MAX PLANCK RICCI FLOW SEMINAR BONN, NOV 2016

RICCI FLOW INVARIANT CURVATURE CONDITIONS

(FOLLOWING B. WILKING, CRELLE 2013)

QUESTION: IF A CLOSED RIEM. MFLD (M^n, g_0) SATISFIES A CURVATURE CONDITION C, DOES ITS RICCI FLOW EVOLUTION g_t, $t \geq 0$, CONTINUE TO SATISFY C?

- CURVATURE CONDITION: $C \subset S^2_B(\Lambda^2 \mathbb{R}^n) \cong S^2_B(so(n))$

O(n) - INVARIANT CONE

SPACE OF ALGEBRAIC CURVATURE OPERATORS

"LOCAL MODELS"

- (M,g) SATISFIES C IF $\forall p \in M$, $\exists i_p : T_pM \xrightarrow{\cong} \mathbb{R}^n$ ISOMETRY SUCH THAT $(i_p)_* R_p \in C$, WHERE FOR $x, y, z, w \in \mathbb{R}^n$:

$$(i_p)_* R_p(x \wedge y, z \wedge w) = R_p\left(i_p^{-1}(x) \wedge i_p^{-1}(y),\ i_p^{-1}(z) \wedge i_p^{-1}(w)\right)$$

NOTE: C IS $O(n)$ - INVARIANT $\rightsquigarrow$ THE ABOVE IS INDEPENDENT OF i_p.

- IN THIS TALK, $R : \underset{\Lambda^2\mathbb{R}^n}{\underset{\shortparallel}{so(n)}} \times \underset{\Lambda^2\mathbb{R}^n}{\underset{\shortparallel}{so(n)}} \longrightarrow \mathbb{R}$ SYMMETRIC BILINEAR FORM

$$R(x \wedge y, z \wedge w) = \langle R(x,y)z, w \rangle$$

AND ITS COMPLEXIFICATION $R : so(n,\mathbb{C}) \times so(n,\mathbb{C}) \longrightarrow \mathbb{C}$, WHICH IS A COMPLEX BILINEAR FORM.

MORE GENERALLY: G = HOLONOMY GROUP OF (M,g)

$\begin{cases} \mathfrak{g} = \text{LIE ALGEBRA OF } G \\ \mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \end{cases}$

$$R : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R} \quad \overset{\text{COMPLEXIFY}}{\rightsquigarrow} \quad R : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathbb{C}$$

E.G., IF (M,g) IS KÄHLER, USE $G = U(n)$ $\mathfrak{g} = \mathfrak{u}(n)$ ETC.

THEOREM. LET $S \subset \mathfrak{g}_{\mathbb{C}}$ BE A SUBSET INVARIANT UNDER THE ADJOINT REPRESENTATION OF $G_{\mathbb{C}}$, $Ad : G_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$. THEN $\forall h \in \mathbb{R}$, THE CURVATURE CONDITION $C(S,h) = \{R \in S^2(\mathfrak{g}) : R(v,\bar{v}) \geq h \ \forall v \in S\}$ IS INVARIANT UNDER RICCI FLOW.

A LIE GROUP WITH LIE ALGEBRA $\mathfrak{g}_{\mathbb{C}}$

<u>NOTE</u>: <u>ALL</u> CURRENTLY KNOWN INVARIANT CURVATURE CONDITIONS ARE OF THE ABOVE FORM $C(S,h)$.

<u>EXAMPLES</u>: USING $\mathfrak{g} = so(n)$ AND $h = 0$.

THIS IS THE HERMITIAN FORM $R : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{C}$

(a) $S = so(n,\mathbb{C})$, $\Rightarrow C = \{R \in S^2(so(n)) \ ; \ R(v,\bar{v}) \geq 0, \forall v \in so(n,\mathbb{C})\}$

$\overset{\star}{=} \{R \in S^2(so(n)) \ ; \ \underline{R} \text{ IS NONNEGATIVE}\}$

THIS IS THE "REAL" CURVATURE OPERATOR $R : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$

$\star$ $\subset$: TRIVIAL; BY <u>RESTRICTION</u>

$\supset$:

$v = X + iY, \quad X, Y \in so(n)$

$\bar{v} = X - iY$

$$R(v,\bar{v}) = R(X+iY, X-iY) = R(X,X) - i\cancel{R(X,Y)} + i\cancel{R(Y,X)} + R(Y,Y)$$
$$= R(X,X) + R(Y,Y) \geq 0.$$

B/C THE REAL FORM IS ≥ 0.

