STRONGLY POSITIVE AND NONNEGATIVE CURVATURE

JOINT WORK WITH R. MENDES

1. STRONGLY POSITIVE/NONNEGATIVE CURVATURE
20 min
2. EXAMPLES
10 min
3. BOCHNER TECHNIQUE
15 min

1. STRONGLY POSITIVE/NONNEGATIVE CURVATURE

- (M, g) RIEM. MFLD.
- $R: \Lambda^2 T_pM \longrightarrow \Lambda^2 T_pM$ CURVATURE OPERATOR

$$\langle R(X\wedge Y), Z\wedge W\rangle = \langle R(X,Y)Z, W\rangle$$

- $Gr_2 T_pM = \{\sigma \in \Lambda^2 T_pM : \underbrace{\sigma\wedge\sigma = 0, \ \|\sigma\| = 1}\}$ 2-GRASSMANNIAN

$\sigma = X\wedge Y$, $X, Y \in T_pM$ orthonormal vector.
i.e., unit decomposable (or rank 1) elements of $\Lambda^2 T_pM$

- $\sec: Gr_2 T_pM \longrightarrow \mathbb{R}$ SECTIONAL CURVATURE FUNCTION

$$\sec(X\wedge Y) = \langle R(X\wedge Y), X\wedge Y\rangle$$

- $\omega \in \Lambda^4 T_pM \rightsquigarrow \omega : \Lambda^2 T_pM \longrightarrow \Lambda^2 T_pM$

$$\langle \omega(X\wedge Y), Z\wedge W\rangle = \omega(X, Y, Z, W)$$

DEFINES INCLUSION $\Lambda^4 T_pM \hookrightarrow Sym^2(\Lambda^2 T_pM)$

- $b : \mathrm{Sym}^2(\Lambda^2 T_pM) \longrightarrow \Lambda^4 T_pM$ BIANCHI MAP (ORTHOGONAL PROJ.)

$$b(S)(X,Y,Z,W) = \frac{1}{3}\Big(\langle S(X\wedge Y), Z\wedge W\rangle + \langle S(Y\wedge Z), X\wedge W\rangle + \langle S(Z\wedge X), Y\wedge W\rangle\Big).$$

- $R+\omega \in \mathrm{Sym}^2(\Lambda^2 T_pM)$ "MODIFIED CURVATURE OPERATOR"

NOTE: $b(R+\omega) = \omega.$ ← does not satisfy the 1st Bianchi identity if $\omega \neq 0$

same sectional curvature function

$$\sec_{R+\omega}(X\wedge Y) = \langle (R+\omega)(X\wedge Y), X\wedge Y\rangle = \langle R(X\wedge Y), X\wedge Y\rangle = \sec(X\wedge Y)$$

DEF: (M,g) HAS STRONGLY POSITIVE (RESP. NONNEGATIVE) CURVATURE IF $\forall p\in M$, $\exists\, \omega_p \in \Lambda^4 T_pM$ SUCH THAT

$$(R_p+\omega_p) : \Lambda^2 T_pM \to \Lambda^2 T_pM$$

IS POSITIVE-DEFINITE (RESP. POSITIVE-SEMIDEFINITE).

$$\boxed{R>0} \implies \boxed{\text{STRONGLY POSITIVE CURVATURE}} \implies \boxed{\sec > 0}$$

analogously for nonnegative curv.

- IF $\dim \leq 3$, ALL ARE EQUIVALENT.
- IF $\dim = 4$, $\sec>0 \implies$ STRONGLY POS. CURV. (THORPE)

PF: (PÜTTMANN) LET $\displaystyle \lambda := \max_{\omega\in\Lambda^4 T_pM} \min \mathrm{Spec}(R+\omega)$, ← first eigenvalue $\lambda_1(R+\omega)$

← is attained b/c $\mathrm{tr}\,\omega = 0$

AND $\omega_{max} \in \Lambda^4 T_pM$ WHERE IT IS ACHIEVED.

LET $E_\lambda = \operatorname{Ker}\left(R + \omega_{max} - \lambda \, \mathrm{Id}\right)$. $\boxed{\lambda\text{-eigenspace of } R+\omega_{max}}$

IF $\exists \sigma \in E_\lambda \cap Gr_2 T_pM$, THEN $\lambda = \sec(\sigma) > 0$ SO $R+\omega_{max} > 0$.

