ON DIFFERENT NOTIONS OF POSITIVITY OF CURVATURE

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Abstract

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We study interactions between the geometry and topology of Riemannian manifolds that satisfy curvature positivity conditions closely related to positive sectional curvature ($\sec > 0$). First, we discuss two notions of *weakly positive curvature*, defined in terms of averages of pairs of sectional curvatures. The manifold $S^2 \times S^2$ is proved to satisfy these curvature positivity conditions, implying it satisfies a property intermediate between $\sec > 0$ and positive Ricci curvature ($\mathrm{Ric} > 0$), and between $\sec > 0$ and $\sec \geq 0$. Combined with surgery techniques, this construction allows to classify (up to homeomorphism) the closed simply-connected 4-manifolds that admit a Riemannian metric for which averages of pairs of sectional curvatures of orthogonal planes are positive. Second, we study the notion of *strongly positive curvature*, which is intermediate between $\sec > 0$ and positive-definiteness of the curvature operator ($R > 0$). We elaborate on joint work with Mendes [14, 15], which yields the classification of simply-connected homogeneous spaces that admit an invariant metric with strongly positive curvature. These methods are then used to study the moduli space of homogeneous metrics with strongly positive curvature on the Wallach flag manifolds and on Berger spheres.

CONTENTS

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CHAPTER 1

INTRODUCTION

The study of closed manifolds with positive sectional curvature ($\sec > 0$) is one of the most challenging areas in Riemannian geometry. Despite being a classical subject, surprisingly little is understood about this class. Besides the theorems of Bonnet-Myers and Synge on the fundamental group, very few topological obstructions are known. In addition, some of the oldest open problems in global Riemannian geometry regard manifolds with $\sec > 0$, including the celebrated Hopf Problems:

Hopf Problem I. *Does $S^2 \times S^2$ admit a Riemannian metric with $\sec > 0$?*

Hopf Problem II. *If M^{2n} has $\sec > 0$, then it has Euler characteristic $\chi(M) > 0$?*

Rendering the subject even more intriguing, examples of closed manifolds that admit a Riemannian metric with $\sec > 0$ are relatively scarce. Apart from spheres and projective spaces, they are only known to occur in dimensions 6, 7, 12, 13 and 24, with infinite families in dimensions 7 and 13. The unifying feature of these examples is the presence of many symmetries, since almost all constructions rely on taking quotients of compact Lie groups. In fact, the use of symmetries has fostered important developments in the area, as outlined in the Grove symmetry program [43].

In this thesis, we study curvature positivity conditions closely related to $\sec > 0$, with an approach mostly guided by the use of symmetries, that also draws inspiration from the above Hopf Problems. Our goal is to contribute to the understanding of $\sec > 0$ by analyzing different curvature positivity conditions that are either slightly *weaker* or slightly *stronger*, and for which certain results might be of easier access.

We introduce two (somewhat dual) notions of *weakly positive curvature*, $\sec^{0+} > 0$ and $\sec^{\perp} > 0$, defined in terms of averages of pairs of sectional curvatures. For $\theta > 0$, given a Riemannian manifold (M, g), and a 2-plane $\sigma \subset T_pM$, consider the quantity

$$\sec^{\theta}(\sigma) = \min_{\substack{\sigma' \in \mathrm{Gr}_2(T_pM) \\ \mathrm{dist}(\sigma,\sigma') \geq \theta}} \tfrac{1}{2}\big(\sec(\sigma) + \sec(\sigma')\big),$$

where dist is a fixed fiberwise distance function on the Grassmannian of 2-planes tangent to M. The positivity $\sec^{\theta} > 0$ can be interpreted as the positivity of the averages of sectional curvatures of any 2-planes separated by an *angle* at least θ. The conditions $\sec^{0+} > 0$ and $\sec^{\perp} > 0$ are respectively related to the limits of $\sec^{\theta} > 0$ as the lower bound θ for the distances considered in the averages becomes arbitrarily small or arbitrarily large. More precisely, M satisfies $\sec^{0+} > 0$ if for all $\theta > 0$, there is a metric g^{θ} on M for which $\sec^{\theta} > 0$; while (M, g) satisfies $\sec^{\perp} > 0$ if it has $\sec^{\theta} > 0$ for the largest possible θ, so that averages of pairs of sectional curvatures of *orthogonal planes* are positive, see Sections 5.3 and 5.4 for details.

Both $\sec^{0+} > 0$ and $\sec^{\perp} > 0$ are clearly *weaker* than $\sec > 0$. These conditions are respectively intermediate between $\sec > 0$ and positive Ricci curvature ($\mathrm{Ric} > 0$), and $\sec > 0$ and positive scalar curvature ($\mathrm{scal} > 0$). Moreover, $\sec^{0+} > 0$ is also intermediate between $\sec > 0$ and $\sec \geq 0$.

The first main result in this thesis, which appeared in [13], is the following:

THEOREM A. *The manifold $S^2 \times S^2$ satisfies $\sec^{0+} > 0$, and hence also $\sec^{\perp} > 0$.*

In addition to its connection with the Hopf Problem I, the proof of this result yields the existence of metrics with $\sec^{\theta} > 0$ arbitrarily close to the standard product metric as $\theta \searrow 0$. This provides a further connection with the lesser known:

LOCAL HOPF PROBLEM I. *Is there a sequence g_n of metrics on $S^2 \times S^2$ with $\sec_{\mathsf{g}_n} > 0$ that converges to the standard product metric?*

Although both the classical and local versions of the Hopf Problem I remain unanswered, Theorem A improves on the curvature positivity conditions $\sec \geq 0$ and $\mathrm{Ric} > 0$ of the standard product metric, via small deformations. It should be mentioned that there is compelling evidence for $S^2 \times S^2$ *not* admitting metrics with $\sec > 0$, since one such metric could have at most a finite isometry group, see Hsiang and Kleiner [52] and Grove and Wilking [45]. At least, this indicates that such metrics are very difficult to find. Thus, Theorem A may be also regarded as an attempt to understand how much positive curvature can be acquired by special deformations of the standard product metric, in search of reasons for why they fail to have $\sec > 0$.

The fact that $S^2 \times S^2$ admits metrics with $\sec^\perp > 0$ is a key ingredient in the proof of our second main result, which uses a surgery stability criterion recently obtained by Hoelzel [51] to construct *many* other 4-manifolds with $\sec^\perp > 0$. These examples actually exhaust the list of homeomorphism types of closed simply-connected 4-manifolds that satisfy $\mathrm{scal} > 0$, so that, combined with the work of Sha and Yang [90], we have the following classification result:

THEOREM B. *Let M^4 be a smoothable closed simply-connected topological 4-manifold. Up to endowing M with different smooth structures, the following are equivalent:*

(i) *M^4 satisfies $\sec^\perp > 0$;*

(ii) *M^4 satisfies $\mathrm{Ric} > 0$;*

(iii) *M^4 satisfies $\mathrm{scal} > 0$.*

The other results in this thesis are about a curvature positivity condition stronger than $\sec > 0$, called *strongly positive curvature*. This term was coined by Grove, Verdiani and Ziller [44], for a concept that stems from the work of Thorpe [96, 97]. Let (M, g) be a Riemannian manifold and $R\colon \wedge^2 T_pM \to \wedge^2 T_pM$ its curvature operator. A 2-plane $\sigma \subset T_pM$ can be seen as an element $X \wedge Y \in \wedge^2 T_pM$, by choosing orthonormal vectors $X, Y \in T_pM$ that span σ. From this viewpoint, sectional curvature is the associated quadratic form $\sec(\sigma) = \langle R(\sigma), \sigma \rangle$, restricted to $\mathrm{Gr}_2(T_pM) \subset \wedge^2 T_pM$. Any 4-form $\omega \in \wedge^4 T_pM$ induces a symmetric operator

$$\omega\colon \wedge^2 T_pM \to \wedge^2 T_pM, \quad \langle \omega(\alpha), \beta \rangle = \langle \omega, \alpha \wedge \beta \rangle.$$

The quadratic form associated to ω vanishes on $\sigma \in \mathrm{Gr}_2(T_pM)$, since $\sigma \wedge \sigma = 0$, so

$$\sec(\sigma) = \langle R(\sigma), \sigma \rangle = \langle (R + \omega)(\sigma), \sigma \rangle.$$

This observation, known as *Thorpe's trick*, implies that if there exists $\omega \in \wedge^4 T_pM$ such that the *modified curvature operator* $R + \omega$ is positive-definite, then $\sec(\sigma) > 0$ for all 2-planes $\sigma \subset T_pM$. The manifold (M, g) is said to have strongly positive curvature if it has a 4-form ω such that $R + \omega$ is positive-definite at all points $p \in M$.