(b) $S = \{v \in so(n,\mathbb{C}) : \operatorname{tr}(v^2) = 0\}$

$$\Rightarrow C = \{R \in S^2(so(n)) \ ; \ R(v,\bar{v}) \geq 0 \ \forall v \in so(n,\mathbb{C}) \text{ S.T. } \operatorname{tr}(v^2) = 0\}$$
$$\overset{\star}{=} \{R \in S^2(so(n)) \ ; \ \underline{R \text{ IS 2-NON NEGATIVE}}\}$$

$\star$ $v = X + iY, \quad X, Y \in so(n)$

$$\operatorname{tr}(v^2) = \operatorname{tr}(X^2 - Y^2 + iXY - iYX) = \operatorname{tr}(X^2) - \operatorname{tr}(Y^2) + 2i\operatorname{tr}(XY) = 0$$

$$\iff \begin{cases} \operatorname{tr}(X^2) = \operatorname{tr}(Y^2) \\ \operatorname{tr}(XY) = 0 \end{cases} \iff \begin{cases} |X| = |Y| \\ \langle X, Y \rangle = 0 \end{cases} \iff \{X,Y\} \text{ ARE } \underline{\text{ORTHONORMAL}}.$$

SO:

$$R(v,\bar{v}) \geq 0, \forall v \in S \iff R(X,X) + R(Y,Y) \geq 0, \ \forall \{X,Y\} \text{ ORTHONORMAL}$$
$$\iff R \text{ IS 2-NONNEGATIVE.}$$

(c) $S = \{ v \in so(n,\mathbb{C}) : \operatorname{rank}(v) = 2,\ v^2 = 0 \}$

$\Rightarrow C = \{ R \in S^2(so(n)) : R$ HAS NONNEGATIVE ISOTROPIC CURVATURE$\}$

(d) $S = \{ v \in so(n,\mathbb{C}) : \operatorname{rank}(v) = 2,\ v^3 = 0 \}$

$\Rightarrow C = \{ R \in S^2(so(n)) : R_{M\times\mathbb{R}}$ HAS NONNEGATIVE ISOTROPIC CURVATURE$\}$

(e) $S = \{ v \in so(n,\mathbb{C}) : \operatorname{rank}(v) = 2 \}$

$\Rightarrow C = \{ R \in S^2(so(n)) : R_{M\times\mathbb{R}^2}$ HAS NONNEGATIVE ISOTROPIC CURVATURE$\}$

($\Leftrightarrow R_M$ HAS NONNEGATIVE COMPLEX CURVATURE) [NI-WOLFSON]

...

(a), (b) : [HAMILTON]

(c), (d), (e) : [BRENDLE-SCHOEN].

NOTE: MANY OTHER CURVATURE NONNEGATIVITY CONDITIONS ARE NOT PRESERVED BY RICCI FLOW:

$\underline{Ric \geq 0}$ IS PRESERVED IF $n=3$ [HAMILTON]
BUT NOT IF $n \geq 4$ [MÁXIMO]

$\underline{sec \geq 0}$ IS PRESERVED IF $n=3$ [HAMILTON]
BUT NOT IF $n \geq 4$ [B. - KRISHNAN]

SKIP

CURIOUS FACT: "$sec \geq 0$" IS OF THE FORM $C(S,h)$:

$$S = \{ v \in so(n),\ \operatorname{rank}(v) = 2 \} = Gr_2 \mathbb{R}^n \subset so(n) \subset so(n,\mathbb{C})$$

$$\Rightarrow C(S) = \{ R \in S^2(so(n)) : sec_R \geq 0 \}.$$

BUT S IS NOT $Ad_{G_\mathbb{C}}$ - INVARIANT! ACTUALLY, THE $Ad_{G_\mathbb{C}}$ - SATURATION OF $S \subset so(n,\mathbb{C})$ IS $S = \{ v \in so(n,\mathbb{C}) : \operatorname{rank}(v) = 2 \}$ FROM ITEM (e) ABOVE...