IF $E_\lambda \cap Gr_2 T_pM = \emptyset$, THEN

$$q : \Lambda^2 T_pM \longrightarrow \Lambda^4 T_pM \cong \mathbb{R}$$
$$\alpha \longmapsto \alpha \wedge \alpha$$

IS S.T. $q(\alpha) \neq 0$, $\forall \alpha \in E_\lambda \setminus \{0\}$.

MOREOVER, $q(E_\lambda \setminus \{0\}) \subset \mathbb{R}_+$ HALF-LINE, $\longleftarrow$ Clear if $\dim E_\lambda = 1$, if $\dim E_\lambda \geq 2$, then $E_\lambda \setminus \{0\}$ is connected, and so is $q(E_\lambda\setminus\{0\}) \subset \mathbb{R}\setminus\{0\}$.

SO

$$\left\langle \left(R+\omega_{max} + \alpha\wedge\alpha\right)(\beta), \beta \right\rangle = \left\langle (R+\omega_{max})(\beta), \beta \right\rangle + \underbrace{q(\alpha)}_{>0} \|\beta\|^2 > \lambda \|\beta\|^2.$$

Contradicting maximality of ω_{max}. $\square$

> **THM** (B.–MENDES). IF $\pi : (\overline{M}, \overline{g}) \to (M, g)$ IS A RIEM. SUBMERSION AND $(\overline{M}, \overline{g})$ HAS STRONGLY POSITIVE CURVATURE, THEN SO DOES (M, g).

PF: LET $\alpha = A^*A$, I.E. $\langle \alpha(X\wedge Y), Z\wedge W\rangle = \langle A_X Y, A_Z W\rangle$.

REARRANGING THE GRAY – O'NEILL FORMULA,

$$\langle R(X\wedge Y); Z\wedge W\rangle = \langle \overline{R}(\overline{X}\wedge\overline{Y}), \overline{Z}\wedge\overline{W}\rangle + 3\langle \alpha(\overline{X}\wedge\overline{Y}), \overline{Z}\wedge\overline{W}\rangle - 3b(\alpha)(\overline{X},\overline{Y},\overline{Z},\overline{W})$$

IF $\overline{R} + \overline{\omega} > 0$, THEN $\omega := \left(\overline{\omega} + 3b(\alpha)\right)\big|_{\Lambda^4 T_pM}$ IS S.T. $R+\omega > 0$. $\square$

Similar argument for Cheeger deform.

2. EXAMPLES

CROSS:
- S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$ HAVE STRONGLY POSITIVE CURVATURE
- $\mathbb{C}aP^2$ DOES NOT!

PF: IF $R+\omega > 0$, AVERAGING ω WOULD GIVE $\overline{\omega} \in \Omega^4(\mathbb{C}aP^2)^{F_4}$ S.T. $R + \overline{\omega} > 0$, AND $\underbrace{\Delta\overline{\omega} = 0}_{\text{because } \mathbb{C}aP^2 \text{ is symm. space}}$. THUS $0 \neq [\overline{\omega}] \in H^4(\mathbb{C}aP^2; \mathbb{R}) = \{0\}$. $\square$ 2

HOMOGENEOUS CLASSIFICATION:

$\boxed{\text{THM (B.-MENDES). ALL SIMPLY-CONNECTED HOMOGENEOUS SPACES WITH } \sec > 0 \text{ ADMIT A HOMOGENEOUS METRIC WITH STRONGLY POSITIVE CURVATURE, EXCEPT } \mathbb{C}a P^2.}$

OTHER KNOWN EXAMPLES OF $\sec > 0$:

- BIQUOTIENTS (ESCHENBURG AND BAZAIKIN SPACES) ✓ (INFINITELY MANY)
  By the approximating argument to $W^7_{k,l}$ or B^{13}
- EXOTIC $T_1 S^4$ (GROVE-VERDIANI-ZILLER) ✓

STRONGLY NONNEGATIVE CURVATURE:

Maybe skip writing down and just read list

By the submersion result

- ALL COMPACT HOMOGENEOUS SPACES G/H AND BIQUOTIENTS $G/\!/H$.
- ALL CONNECTED SUMS OF 2 CROSS (W/ ANY ORIENTATION) [CHEEGER]
- ALL COHOMOGENEITY ONE MANIFOLDS W/ CODIMENSION 2 SINGULAR ORBITS [GROVE-ZILLER]
- ALL VECTOR BUNDLES AND SPHERE BUNDLES OVER S^4 AND S^5, IN PARTICULAR, ALL MILNOR SPHERES (20 OF THE 28 EXOTIC 7-SPHERES) [GROVE-ZILLER]
- ... MANY VECTOR BUNDLES AND SPHERE BUNDLES OVER $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ [GROVE-ZILLER]
  E.g. all of rank ≥ 6
- A REPRESENTATIVE OF ANY CLASS OF STABLE VECTOR BUNDLES OVER ANY CROSS [RIGAS, GONZÁLEZ-ÁLVARO[4]]
  Student of Luis Guijarro

$\boxed{\text{ACCORDING TO BURKHARD: "ALL KNOWN EXAMPLES OF } \sec \geq 0\text{"}}$

UPSHOT SO FAR:

"The message I hoped to pass along so far is:"

$\boxed{\begin{array}{c} \sec > 0 \\ (\sec \geq 0) \end{array}} \quad \underset{\text{VERY MUCH ANALOGOUS}}{\sim} \quad \boxed{\text{STRONGLY POSITIVE (NONNEGATIVE) CURVATURE}}$

HOW TO USE THIS?

$\boxed{\text{SEARCH FOR OBSTRUCTIONS!}}$

→ EITHER DISTINGUISH THE 2 THEORIES, OR GET MORE ON $\sec > 0$

3. BOCHNER TECHNIQUE

- (M,g) CLOSED RIEM. MFLD., $\dim M = n$.

- ρ REPRESENTATION OF $SO(n)$, $Spin(n)$

- $E \to M$ VECTOR BUNDLE ASSOCIATED TO PRINCIPAL FRAME BUNDLE BY ρ.

- WEITZENBÖCK FORMULA: $\Delta = \nabla^*\nabla + c_\rho K^\rho(R)$, $c_\rho \in \mathbb{R}$ CONSTANT

operators on sections of E

$c_\rho > 0$ on $\otimes^k TM^*$, such as k-forms but $c_\rho < 0$ on vector fields

SKIP:

$$R = \sum_{a,b} R_{ab} X_a \otimes X_b, \quad \{X_a\} \text{ BASIS OF } \mathfrak{so}(n) \cong \Lambda^2 \mathbb{R}^n$$

$$K^\rho(R) = -\sum_{a,b} R_{ab}\, d\rho(X_a) \circ d\rho(X_b)$$

$\rho : SO(n) \to Aut(E)$
$d\rho : \mathfrak{so}(n) \to End(E)$
$d\rho(X_a) \circ d\rho(X_b) \in End(E)$

In general, $K^\rho(R) \geq 0$ and $\ker K^\rho(R) = \rho_0$, the trivial isotypic comp. of ρ.

- IF ρ HAS NO TRIVIAL FACTORS, $R > 0 \Rightarrow K^\rho(R) > 0$

$d\rho(X_a) \in End(E)$ is skew symm, so $d\rho(X_a)^2 \leq 0$.

- $K^\rho : Sym^2(\Lambda^2 \mathbb{R}^n) \to Sym^2(E)$ IS $SO(n)$-EQUIVARIANT

$$Sym^2(\Lambda^2 \mathbb{R}^n) \cong \ker(b) \oplus \Lambda^4 \mathbb{R}^n$$

$A \in SO(n)$ $A \cdot R = \sum_{ab} R_{ab}\, Ad(A)X_a \otimes Ad(A) X_b$
$\phi \in End(E)$ $A \cdot \phi = \rho(A)\, \phi\, \rho(A^{-1})$

- TWO PROTOTYPES TO GET VANISHING THEOREMS:

EXAMPLE: $\rho = \mathbb{R}^n$, $c_\rho = 2$, $K^\rho(R) = Ric$
$R + \omega > 0 \Rightarrow Ric > 0 \Rightarrow$ HARMONIC 1-FORMS VANISH

① FIND ρ S.T. $K^\rho(\omega) = 0$, $\forall \omega \in \Lambda^4 \mathbb{R}^n$ "Free Vanishing Thm"

$\rightsquigarrow$ IF $R + \omega > 0$, THEN $K^\rho(R) = K^\rho(R + \omega) > 0$ SO $\Delta\alpha = 0 \xRightarrow{c_\rho > 0} \alpha = 0$

② CHARACTERIZE THE SET $\mathcal{D} = \{\alpha \in E : \langle K^\rho(\omega)\alpha, \alpha\rangle = 0, \forall \omega \in \Lambda^4 \mathbb{R}^n\}$

$\rightsquigarrow$ IF $R + \omega > 0$, THEN $\alpha \in \mathcal{D}$, $\Delta\alpha = 0 \Rightarrow \alpha = 0$. "Refined version"

EXAMPLE: $\rho = \Lambda^2 \mathbb{R}^n$, $K^\rho(\omega) = 4\omega$, $\forall \omega \in \Lambda^4 \mathbb{R}^n$ SO $\langle K^\rho(\omega)\alpha, \alpha\rangle = 4\langle \omega, \alpha \wedge \alpha\rangle$

HENCE $\mathcal{D} = Gr_2 T_pM$ ARE "DECOMPOSABLE 2-FORMS", I.E. 2-PLANES

More precisely: $\rho = \Lambda^2\mathbb{R}^n$, $K^\rho(R) = \underset{\substack{\uparrow \\ \text{Multiples} \\ \text{of Id} \\ \mathbb{R}\,\mathfrak{q}\otimes\mathfrak{q}}}{(2n-4)R_{\mathcal{U}}} + \underset{\substack{\uparrow \\ \text{Traceless} \\ \text{Ricci} \\ \mathfrak{q}\otimes \mathrm{Sym}^2_0(\mathbb{R}^n)}}{(n-4)R_{\mathcal{L}}} - \underset{\substack{\uparrow \\ \text{Weyl}}}{2R_W} + \underset{\substack{\uparrow \\ \text{4-forms.}}}{4R_{\Lambda^4}}$

THM (B.-MENDES). (M^n, g) CLOSED WITH STRONGLY POSITIVE CURVATURE. IF $\alpha \in \Omega^2(M)$ SATISFIES $\Delta\alpha = 0$ AND $\alpha\wedge\alpha \equiv 0$, THEN $\alpha = 0$.

PF: $0 = \langle \Delta\alpha, \alpha\rangle = \langle \nabla^*\nabla\alpha, \alpha\rangle + 2\langle K^\rho(R+\omega)\alpha,\alpha\rangle - 2\langle K^\rho(\omega)\alpha,\alpha\rangle$

$$= |\nabla\alpha|^2 + 2\langle \underbrace{K^\rho(R+\omega)}_{>0}\alpha,\alpha\rangle - 8\langle \omega, \underbrace{\alpha\wedge\alpha}_{0}\rangle$$

THUS $\alpha = 0$.

Same if $(R+\omega) \geq 0$ and $(R+\omega)_p > 0$, or replace $\alpha = 0$ by $\nabla\alpha = 0$ if $(R+\omega) \geq 0$

• "ANALOGOUS" RESULTS FOR STRONGLY NONNEGATIVE CURVATURE. □

COR. (M^n, g) CLOSED WITH 2-DIM. FOLIATION $\mathcal{F}$ S.T. $\mathcal{F}$ AND $\mathcal{F}^\perp$ HAVE MINIMAL LEAVES. THEN (M, g) DOES NOT HAVE STRONGLY POSITIVE CURV.

PF: $\chi_{\mathcal{F}}$ CHARACTERISTIC 2-FORM OF $\mathcal{F}$. $\Delta\chi_{\mathcal{F}} = 0 \Leftrightarrow \mathcal{F}$ AND $\mathcal{F}^\perp$ MINIMAL

THM (B.-MENDES). (M^n, g) CLOSED WITH STRONGLY POSITIVE CURVATURE AND GEOMETRICALLY FORMAL IN DEGREE 2. THEN $\forall c \in H^2(M,\mathbb{R}) \setminus \{0\}$, $c \cup c \neq 0$. IN PARTICULAR, $b_2(M) \neq 0 \Rightarrow b_4(M) \neq 0$.