Strongly positive curvature is clearly an intermediate condition between $\sec > 0$ and positive-definiteness of the curvature operator ($R > 0$). By the work of Böhm and Wilking [17], manifolds with $R > 0$ are known to be diffeomorphic to spherical space forms. Thus, strongly positive curvature may be relevant in understanding the gap between this well-understood class and the intriguing class of manifolds with $\sec > 0$. Furthermore, strongly positive curvature and $\sec > 0$ are equivalent in dimensions ≤ 4, providing an interesting viewpoint on the Hopf Problem I. There is an important computational advantage to studying positive-definiteness of modified curvature operators instead of $\sec > 0$, since the latter is a highly nonlinear problem, while the former is linear.

Although it was implicitly used by others authors, strongly positive curvature and the analogously defined *strongly nonnegative curvature* have not been systematically studied until the joint work with Mendes [15]. Propelled by the discovery that Riemannian submersions preserve these conditions [15, Thm. A], we obtain the following classification result in the spirit of the symmetry program, which appears in [14, 15].

Theorem C. *All simply-connected homogeneous spaces with* $\sec > 0$ *admit a homogeneous metric with strongly positive curvature, except for the Cayley plane* $\mathbb{C}\mathrm{a}P^2$.

A detailed account on the proof of the above classification is provided, supplying several details omitted in [14, 15]. We also discuss in depth the concept of *strongly fat* homogeneous bundles, which plays a pivotal role in this proof. As a consequence of this classification, we obtain a complete description of the moduli space of homogeneous metrics with strongly positive and nonnegative curvature on certain homogeneous spaces, called Wallach flag manifolds [14]. More precisely, we have:

Theorem D. *A homogenous metric on the Wallach flag manifolds* W^6, W^{12}, *or* W^{24} *has strongly nonnegative curvature if and only if it has* $\sec \geq 0$. *Furthermore, a homogenous metric on* W^6 *or* W^{12} *has strongly positive curvature if and only if it has* $\sec > 0$, *and a homogenous metric on* W^{24} *has strongly positive curvature if and only if it has* $\sec > 0$ *and does not submerge onto* $\mathbb{C}\mathrm{a}P^2$.

In a similar spirit, we analyze the moduli spaces of Berger metrics with strongly positive and nonnegative curvature, proving the following new result:

Theorem E. *The Berger spheres* $\left(S^{2n+1}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\right)$ *and* $\left(S^{4n+3}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\right)$ *have strongly positive curvature for all* $0 < \lambda \leq 1$. *The Berger sphere* $\left(S^{15}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\right)$ *has strongly positive curvature if and only if* $0 < \lambda < \lambda_* \cong 1.184$, *where* λ_* *is the largest real root of* $p(\lambda) = 289\lambda^3 - 612\lambda^2 + 360\lambda - 48$.

We recall that the Berger metrics $\lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$ above are known to have $\sec > 0$ if and only if $0 < \lambda < \frac{4}{3}$. A particularly interesting consequence of the above is that there exist homogeneous spaces with $\sec > 0$ that do not have strongly nonnegative curvature, namely $\left(S^{15}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\right)$ with $\lambda_* < \lambda < \frac{4}{3}$. Previously, the only known examples of homogeneous spaces with $\sec > 0$ that failed to have strongly positive curvature had strongly nonnegative curvature (the Cayley plane $\mathbb{C}\mathrm{a}P^2$, the Berger space B^{13}, and the Wallach flag manifold W^{24}).

This thesis is organized as follows. Chapter 2 recalls basic notions from Riemannian geometry, establishing the conventions and notation used throughout the text. In Part I (Chapters 3 and 4), we study two metric deformation techniques, *first-order deformations* and *Cheeger deformations*. These chapters mainly discuss the work of Strake [94, 95] and Müter [72] respectively, supplementing it with new observations, with a treatment partly inspired by [2, Chap. 6]. In Part II (Chapters 5, 6 and 7), we study the two notions $\sec^{0+} > 0$ and $\sec^{\perp} > 0$ of weakly positive curvature. The basic definitions and results are given in Chapter 5, while Chapters 6 and 7 contain the proofs of Theorems A and B respectively. In Part III (Chapters 8, 9 and 10), we study strongly positive curvature. Foundational results on modified curvature operators and homogeneous spaces are given in Chapter 8. The notion of strongly fat homogeneous bundles is introduced in Chapter 9, leading to the proof of the *strong* Wallach Theorem (Theorem 9.5). This is the main tool in the proof of Theorem C, given in Chapter 9. Finally, Theorems D and E are proved in Chapter 10.

CHAPTER 2

PRELIMINARIES AND NOTATION

In this chapter, we establish some basic notation and terminology from Riemannian geometry that is used throughout this thesis. All manifolds are supposed to be C^∞-smooth and finite-dimensional.

2.1 Manifolds and bundles

Let M be a manifold with $\dim M = n$. We denote by $\otimes^k TM$ the kth tensor power of the tangent bundle TM and by $\vee^k TM$ and $\wedge^k TM$ the subbundles formed by symmetric and skewsymmetric k-tensors, respectively. We convention that the space of smooth sections of these bundles is denoted by the same symbol as the bundle, e.g., we write $X \in TM$ for a smooth vector field $X \colon M \to TM$. The same conventions apply for the cotangent bundle TM^*, as well as $\otimes^k TM^*$, $\vee^k TM^*$ and $\wedge^k TM^*$. We denote by $\mathrm{Gr}_2 TM$ the Grassmannian bundle of 2-planes tangent to M, whose fiber over the point $p \in M$ is the Grassmannian of 2-planes on T_pM, denoted

$$\mathrm{Gr}_2(T_pM) = \{\sigma \subset T_pM : \dim \sigma = 2\}. \tag{2.1}$$

A Riemannian metric $\mathsf{g} \in \vee^2 TM^*$ is a positive-definite symmetric tensor, usually denoted g and simply referred to as *metric*. When the choice of metric g on M is unambiguous, it is sometimes denoted $\langle \cdot, \cdot \rangle$ and the corresponding norm is denoted

$\|\cdot\|$. Furthermore, in this case we identify TM and TM^* via the isomorphisms

$$(\cdot)^\flat \colon TM \to TM^* \quad \text{and} \quad (\cdot)^\# \colon TM^* \to TM, \tag{2.2}$$

where $X^\flat = \langle X, \cdot \rangle$ and $\alpha = \langle \alpha^\#, \cdot \rangle$, for $X \in TM$ and $\alpha \in TM^*$. We implicitly use the identifications of the corresponding tensor powers induced by (2.2), that allow to *raise* and *lower* indices of tensors. Furthermore, we use the same symbols $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ as above to denote the corresponding objects on such tensor powers. In the presence of a metric, we also identify $\mathrm{Gr}_2 TM$ as a subbundle of $\wedge^2 TM$, by identifying each (oriented) 2-plane $\sigma \in \mathrm{Gr}_2(T_pM)$ with the element $X \wedge Y \in \wedge^2 T_pM$, where $\{X, Y\}$ is an (oriented) orthonormal basis of σ. In this way, we have, up to a double covering,

$$\mathrm{Gr}_2(T_pM) = \{X \wedge Y \in \wedge^2 T_pM : \|X \wedge Y\| = 1\}. \tag{2.3}$$

Henceforth, we do not make any distinctions between (2.1) and its oriented double covering (2.3), since all the notions we consider on 2-planes are independent of the choice of orientation.