The smallest $Ad_{G_\mathbb{C}}$ - invariant subset of $so(n,\mathbb{C})$ containing S

NOTE: THIS MEANS THAT $sec_M \geq 0 \Rightarrow M\times\mathbb{R}^2$ HAS NONNEGATIVE ISOTROPIC CURVATURE AND THE RICCI FLOW EVOLUTION OF THE EXAMPLES IN [B. - KRISHNAN] PRESERVES THIS LATTER CURVATURE CONDITION...

PROOF OF THEOREM: BY HAMILTON'S MAXIMUM PRINCIPLE, IF A TENSOR $F \in \Gamma(E)$ EVOLVES BY A DIFFUSION-REACTION EQUATION

$$\frac{\partial F}{\partial t} = \Delta F + \Phi(F)$$

AND (i) $C \subset \Gamma(E)$ IS CLOSED,

(ii) C IS INVARIANT UNDER PARALLEL TRANSLATION,

(iii) $C \cap E_p$ IS CONVEX $\forall p \in M$,

(iv) C IS INVARIANT UNDER THE REACTION ODE $\frac{dF}{dt} = \Phi(F)$,

THEN C IS ALSO INVARIANT UNDER THE DIFFUSION-REACTION PDE.

RECALL EVOLUTION OF CURVATURE OPERATOR ALONG RICCI FLOW:

$$\frac{\partial R}{\partial t} = \Delta R + \underbrace{R^2 + R^\#}_{\Phi(R)}.$$

[WHERE:
$R^\# = ad \circ (R \wedge R) \circ ad^*$
$ad : \Lambda^2 V \to V$
$x \wedge y \mapsto [x,y] = ad_x y$]

CAN BE ASSUMED CLOSED, BY REPLACING IT WITH ITS CLOSURE

TO SHOW $C = C(S,0)$ IS INVARIANT UNDER $\frac{dR}{dt} = R^2 + R^\#$, WE PROVE:

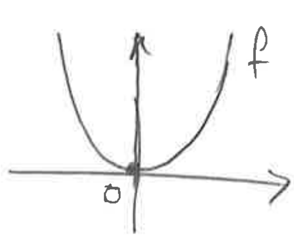
$$\boxed{\text{CLAIM: IF } R \in C,\ R(v,\bar{v}) = 0 \text{ FOR SOME } v \in S, \text{ THEN } R^2(v,\bar{v}) + R^\#(v,\bar{v}) \geq 0.}$$

CLEARLY, $R^2(v,\bar{v}) = \langle R^2(v), \bar{v} \rangle \underset{R \text{ SYMMETRIC}}{=} \langle Rv, R\bar{v} \rangle \underset{R \text{ REAL}}{=} \langle Rv, \overline{Rv} \rangle \geq 0.$

REMAINS TO SHOW $2R^\#(v,\bar{v}) = -\operatorname{tr}(ad(v)\, R\, ad(\bar{v})\, R) \geq 0$

SINCE S IS $Ad_{G_{\mathbb{C}}}$-INVARIANT, $\forall t \in \mathbb{R}$, $x \in \mathfrak{g}_{\mathbb{C}}$,

$$0 \leq R\big(Ad(\exp(tx))v,\ Ad(\exp(t\bar{x}))\bar{v}\big) = f(t)$$



WITH EQUALITY WHEN $t = 0$.

RECALL: $Ad(\exp(tx))\,v = \exp(t\, ad(x) v)$, SO:

$$\frac{d}{dt} Ad(\exp(tx))v \Big|_{t=0} = ad(x)v = [x,v], \quad \frac{d^2}{dt^2} Ad(\exp(tx))v\Big|_{t=0} = ad(x)ad(x)v = [x,[x,v]].$$

SO, DIFFERENTIATING TWICE, WE GET $0 \le f''(0)$:

$$0 \le 2R\big(\mathrm{ad}(x)v, \mathrm{ad}(\bar{x})\bar{v}\big) + R\big(\mathrm{ad}(x)\mathrm{ad}(x)v, \bar{v}\big) + R\big(v, \mathrm{ad}(\bar{x})\mathrm{ad}(\bar{x})\bar{v}\big)$$

