAGONAL WHENEVER g IS DIAGONAL, THEN DIAGONAL METRICS SHOULD EVOLVE THROUGH OTHER DIAGONAL g_t



VELOCITY OF THE FLOW IS TANGENT TO SUBMFLD AT ALL POINTS, SO TRAJECTORIES SHOULD "STAY IN."

- <u>OTHER QUESTIONS FOR THE FUTURE</u>:
  - LONG TERM BEHAVIOUR
  - UNDERSTAND SINGULARITIES
  - CONVERGENCE TO SOLITONS
  - EXOTIC KERVAIRE SPHERES