2.2 Curvature

Let g be a Riemannian metric on M. The Levi-Civita connection of g is denoted ∇, and is given by the Koszul formula:

$$\begin{aligned}\langle \nabla_X Y, Z \rangle = \tfrac{1}{2}\Big(& X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ & + \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle + \langle [Z,Y], X\rangle \Big).\end{aligned} \tag{2.4}$$

We convention that the curvature tensor of (M, g) is given by

$$R_{\mathsf{g}}(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z. \tag{2.5}$$

We also make frequent use of the curvature operator, denoted by the same symbol,

$$R_{\mathrm{g}}\colon \wedge^2 TM \to \wedge^2 TM, \quad \langle R_{\mathrm{g}}(X\wedge Y), Z\wedge W\rangle = \langle R_{\mathrm{g}}(X,Y)Z, W\rangle, \tag{2.6}$$

which is a self-adjoint operator. The sectional curvature of (M,g) is the quadratic form associated to (2.6) restricted to $\mathrm{Gr}_2 TM \subset \wedge^2 TM$, that is, the function

$$\sec_{\mathrm{g}}\colon \mathrm{Gr}_2 TM \to \mathbb{R}, \quad \sec_{\mathrm{g}}(\sigma) = \sec_{\mathrm{g}}(X\wedge Y) = \langle R_{\mathrm{g}}(X,Y)X, Y\rangle, \tag{2.7}$$

where $\sigma = X\wedge Y \in \mathrm{Gr}_2(T_pM)$. The Riemannian manifold (M,g) is said to have positive sectional curvature, or $\sec_{\mathrm{g}} > 0$, if the above is a positive function, i.e., $\sec_{\mathrm{g}}(\sigma) > 0$ for all 2-planes $\sigma \in \mathrm{Gr}_2 TM$, and analogously for nonnegative sectional curvature. We say that $\sigma \in \mathrm{Gr}_2 TM$ is a flat plane if $\sec_{\mathrm{g}}(\sigma) = 0$, and denote the set of flat planes by $\sec_{\mathrm{g}}^{-1}(0) \subset \mathrm{Gr}_2 TM$.

The Ricci curvature of (M,g) is denoted

$$\mathrm{Ric}_{\mathrm{g}} \in \vee^2 TM, \quad \mathrm{Ric}_{\mathrm{g}}(X,Y) = \sum_{i=1}^{n} \langle R_{\mathrm{g}}(X,e_i)Y, e_i\rangle, \tag{2.8}$$

where $X, Y \in T_pM$ and $\{e_i\}$ is an orthonormal basis of T_pM. In particular, if $X \in T_pM$ is a unit vector, we may assume that $X = e_n$ and hence

$$\mathrm{Ric}_{\mathrm{g}}(X) := \mathrm{Ric}_{\mathrm{g}}(X,X) = \sum_{i=1}^{n-1} \sec_{\mathrm{g}}(X\wedge e_i). \tag{2.9}$$

Finally, the scalar curvature of (M,g) is the function

$$\mathrm{scal}_{\mathrm{g}}\colon M \to \mathbb{R}, \quad \mathrm{scal}(p) = \sum_{i=1}^{n} \mathrm{Ric}_{\mathrm{g}}(e_i) = \sum_{i=1}^{n}\sum_{j=1}^{n-1} \sec_{\mathrm{g}}(e_i\wedge e_j). \tag{2.10}$$

Other notions of curvature arise from the complexification $TM^{\mathbb{C}} := TM \otimes \mathbb{C}$. A

Riemannian metric $\mathsf{g} \in \vee^2 TM$ can be extended in two ways to $\vee^2 TM^{\mathbb{C}}$; namely, as a complex bilinear form or as a Hermitian inner product. We denote these respectively by $\langle \cdot, \cdot \rangle$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, and note that $\langle\!\langle X, Y \rangle\!\rangle = \langle X, \overline{Y} \rangle$ for all $X, Y \in TM^{\mathbb{C}}$. We denote by the same symbols the corresponding objects on tensor powers of $TM^{\mathbb{C}}$. The curvature operator (2.6) extends to a complex linear operator on $\wedge^2 TM^{\mathbb{C}}$. The associated real quadratic form restricted to $\mathrm{Gr}_2 TM^{\mathbb{C}} \subset \wedge^2 TM^{\mathbb{C}}$, that is, the function

$$\sec_{\mathsf{g}}^{\mathbb{C}} \colon \mathrm{Gr}_2 TM^{\mathbb{C}} \to \mathbb{R}, \quad \sec_{\mathsf{g}}^{\mathbb{C}}(\sigma) = \sec_{\mathsf{g}}^{\mathbb{C}}(X \wedge Y) = \langle\!\langle R_{\mathsf{g}}(X, Y)X, Y \rangle\!\rangle, \tag{2.11}$$

where $\sigma = X \wedge Y \in \mathrm{Gr}_2 TM^{\mathbb{C}}$, is called complex sectional curvature. Note that $\sec_{\mathsf{g}}^{\mathbb{C}}(X \wedge Y) = \langle R_{\mathsf{g}}(X, Y)\overline{X}, \overline{Y} \rangle$. The restriction of (2.11) to $\mathrm{Gr}_2 TM$ coincides with the sectional curvature function (2.7). A 2-plane $\sigma \in \mathrm{Gr}_2 TM^{\mathbb{C}}$ is called isotropic if $\langle X, X \rangle = 0$ for all $X \in \sigma$. It is easy to see that $\sigma \in \mathrm{Gr}_2 TM^{\mathbb{C}}$ is isotropic if and only if $\sigma = (X + iY) \wedge (Z + iW)$, where $X, Y, Z, W \in TM$ are (real) orthonormal vectors. For such an isotropic plane, by the Bianchi identity $\mathfrak{b}(R) = 0$ (see Section 2.3),

$$\begin{aligned} \sec_{\mathsf{g}}^{\mathbb{C}}(\sigma) &= \sec_{\mathsf{g}}(X \wedge Z) + \sec_{\mathsf{g}}(X \wedge W) \\ &\quad + \sec_{\mathsf{g}}(Y \wedge Z) + \sec_{\mathsf{g}}(Y \wedge W) - 2 \langle R_{\mathsf{g}}(X, Y)Z, W \rangle. \end{aligned} \tag{2.12}$$

The Riemannian manifold (M, g) is said to have positive isotropic curvature if $\sec_{\mathsf{g}}^{\mathbb{C}}(\sigma) > 0$ for all isotropic 2-planes $\sigma \in \mathrm{Gr}_2 TM^{\mathbb{C}}$, and analogously for nonnegative isotropic curvature.

Without reference to a specific Riemannian metric, we say that a manifold M satisfies a certain curvature condition if there exists a metric g on M such that (M, g) satisfies such condition. For example, we say that M *has positive-definite curvature operator*, or satisfies $R > 0$, if there exists a metric g on M such that (2.6) is a positive-definite operator, that is, $\langle R_{\mathsf{g}}(\alpha), \alpha \rangle > 0$ for all $\alpha \in \wedge^2 TM$, $\alpha \neq 0$; and we say that M *has positive sectional curvature*, or *satisfies* $\sec > 0$, if there exists a metric g on M such that $\sec_{\mathsf{g}} > 0$.