PF: IF NOT, TAKE $0 \neq \alpha \in c$, $\Delta\alpha = 0$. THEN $\Delta(\alpha\wedge\alpha) = 0$ AND $\alpha\wedge\alpha \in c\cup c = 0$, SO $\alpha\wedge\alpha \equiv 0$. THUS $\alpha = 0$ (so $c = 0$) BY THM 1. □

THM (B.-MENDES). (M^4, g) CLOSED, SIMPLY-CONNECTED, WITH $\sec > 0$, S.T. $\forall p \in M$, $R_p : \Lambda^2 T_pM \to \Lambda^2 T_pM$ IS NOT POSITIVE-DEFINITE. THEN $M \cong \#^k \mathbb{CP}^2$.

COR. IF $S^2 \times S^2$ (OR $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, OR ANY INDEFINITE 4-MANIFOLD) HAS A METRIC g WITH $\sec > 0$, THEN $\exists N$ 2-SIDED HYPERSURFACE SUCH THAT $R|_N$ IS POSITIVE-DEFINITE.

PF: M^4, $\sec > 0 \Rightarrow \exists f : M \to \mathbb{R}$, $R + f\,\mathrm{vol} > 0$.

BY HYPOTHESIS, $f(p) \neq 0$, $\forall p \in M$, SO ASSUME $f > 0$.

IF $\alpha \in \Omega^2(M)$ SATISFIES $\Delta\alpha = 0$, $*\alpha = -\alpha$, THEN

$$0 = \langle \Delta\alpha, \alpha\rangle = |\nabla\alpha|^2 + 2\langle K^\rho(R+\omega)\alpha,\alpha\rangle - 8\langle \omega, \alpha\wedge\alpha\rangle$$

$\boxed{\int_M \alpha\wedge *\alpha = \int_M |\alpha|^2\,\mathrm{vol}}$

$$\equiv |\nabla\alpha|^2 + 2\langle K^\rho(\underbrace{R+f\,\mathrm{vol}}_{>0})\alpha,\alpha\rangle + \underbrace{8\int_M f|\alpha|^2}_{>0}.$$

SO $\alpha = 0$, HENCE $b_2^-(M) = 0$.

BY DONALDSON - FREEDMAN, $M \cong \#^k \mathbb{CP}^2$.

$\square$

PF (COR): LET $N = f^{-1}(0)$.

By Sard's Thm, can perturb $f \in C^\infty(M)$ so that $0 \in \mathbb{R}$ is a regular value.

WORK IN PROGRESS: SPLIT $M^4 = M_+ \cup_N M_-$, $M_+ = \{f \geq 0\}$, $M_- = \{f \leq 0\}$

AND ANALYZE EACH "PIECE" $M_\pm$ ADAPTING THE ABOVE TO THE CASE WITH BOUNDARY.

ISSUE: NO CONTROL ON II_N.

THM (B.-MENDES). (M^n, g) CLOSED WITH STRONGLY POSITIVE CURVATURE.

IF $h \in \mathrm{Sym}^k(M)$ SATISFIES $\Delta h = 0$, THEN

- $h = 0$, IF $2 \nmid k$
- $h = \underbrace{g \odot \cdots \odot g}_{k/2}$, IF $2 \mid k$.

PF: $\rho = \mathrm{Sym}^k \mathbb{R}^n$, $K^\rho(w) = 0$ $\forall w \in \Lambda^4\mathbb{R}^n$ BECAUSE $\Lambda^4\mathbb{R}^n \not\subset \mathrm{Sym}^2(\mathrm{Sym}^k\mathbb{R}^n)$

AND $\mathrm{span}\{\underbrace{g \odot \cdots \odot g}_{k/2}\} \subset \mathrm{Sym}^2(\mathrm{Sym}^k\mathbb{R}^n)$ IS THE TRIVIAL ISOTYPIC COMPONENT.

$\square$