REPLACING $x \leftrightsquigarrow ix$, $\bar{x} \leftrightsquigarrow -i\bar{x}$ THE 1^{st} TERM DOESN'T CHANGE, 2^{nd} & 3^{rd} TERMS CHANGE SIGN

ADDING TOGETHER THESE 2 INEQUALITIES, 2^{nd} & 3^{rd} TERMS CANCEL:

$$0 \le R\big(\mathrm{ad}(x)v, \mathrm{ad}(\bar{x})\bar{v}\big)$$

$\mathrm{ad}(x)y = [x,y]$ $$= R\big(\mathrm{ad}(v)x, \mathrm{ad}(\bar{v})\bar{x}\big) \qquad \forall x \in \mathfrak{g}_{\mathbb{C}}$$

$\mathrm{ad}(x)^* = -\mathrm{ad}(x)$ $$= -\langle \mathrm{ad}(\bar{v}) R\, \mathrm{ad}(v)x, \bar{x}\rangle$$

SO $-\mathrm{ad}(\bar{v})R\,\mathrm{ad}(v)$ AND $-\mathrm{ad}(v)R\,\mathrm{ad}(\bar{v})$ ARE <u>NONNEGATIVE</u> HERMITIAN OPERATORS ON $\mathfrak{g}_{\mathbb{C}}$. FINALLY, $\mathrm{tr}\big(\underbrace{-\mathrm{ad}(v)R\,\mathrm{ad}(\bar{v})}R\big) \ge 0$ BECAUSE R IS NONNEGATIVE ON THE IMAGE OF $\mathrm{ad}(v)$, WHICH CONTAINS THE IMAGE OF THE NONNEGATIVE OPERATOR $-\mathrm{ad}(v)R\,\mathrm{ad}(\bar{v})$. $\square$

! FOR GENERAL $h \neq 0$, DO NOT HAVE SCALE-INVARIANCE OF $C(S,h)$ AND DO NOT KNOW IF $\inf_{v \in S} R(v,\bar{v})$ IS ATTAINED. NEED TO STUDY

$$\frac{dR}{dt} = R^2 + R^\# + \varepsilon\,\mathrm{Id}, \qquad \varepsilon > 0$$

AND SHOW IF $R(0) \in C(S,h)$, THEN $R(t) \in C(S, h - \varepsilon t)$, AND THEN LET $\varepsilon \downarrow 0$.

BY <u>CONTRADICTION</u>, SUPPOSE $\exists t_i \downarrow 0$ WITH $R(t_i) \notin C(S, h - \varepsilon t_i)$, IE. $\exists v_i$ WITH $R(t_i)(v_i, \bar{v}_i) < h - \varepsilon t_i$. IF $\{v_i\}$ IS BOUNDED, GET $v_i \to v \in S$ AND $R(0)(v,\bar{v}) = h$. BY CLAIM, $R'(0)(v,\bar{v}) \ge 0$, CONTRADICTION!

IF $|v_i| \to \infty$, THEN $\frac{v_i}{|v_i|} \to w \in S$ WITH $R(w,\bar{w}) \le 0$ AND

$$w \in \partial_\infty S = \{x \in \mathfrak{g}_{\mathbb{C}} : \exists \lambda_i \in \mathbb{R},\ v_i \in S,\ \lambda_i \to 0,\ \lambda_i v_i \to x\}.$$

↖ "BOUNDARY OF S AT INFINITY".

$\partial_\infty S$ IS SCALE-INVARIANT AND $Ad_{G_{\mathbb{C}}}$- INVARIANT, SO $R \in C(S,h) \Rightarrow R \in C(\partial_\infty S, 0)$. THUS $R(w,\bar{w}) = 0$ AND BY THE CLAIM $R'(0)(w,\bar{w}) \geq 0$, HENCE FOR LARGE i, $v_i = |v_i| u_i$ AND $R'(t)(u_i, \bar{u_i}) > \varepsilon/2$, $t \in [0,\delta]$ SO $R(t_i)(v_i, \bar{v_i}) \geq R(0)(v_i, \bar{v_i}) \geq h$ FOR LARGE i, A CONTRADICTION. $\square$