2.3 Pointwise curvature conditions

Let V be an n-dimensional real vector space, endowed with an inner product. Let

$$S(\wedge^2 V) := \big\{ R\colon \wedge^2 V \to \wedge^2 V : \langle R(\alpha), \beta\rangle = \langle R(\beta), \alpha\rangle \text{ for all } \alpha, \beta \in \wedge^2 V \big\}$$

be the space of symmetric linear operators on $\wedge^2 V$, endowed with the inner product $\langle R, S\rangle = \operatorname{tr} RS$. We identify every $\omega \in \wedge^4 V$ as an operator $\omega \in S(\wedge^2 V)$ by setting

$$\langle \omega(\alpha), \beta\rangle = \langle \omega, \alpha \wedge \beta\rangle, \qquad \text{for all } \alpha, \beta \in \wedge^2 V, \tag{2.13}$$

that is, $\langle \omega(X \wedge Y), Z \wedge W\rangle := \omega(X, Y, Z, W)$. This determines a linear isometry $\wedge^4 V \subset S(\wedge^2 V)$, which is regarded as an inclusion. The orthogonal projection operator $\mathfrak{b}\colon S(\wedge^2 V) \to \wedge^4 V$ onto this subspace is the *Bianchi map* (see [12, §1.G]), that maps each $R \in S(\wedge^2 V)$ to the 4-form $\mathfrak{b}(R) \in \wedge^4 V$ given by

$$\mathfrak{b}(R)(X,Y,Z,W) = \tfrac{1}{3}\Big(\langle R(X\wedge Y), Z\wedge W\rangle + \langle R(Y\wedge Z), X\wedge W\rangle + \langle R(Z\wedge X), Y\wedge W\rangle\Big).$$

The kernel of the Bianchi map is denoted

$$S_{\mathfrak{b}}(\wedge^2 V) := \ker \mathfrak{b} = \big\{ R \in S(\wedge^2 V) : \mathfrak{b}(R) = 0 \big\}. \tag{2.14}$$

The elements $R \in S_{\mathfrak{b}}(\wedge^2 V)$ are called *algebraic curvature operators*, since on a Riemannian manifold (M, g), the curvature operator (2.6) satisfies $R \in S_{\mathfrak{b}}(\wedge^2 T_p M)$ for all $p \in M$. By the above, there is an orthogonal direct sum decomposition

$$S(\wedge^2 V) = S_{\mathfrak{b}}(\wedge^2 V) \oplus \wedge^4 V.$$

This is a decomposition of $\mathsf{O}(n)$-representations, since the Bianchi map is equivariant with respect to the natural $\mathsf{O}(n)$-actions on $S(\wedge^2 V)$ and $\wedge^4 V$, given by

$$\begin{aligned} &A\cdot R := A^{-1} R\, A, \\ &A\cdot (X\wedge Y\wedge Z\wedge W) := A\,X\wedge A\,Y\wedge A\,Z\wedge A\,W, \end{aligned} \qquad A\in\mathsf{O}(n). \tag{2.15}$$

The $\mathsf{O}(n)$-representation on $\wedge^4 V$ is irreducible, while the $\mathsf{O}(n)$-representation on $S_{\mathfrak{b}}(\wedge^2 V)$ splits as sum of three irreducible subrepresentations, that are related to the Ricci and Weyl tensors, see [12, Thm. 1.114]

The so-called *pointwise* curvature conditions on a Riemannian manifold (M,g) may be expressed as $\mathsf{O}(n)$-invariant cones in $S_{\mathfrak{b}}(\wedge^2 V)$. Namely, given one such cone $C\subset S_{\mathfrak{b}}(\wedge^2 V)$, we say that (M,g) *satisfies* C if, for all $p\in M$ and all linear isometries $\iota\colon V\to T_pM$, the pull-back by ι of the curvature operator $R\in S_{\mathfrak{b}}(\wedge^2 T_pM)$ is such that $\iota^*(R)\in C$. Consider, for example, the open $\mathsf{O}(n)$-invariant convex cones

$$\begin{aligned} C_{R>0} &:= \big\{R\in S_{\mathfrak{b}}(\wedge^2 V) : \langle R(\alpha),\alpha\rangle > 0 \text{ for all } \alpha\in\wedge^2 V, \alpha\neq 0\big\}, \\ C_{\sec>0} &:= \big\{R\in S_{\mathfrak{b}}(\wedge^2 V) : \langle R(\sigma),\sigma\rangle > 0 \text{ for all } \sigma\in \mathrm{Gr}_2(V)\big\}. \end{aligned}$$

Then, in the above sense, (M,g) satisfies $C_{R>0}$ if and only if $R_{\mathsf{g}}>0$, and it satisfies $C_{\sec>0}$ if and only if $\sec_{\mathsf{g}}>0$. In this context, if two $\mathsf{O}(n)$-invariant cones C_1, C_2 in $S_{\mathfrak{b}}(\wedge^2 V)$ satisfy $C_1\subset C_2$, we say that condition C_1 implies C_2. For instance, clearly $C_{R>0}\subset C_{\sec>0}$, corresponding to the fact that $R>0$ implies $\sec>0$.

2.4 Lie groups

A bi-invariant Riemannian metric on a Lie group G is denoted by Q, which is identified with the positive-definite symmetric bilinear form $Q\in\vee^2\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Since Q is bi-invariant, that is, left and right translations in G are isometries, we have that $Q([X,Y],Z) = Q(X,[Y,Z])$ for all $X,Y,Z\in\mathfrak{g}$. The curvature operator of (G, Q) can be computed as (see [2, Prop. 2.26] or [75, §3.4]),

$$R_{\mathsf{G}} \colon \wedge^2 \mathfrak{g} \to \wedge^2 \mathfrak{g}, \quad \langle R_{\mathsf{G}}(X \wedge Y), Z \wedge W \rangle = \tfrac{1}{4} Q([X, Y], [Z, W]). \tag{2.16}$$

Thus, (G, Q) has positive-semidefinite curvature operator $R_{\mathsf{G}} \geq 0$. In particular,

$$\sec_Q(X \wedge Y) = \tfrac{1}{4} \big\| [X, Y] \big\|_Q^2, \tag{2.17}$$

and $\sec_Q \geq 0$. It is well-known that $\mathsf{SU}(2)$ endowed with a bi-invariant metric is isometric to the round sphere S^3, and $\pi \colon \mathsf{SU}(2) \to \mathsf{SO}(3)$ is a double covering map, corresponding to $S^3 \to \mathbb{R}P^3$. In particular, $(\mathsf{SU}(2), Q)$ and $(\mathsf{SO}(3), Q)$ have $\sec_Q > 0$.

2.5 Product manifolds

Let $M = M_1 \times M_2$ be a product manifold. Given $p \in M$, we write $p = (p_1, p_2)$, where $p_i \in M_i$, and $T_pM = T_{p_1}M_1 \oplus T_{p_2}M_2$. Given $X \in T_pM$, we have $X = (X_1, X_2)$, where $X_i \in T_{p_i}M_i$. A product metric on M is a metric of the form $\mathsf{g} = \mathsf{g}_1 \oplus \mathsf{g}_2$, where g_i is a metric on M_i. In other words,

$$\mathsf{g}(X, Y) = \mathsf{g}_1(X_1, Y_1) + \mathsf{g}_2(X_2, Y_2).$$

Routine computations show that the curvature operator of (M, g) is given by

$$\mathsf{g}(R(X \wedge Y), Z \wedge W) = \mathsf{g}_1\big(R_1(X_1 \wedge Y_1), Z_1 \wedge W_1\big) + \mathsf{g}_2\big(R_2(X_2 \wedge Y_2), Z_2 \wedge W_2\big), \tag{2.18}$$

where $R_i \colon \wedge^2 TM_i \to \wedge^2 TM_i$ is the curvature operator of (M_i, g_i). In particular, the sectional curvature of $X \wedge Y \in \mathrm{Gr}_2(T_pM)$, with $\|X \wedge Y\|_{\mathsf{g}}^2 = 1$, is

$$\sec_{\mathsf{g}}(X \wedge Y) = \sec_{\mathsf{g}_1}(X_1 \wedge Y_1) \, \big\| X_1 \wedge Y_1 \big\|_{\mathsf{g}_1}^2 + \sec_{\mathsf{g}_2}(X_2 \wedge Y_2) \, \big\| X_2 \wedge Y_2 \big\|_{\mathsf{g}_2}^2. \tag{2.19}$$

We say that a 2-plane $X \wedge Y \in \mathrm{Gr}_2(T_pM)$ is a *mixed plane* if either $X_1 = 0$ and $Y_2 = 0$, or $X_2 = 0$ and $Y_1 = 0$. In other words, mixed planes are spanned by a pair of vectors with one tangent to each of the factors M_1 and M_2. It follows from (2.19) that mixed planes have zero sectional curvature. Furthermore, if $X \wedge Y \in \mathrm{Gr}_2(T_pM)$ satisfies $X_j = Y_j = 0$, then $\sec_{\mathsf{g}}(X \wedge Y) = \sec_{\mathsf{g}_i}(X_i \wedge Y_i)$, where $\{i, j\} = \{1, 2\}$.

2.6 Immersions and submersions

An immersion $i\colon M \to \overline{M}$ between Riemannian manifolds (M, g) and $(\overline{M}, \overline{\mathsf{g}})$ is an *isometric immersion* if, for all $p \in M$, $\mathrm{d}i(p)$ is a linear isometry from T_pM into $T_{i(p)}\overline{M}$. When $i\colon M \to \overline{M}$ is injective, we identify it as an inclusion map, and write $T_p\overline{M} = T_pM \oplus T_pM^\perp$ for all $p \in M$, where $T_pM^\perp$ is the $\overline{\mathsf{g}}$-orthogonal complement to T_pM. If $X \in T_p\overline{M}$ and $p \in M$, we write $X^\top \in T_pM$ and $X^\perp \in T_pM^\perp$ for its tangent and normal components. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connections of (M, g) and $(\overline{M}, \overline{\mathsf{g}})$. The *second fundamental form* $\mathrm{I\!I}$ of the isometric immersion $i\colon M \to \overline{M}$ is

$$\mathrm{I\!I}\colon TM \times TM \to TM^\perp, \quad \mathrm{I\!I}(X, Y) = \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y = (\overline{\nabla}_{\overline{X}}\overline{Y})^\perp, \tag{2.20}$$

where $\overline{X}, \overline{Y} \in T\overline{M}$ denote local extensions of $X, Y \in TM$. The relation between the curvature operators R and $\overline{R}$ of (M, g) and $(\overline{M}, \overline{\mathsf{g}})$ is given by the *Gauss equation* (see [12, Chap. 1]),

$$\begin{aligned}\mathsf{g}(R(X \wedge Y), Z \wedge W) = \overline{\mathsf{g}}\big(&\overline{R}(X \wedge Y), Z \wedge W\big) \\ &+ \overline{\mathsf{g}}\big(\mathrm{I\!I}(X, Z), \mathrm{I\!I}(Y, W)\big) - \overline{\mathsf{g}}\big(\mathrm{I\!I}(X, W), \mathrm{I\!I}(Y, Z)\big),\end{aligned} \tag{2.21}$$

in particular, the sectional curvatures of (M, g) and $(\overline{M}, \overline{\mathsf{g}})$ satisfy

$$\sec_{\mathsf{g}}(X \wedge Y) = \sec_{\overline{\mathsf{g}}}(X \wedge Y) + \big\langle \mathrm{I\!I}(X, X), \mathrm{I\!I}(Y, Y)\big\rangle - \big\|\mathrm{I\!I}(X, Y)\big\|^2. \tag{2.22}$$

If $\mathrm{I\!I}$ vanishes identically, then M is called *totally geodesic.*

A submersion $\pi\colon \overline{M} \to M$ between Riemannian manifolds $(\overline{M}, \overline{\mathsf{g}})$ and (M, g) is a *Riemannian submersion* if, for all $p \in \overline{M}$, $\mathrm{d}\pi(p)$ is a linear isometry from $(\ker \mathrm{d}\pi(p))^{\perp}$ onto $T_{\pi(p)}M$, where $(\ker \mathrm{d}\pi(p))^{\perp}$ is the $\overline{\mathsf{g}}$-orthogonal complement to $\ker \mathrm{d}\pi(p)$. We denote the *vertical* and *horizontal spaces* respectively by

$$\mathcal{V}_p := \ker \mathrm{d}\pi(p) \quad \text{and} \quad \mathcal{H}_p := (\ker \mathrm{d}\pi(p))^{\perp} = \big\{X \in T_p\overline{M} : \overline{\mathsf{g}}(X, \mathcal{V}_p) = 0\big\}. \tag{2.23}$$

If $X \in T_p\overline{M}$, we write $X^{\mathcal{V}}$ and $X^{\mathcal{H}}$ for its vertical and horizontal components; and if $X \in T_{\pi(p)}M$, we denote by $\overline{X} \in T_p\overline{M}$ its horizontal lift, i.e., the unique horizontal vector $\overline{X} \in T_p\overline{M}$ such that $\mathrm{d}\pi(p)\overline{X} = X$. Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections of $(\overline{M}, \overline{\mathsf{g}})$ and (M, g). The two fundamental tensors T and A of the Riemannian submersion $\pi\colon \overline{M} \to M$ are the $(1,2)$-tensors on $\overline{M}$ given by

$$\begin{aligned} T_X Y &= \big(\overline{\nabla}_{X^{\mathcal{V}}} Y^{\mathcal{V}}\big)^{\mathcal{H}} + \big(\overline{\nabla}_{X^{\mathcal{V}}} Y^{\mathcal{H}}\big)^{\mathcal{V}}, \\ A_X Y &= \big(\overline{\nabla}_{X^{\mathcal{H}}} Y^{\mathcal{H}}\big)^{\mathcal{V}} + \big(\overline{\nabla}_{X^{\mathcal{H}}} Y^{\mathcal{V}}\big)^{\mathcal{H}}. \end{aligned} \tag{2.24}$$

If $X, Y \in TM$ are vector fields, then

$$A_{\overline{X}}\overline{Y} = \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla_X Y} = \tfrac{1}{2}[\overline{X}, \overline{Y}]^{\mathcal{V}} \tag{2.25}$$

so that, for all $V \in \mathcal{V}_p$, we have

$$\big\langle A_{\overline{X}}\overline{Y}, V\big\rangle = \tfrac{1}{2}\big\langle [\overline{X}, \overline{Y}], V\big\rangle = \tfrac{1}{2}V^{\flat}([\overline{X}, \overline{Y}]) = -\tfrac{1}{2}\mathrm{d}(V^{\flat})(\overline{X}, \overline{Y}). \tag{2.26}$$

Note that the left and right ends of (2.26) are tensorial on $\overline{X}$ and $\overline{Y}$. The relation between the curvature operators $\overline{R}$ and R of $(\overline{M}, \overline{\mathsf{g}})$ and (M, g) is given by the *Gray-*

O'Neill formula (see [12, Chap. 9]),

$$
\begin{aligned}
\mathsf{g}(R(X \wedge Y), Z \wedge W) = \overline{\mathsf{g}}\big(\overline{R}(\overline{X} \wedge \overline{Y}), \overline{Z} \wedge \overline{W}\big) + 2\,\overline{\mathsf{g}}\big(A_{\overline{X}}\overline{Y}, A_{\overline{Z}}\overline{W}\big) \\
- \overline{\mathsf{g}}\big(A_{\overline{Y}}\overline{Z}, A_{\overline{X}}\overline{W}\big) + \overline{\mathsf{g}}\big(A_{\overline{X}}\overline{Z}, A_{\overline{Y}}\overline{W}\big),
\end{aligned}
\tag{2.27}
$$

in particular, the sectional curvatures of $(\overline{M}, \overline{\mathsf{g}})$ and (M, g) satisfy

$$
\sec_{\mathsf{g}}(X \wedge Y) = \sec_{\overline{\mathsf{g}}}\left(\overline{X} \wedge \overline{Y}\right) + 3\left\|A_{\overline{X}}\overline{Y}\right\|^2. \tag{2.28}
$$

PART I

METRIC DEFORMATION TECHNIQUES

CHAPTER 3

FIRST-ORDER DEFORMATION

A *first-order deformation* of a Riemannian metric $\mathtt{g}$ is a (linear) path

$$\mathtt{g}_s := \mathtt{g} + s\,\mathtt{h}, \quad s \in \mathbb{R}, \tag{3.1}$$

where $\mathtt{h} \in \vee^2 TM^*$ is a symmetric 2-tensor. Since $\mathtt{g}$ is positive-definite, there exists $\varepsilon > 0$ such that $\mathtt{g}_s$ remains positive-definite if $|s| < \varepsilon$, and is hence a Riemannian metric. The first-order behavior of sectional curvature under such deformations was originally studied by Berger [9] and Bourguignon, Deschamps and Sentenac [18, 19], and later by Ehrlich [33–35] and Strake [94, 95]. In what follows, we discuss some of these results providing detailed proofs.

3.1 Berger-Ebin Slice Theorem

An important question to address when considering deformations of metrics is whether they are nontrivial *up to isometries*. There is a natural action of the diffeomorphism group $\mathrm{Diff}(M)$ on $\vee^2 TM^*$ by pull-back. The orbit of a Riemannian metric $\mathtt{g} \in \vee^2 TM^*$ consists of other Riemannian metrics $f^*(\mathtt{g}) \in \vee^2 TM^*$, and the diffeomorphism $f\colon M \to M$ is clearly an isometry between $(M, \mathtt{g})$ and $(M, f^*\mathtt{g})$. Thus, in order for a metric deformation to be *geometric*, it has to deform metrics up to diffeomorphisms, i.e., deform Riemannian structures up to reparametrizations of M. Due to technical difficulties to work on the orbit space of the $\mathrm{Diff}(M)$-action on $\vee^2 TM^*$, it is more convenient to consider deformations of metric tensors, such as

(3.1), and use other means to ensure that it is not contained in a Diff(M)-orbit. This can be achieved through the work of Ebin [32], and Berger and Ebin [10], as follows.

Fix a metric $\mathtt{g}$ on the closed smooth manifold M, and consider the connection $\nabla\colon TM^* \to \otimes^2 TM^*$ induced by its Levi-Civita connection. The L^2-inner product

$$\langle \mathtt{t}_1, \mathtt{t}_2 \rangle := \int_M \mathtt{g}\big(\mathtt{t}_1(x), \mathtt{t}_2(x)\big)\, \mathrm{vol}_{\mathtt{g}}$$

provides a notion of *formal adjoint* operator ∇^* to ∇. Define the operator

$$\delta\colon \vee^2\, TM^* \to TM^*, \quad \delta := \nabla^*|_{\vee^2 TM^*}.$$

Routine computations show that the formal adjoint $\delta^*\colon TM^* \to \vee^2 TM^*$ of δ satisfies

$$(\delta^*\alpha)(X,Y) = \tfrac{1}{2}\big(X(\alpha(Y)) + Y(\alpha(X))\big) = \tfrac{1}{2}(\mathcal{L}_{\alpha^\#}\mathtt{g})(X,Y), \tag{3.2}$$

where $\mathcal{L}$ denotes the Lie derivative. Indeed, δ is the composition of ∇^* with the inclusion $\vee^2 TM^* \subset \otimes^2 TM^*$, hence its formal adjoint is the composition of the symmetrization operator, i.e., the projection $\otimes^2 TM^* \to \vee^2 TM^*$, with the connection ∇. In particular, from (3.2), one sees that the range $\mathrm{Im}\,\delta^*$ is the tangent space to the Diff(M)-orbit through $\mathtt{g}$. More precisely, if $\eta_s \in \mathrm{Diff}(M)$ is a family of diffeomorphisms with $\eta_0 = \mathrm{Id}$ and $\frac{\mathrm{d}}{\mathrm{d}s}\eta_s\big|_{s=0} = V$, then we have $\frac{\mathrm{d}}{\mathrm{d}s}\eta_s^*(\mathtt{g})\Big|_{s=0} = \mathcal{L}_V \mathtt{g} = 2\,\delta^*(V^\flat)$. This observation is indicative[1] of the following splitting proved by Berger and Ebin [10]:

$$\vee^2\, TM^* = \ker\delta \oplus \mathrm{Im}\,\delta^*. \tag{3.3}$$

In the above, the subspace $\ker\delta$ is tangent to the *slice* for the Diff(M)-action on $\vee^2 TM^*$, see Ebin [32], and the subspace $\mathrm{Im}\,\delta^*$ is tangent to the Diff(M)-orbit through

[1] Recall that the above adjoints are only *formal*. The proof of this statement regarding smooth sections requires the study of certain differential operators with injective symbol, see [10, Thm. 4.1].

$\mathtt{g}$, as indicated before.

Thus, to first-order, the deformation (3.1) is *geometric* if and only if $\mathtt{h} \in \ker \delta$. Although this condition is not strictly necessary for many of the results in the remainder of this chapter, it may be assumed that all metric deformations considered here have velocity in $\ker \delta$, i.e., induce nontrivial deformations of the unparametrized Riemannian structure.

3.2 First variation formulas

Denote by ∇^s the Levi-Civita connection of $\mathtt{g}_s$, and let $C \in TM \otimes TM^* \otimes TM^*$ be its first variation, i.e., C is the $(1,2)$-tensor such that

$$\nabla^s_X Y = \nabla_X Y + s\, C(X,Y) + O(s^2), \tag{3.4}$$

where $\nabla = \nabla^0$ is the Levi-Civita connection of $\mathtt{g}$. By lowering indices using $\mathtt{g}$, we also consider C as the $(0,3)$-tensor $C \in \otimes^3 TM^*$, given by

$$C(X,Y,Z) := Z^\flat\big(C(X,Y)\big) = \mathtt{g}(C(X,Y),Z). \tag{3.5}$$

In what follows, we assume that $X,Y,Z \in T_pM$ are $\mathtt{g}$-orthonormal vectors, and we also denote by X, Y and Z local coordinate fields, that is, extensions of these vectors to a neighborhood of $p \in M$ such that

$$[X,Y] = [X,Z] = [Y,Z] = 0 \quad \text{and} \quad (\nabla X)_p = (\nabla Y)_p = (\nabla Z)_p = 0. \tag{3.6}$$

In particular, notice that for any $(0,2)$-tensor $\mathtt{t} \in \otimes^2 TM^*$, we have that at $p \in M$,

$$X\big(\mathtt{t}(Y,Z)\big) = (\nabla \mathtt{t})(X,Y,Z) - \mathtt{t}(\nabla_X Y, Z) - \mathtt{t}(Y, \nabla_X Z) = \nabla_X \mathtt{t}(Y,Z) \tag{3.7}$$

Lemma 3.1. *In the above notation, the first variation C of ∇^s is given by*

$$C(X,Y,Z) = \tfrac{1}{2}\big(X\mathtt{h}(Y,Z) + Y\mathtt{h}(X,Z) - Z\mathtt{h}(X,Y)\big). \tag{3.8}$$

In particular, using (3.7)*, we have*

$$\begin{aligned} C(X,X,Y) &= \nabla_X \mathtt{h}(X,Y) - \tfrac{1}{2}\nabla_Y \mathtt{h}(X,X) \\ C(Y,X,Y) &= \tfrac{1}{2}\nabla_X \mathtt{h}(Y,Y). \end{aligned} \tag{3.9}$$

Proof. Using (3.1), (3.4), (3.5) and (3.6), we have:

$$\begin{aligned} \mathtt{g}_s(\nabla^s_X Y, Z) &= \mathtt{g}(\nabla^s_X Y, Z) + s\,\mathtt{h}(\nabla^s_X Y, Z) \\ &= \mathtt{g}(\nabla_X Y, Z) + s\,\mathtt{g}(C(X,Y),Z) + s\,\mathtt{h}(\nabla_X Y, Z) + O(s^2) \\ &= \mathtt{g}(\nabla_X Y, Z) + s\,C(X,Y,Z) + O(s^2). \end{aligned} \tag{3.10}$$

On the other hand, by (3.1) and the Koszul formula (2.4), we have:

$$\begin{aligned} \mathtt{g}_s(\nabla^s_X Y, Z) &= \tfrac{1}{2}\big(X\mathtt{g}_s(Y,Z) + Y\mathtt{g}_s(X,Z) - Z\mathtt{g}_s(X,Y)\big) \\ &= \mathtt{g}(\nabla_X Y, Z) + s\,\tfrac{1}{2}\big(X\mathtt{h}(Y,Z) + Y\mathtt{h}(X,Z) - Z\mathtt{h}(X,Y)\big). \end{aligned} \tag{3.11}$$

The desired expression (3.8) for C follows from comparing (3.10) and (3.11); while (3.9) follows directly from (3.7) and (3.8). $\square$

Proposition 3.2. *The first variation of the sectional curvature of $\mathtt{g}_s$ is given by:*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\sec_{\mathtt{g}_s}(X\wedge Y)\Big|_{s=0} = {} & \nabla_X\nabla_Y \mathtt{h}(X,Y) - \tfrac{1}{2}\nabla_X\nabla_X \mathtt{h}(Y,Y) - \tfrac{1}{2}\nabla_Y\nabla_Y \mathtt{h}(X,X) \\ & + \mathtt{h}(R(X,Y)X,Y) - \sec_{\mathtt{g}}(X\wedge Y)\big(\mathtt{h}(X,X) + \mathtt{h}(Y,Y)\big), \end{aligned}$$

where $\mathtt{g}_s = \mathtt{g} + s\,\mathtt{h}$.

Proof. From (2.5) and (3.1), we have that the curvature operator R_s of $\mathtt{g}_s$ satisfies:

$$\begin{aligned}\mathtt{g}_s(R_s(X,Y)X,Y) &= \mathtt{g}(R_s(X,Y)X,Y) + s\,\mathtt{h}(R_s(X,Y)X,Y)\\ &= \mathtt{g}(\nabla^s_Y\nabla^s_X X,Y) - \mathtt{g}(\nabla^s_X\nabla^s_Y X,Y) + s\,\mathtt{h}(R(X,Y)X,Y) + O(s^2).\end{aligned}$$

Expanding the first term in the above expression, and using (3.6) and (3.7), we have

$$\begin{aligned}\mathtt{g}(\nabla^s_Y\nabla^s_X X,Y) &= \mathtt{g}\big(\nabla^s_Y(\nabla_X X + s\,C(X,X)),Y\big) + O(s^2)\\ &= \mathtt{g}(\nabla_Y\nabla_X X,Y) + s\,\mathtt{g}\big(C(Y,\nabla_X X) + \nabla_Y C(X,X),Y\big) + O(s^2)\\ &= \mathtt{g}(\nabla_Y\nabla_X X,Y) + s\,\mathtt{g}(\nabla_Y C(X,X),Y) + O(s^2)\\ &= \mathtt{g}(\nabla_Y\nabla_X X,Y) + s\,\nabla_Y C(X,X,Y) + O(s^2),\end{aligned}$$

and, analogously for the second term,

$$\mathtt{g}(\nabla^s_X\nabla^s_Y X,Y) = \mathtt{g}(\nabla_X\nabla_Y X,Y) + s\,\nabla_X C(Y,X,Y) + O(s^2).$$

Thus, by (3.9), the first variation of the unnormalized sectional curvature of $\mathtt{g}_s$ is:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathtt{g}_s(R_s(X,Y)X,Y)\Big|_{s=0} &= \nabla_X\nabla_Y\mathtt{h}(X,Y) - \tfrac{1}{2}\nabla_X\nabla_X\mathtt{h}(Y,Y) - \tfrac{1}{2}\nabla_Y\nabla_Y\mathtt{h}(X,X)\\ &\quad + \mathtt{h}(R(X,Y)X,Y).\end{aligned}$$

Finally, the desired expression follows from the fact that, by (2.7),

$$\sec_{\mathtt{g}_s}(X\wedge Y)\,\|X\wedge Y\|^2_{\mathtt{g}_s} = \mathtt{g}_s(R_s(X,Y)X,Y),$$

where $\|\cdot\|_{\mathtt{g}_s}$ indicates the norm on $\wedge^2 TM$ induced by $\mathtt{g}_s$, $\|X\wedge Y\|^2_0 = 1$, and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\|X\wedge Y\|^2_{\mathtt{g}_s}\Big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathtt{g}_s(X,X)\,\mathtt{g}_s(Y,Y) - \mathtt{g}_s(X,Y)^2\big)\Big|_{s=0}\\ &= \mathtt{h}(X,X)\,\mathtt{g}(Y,Y) + \mathtt{g}(X,X)\,\mathtt{h}(Y,Y) - 2\,\mathtt{h}(X,Y)\,\mathtt{g}(X,Y) \quad (3.12)\end{aligned}$$

$$= \mathtt{h}(X,X) + \mathtt{h}(Y,Y),$$

since $X, Y \in T_pM$ are $\mathtt{g}$-orthonormal. $\square$

In the next chapters, we are particularly interested in deforming metrics with $\sec_{\mathtt{g}} \geq 0$. The above first variation formula for the sectional curvature has a slight simplification in this context, if we consider a flat 2-plane $X \wedge Y \in \mathrm{Gr}_2(T_pM)$.

Corollary 3.3. *Let $(M, \mathtt{g})$ be a Riemannian manifold with $\sec_{\mathtt{g}} \geq 0$, and let $X, Y \in T_pM$ be $\mathtt{g}$-orthonormal vectors that span a flat plane, i.e., $\sec_{\mathtt{g}}(X \wedge Y) = 0$. Then, the first variation of $\sec_{\mathtt{g}_s}(X \wedge Y)$ is given by*

$$\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathtt{g}_s}(X \wedge Y)\Big|_{s=0} = \nabla_X \nabla_Y \mathtt{h}(X,Y) - \tfrac{1}{2}\nabla_X \nabla_X \mathtt{h}(Y,Y) - \tfrac{1}{2}\nabla_Y \nabla_Y \mathtt{h}(X,X). \quad (3.13)$$

Proof. Consider the Jacobi operator

$$J_X \colon T_pM \to T_pM, \quad J_X(Z) := R(X,Z)X.$$

As $R \colon \wedge^2 TM \to \wedge^2 TM$ is self-adjoint, we have that J_X is also self-adjoint, since

$$\mathtt{g}(J_X(Z), W) = \mathtt{g}(R(X,Z)X, W) = \mathtt{g}(R(X,W)X, Z) = \mathtt{g}(J_X(W), Z).$$

Moreover, $\mathtt{g}(J_X(Z), Z) = \mathtt{g}(R(X,Z)X, Z) \geq 0$ for all $Z \in T_pM$, since $\sec_{\mathtt{g}} \geq 0$. Thus, J_X is a self-adjoint positive-semidefinite operator. Denote by $\{e_i\}$ a $\mathtt{g}$-orthonormal basis of eigenvectors of J_X, and by $\lambda_i \geq 0$ the corresponding eigenvalues, so that $J_X(e_i) = \lambda_i\, e_i$. Writing $Y = \sum_i y_i\, e_i$, we have that

$$0 = \sec_{\mathtt{g}}(X \wedge Y) = \mathtt{g}(J_X(Y), Y) = \mathtt{g}\left(\textstyle\sum_i y_i\, \lambda_i\, e_i, \sum_j y_j\, e_j\right) = \textstyle\sum_i \lambda_i\, y_i^2,$$

hence $y_i = 0$ whenever $\lambda_i > 0$. Therefore, $Y \in \ker J_X$, that is,

$$J_X(Y) = R(X, Y)X = 0. \tag{3.14}$$

Thus, Proposition 3.2 and (3.14) give the desired formula (3.13). $\square$

Another important specialization is to consider first-order *conformal* deformations, that is, first-order deformations (3.1) where $\mathtt{h}$ is of the form $\mathtt{h} = \phi\,\mathtt{g}$ for some smooth function $\phi\colon M \to \mathbb{R}$. Notice that, in this case, $\mathtt{g}_s = (1 + s\,\phi)\mathtt{g}$ is a (linear) path of Riemannian metrics in the conformal class of $\mathtt{g}$, for $|s|$ sufficiently small.

Corollary 3.4. *Let $(M, \mathtt{g})$ be a Riemannian manifold with $\sec_{\mathtt{g}} \geq 0$, and let $X, Y \in T_pM$ be $\mathtt{g}$-orthonormal vectors that span a flat plane, i.e., $\sec_{\mathtt{g}}(X \wedge Y) = 0$. Consider the first-order conformal deformation of $\mathtt{g}$ given by (3.1) with $\mathtt{h} = \phi\,\mathtt{g}$. Then, the first variation of $\sec_{\mathtt{g}_s}(X \wedge Y)$ is given by*

$$\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathtt{g}_s}(X \wedge Y)\Big|_{s=0} = -\tfrac{1}{2}\operatorname{Hess}\phi(X, X) - \tfrac{1}{2}\operatorname{Hess}\phi(Y, Y). \tag{3.15}$$

Proof. We apply Corollary 3.3 with $\mathtt{h} = \phi\,\mathtt{g}$. Using (3.6) and (3.7), it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathtt{g}_s}(X \wedge Y)\Big|_{s=0} &= \nabla_X\nabla_Y\phi\,\mathtt{g}(X, Y) - \tfrac{1}{2}\nabla_X\nabla_X\phi\,\mathtt{g}(Y, Y) - \tfrac{1}{2}\nabla_Y\nabla_Y\phi\,\mathtt{g}(X, X)\\ &= X(Y(\phi))\,\mathtt{g}(X, Y) - \tfrac{1}{2}X(X(\phi))\,\mathtt{g}(Y, Y) - \tfrac{1}{2}Y(Y(\phi))\,\mathtt{g}(X, X)\\ &= -\tfrac{1}{2}\operatorname{Hess}\phi(X, X) - \tfrac{1}{2}\operatorname{Hess}\phi(Y, Y),\end{aligned}$$

since X and Y are local coordinate fields that extend the $\mathtt{g}$-orthonormal vectors $X, Y \in T_pM$; in particular, at $p \in M$, $\operatorname{Hess}\phi(X, X) = X(X(\phi)) - \mathrm{d}\phi(\nabla_X X) = X(X(\phi))$, and analogously $\operatorname{Hess}\phi(Y, Y) = Y(Y(\phi))$. $\square$

3.3 First-order Lemma

In order to apply the above first variation formulas to improve curvature, we often use the following auxiliary result:

Lemma 3.5. *Let* $f\colon [0,S]\times K \to \mathbb{R}$ *be a smooth function, where* $S>0$ *and* K *is a compact subset of a manifold. Assume that* $f(0,x)\geq 0$ *for all* $x\in K$*, and* $\frac{\partial f}{\partial s}(0,x)>0$ *if* $f(0,x)=0$*. Then there exists* $s_*>0$ *such that* $f(s,x)>0$ *for all* $x\in K$ *and* $0<s<s_*$*.*

Proof. Since f is smooth and K is compact, there exist $\varepsilon_1>0$ and $\varepsilon_2>0$ such that $\frac{\partial f}{\partial s}(0,x)>\varepsilon_2$ if $f(0,x)<\varepsilon_1$. In particular, we have that $K=K_1\cup K_2$, where

$$K_1 := \big\{x\in K : f(0,x)\geq \varepsilon_1\big\}, \quad \text{and} \quad K_2 := \left\{x\in K : \frac{\partial f}{\partial s}(0,x)\geq \varepsilon_2\right\}. \tag{3.16}$$

By continuity of f, there exists $s_1>0$ such that $f(s,x)\geq \frac{1}{2}\varepsilon_1>0$ for all $0<s<s_1$ and $x\in K_1$. Setting

$$m := \max_{\substack{s\in[0,S]\\ x\in K}} \left|\frac{1}{2}\frac{\partial^2 f}{\partial s^2}(s,x)\right|,$$

we have, by the Taylor polynomial of $f(s,x)$ at $s=0$, that for all $(s,x)\in[0,S]\times K$,

$$f(s,x)\geq f(0,x)+\frac{\partial f}{\partial s}(0,x)\, s - m\, s^2.$$

In particular, if $x\in K_2$ and $0<s<\varepsilon_2/m$, it follows that

$$f(s,x)\geq \varepsilon_2\, s - m\, s^2 > 0.$$

Thus, setting $s_2 := \varepsilon_2/m$ and $s_* := \min\{s_1,s_2\}$, we have that $f(s,x)>0$ for all $0<s<s_*$ and $x\in K$, concluding the proof. ∎

In particular, we have the following tool to deform metrics with sec ≥ 0 into metrics with $\sec > 0$, cf. Strake [94, §1.A].

Corollary 3.6. *Let* (M, g) *be a compact Riemannian manifold with* $\sec_{\mathrm{g}} \geq 0$. *Let* g_s *be a first-order deformation, such that* $\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathrm{g}_s}(\sigma)\big|_{s=0} > 0$ *for all flat planes* σ, *i.e., for all* $\sigma \in \sec_{\mathrm{g}}^{-1}(0)$. *Then, there exists* $s_* > 0$ *such that* $\sec_{\mathrm{g}_s} > 0$ *for all* $0 < s < s_*$.

Proof. Use Lemma 3.5 with $f\colon [0, S] \times \mathrm{Gr}_2 TM \to \mathbb{R}$, $f(s, \sigma) := \sec_{\mathrm{g}_s}(\sigma)$. □

3.4 Obstructions to first-order deformations

We now discuss (several) obstructions to using the above tool. The following is observed in Strake [94, §1.D].

Lemma 3.7. *Let* $(\overline{M}, \overline{\mathrm{g}})$ *be a Riemannian manifold with a totally geodesic immersed submanifold* $i\colon M \to \overline{M}$. *Let* $\overline{\mathrm{g}}_s = \overline{\mathrm{g}} + s\,\overline{\mathrm{h}}$ *be a first-order deformation of* $\overline{\mathrm{g}}$, *and set* $\mathrm{g}_s := i^*(\overline{\mathrm{g}}_s)$. *Then, for all* $X \wedge Y \in \mathrm{Gr}_2(T_pM)$ *and* $p \in M$,

$$\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathrm{g}_s}(X \wedge Y)\Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} \sec_{\overline{\mathrm{g}}_s}(X \wedge Y)\Big|_{s=0}. \tag{3.17}$$

Proof. Denote by $\mathrm{I\!I}_s$ the second fundamental form of (M, g_s) inside $(\overline{M}, \overline{\mathrm{g}}_s)$ and consider its first variation

$$\mathrm{I\!I}_s = \mathrm{I\!I}_0 + s\,\mathrm{I\!I}_0' + O(s^2).$$

Since (M, g) is totally geodesic in $(\overline{M}, \overline{\mathrm{g}})$, we have that $\mathrm{I\!I}_0 = 0$. Therefore,

$$\begin{aligned}
\langle \mathrm{I\!I}_s(X,X), \mathrm{I\!I}_s(Y,Y) \rangle_{\overline{\mathrm{g}}_s} &= \overline{\mathrm{g}}\big(\mathrm{I\!I}_s(X,X), \mathrm{I\!I}_s(Y,Y)\big) + s\,\overline{\mathrm{h}}(\mathrm{I\!I}_s(X,X), \mathrm{I\!I}_s(Y,Y)) \\
&= \overline{\mathrm{g}}\big(\mathrm{I\!I}_0(X,X), \mathrm{I\!I}_0(Y,Y)\big) + s\,\overline{\mathrm{g}}\big(\mathrm{I\!I}_0'(X,X), \mathrm{I\!I}_0(Y,Y)\big) \\
&\quad + s\,\overline{\mathrm{g}}\big(\mathrm{I\!I}_0(X,X), \mathrm{I\!I}_0'(Y,Y)\big) + s\,\overline{\mathrm{h}}(\mathrm{I\!I}_0(X,X), \mathrm{I\!I}_0(Y,Y)) + O(s^2) \\
&= O(s^2).
\end{aligned}$$

and, analogously, $\left\| \mathrm{I\!I}_s(X,Y) \right\|^2_{\overline{\mathrm{g}}_s} = O(s^2)$. Moreover, by (3.12), we also have

$$\|X \wedge Y\|^2_{\overline{\mathrm{g}}_s} = 1 + s\left(\overline{\mathrm{h}}(X,X) + \overline{\mathrm{h}}(Y,Y)\right) + O(s^2).$$

Thus, by the Gauss equation (2.22), differentiating at $s=0$ the expression

$$\sec_{\mathrm{g}_s}(X \wedge Y) - \sec_{\overline{\mathrm{g}}_s}(X \wedge Y) = \frac{\left\langle \mathrm{I\!I}_s(X,X), \mathrm{I\!I}_s(Y,Y) \right\rangle_{\overline{\mathrm{g}}_s} - \left\| \mathrm{I\!I}_s(X,Y) \right\|^2_{\overline{\mathrm{g}}_s}}{\|X \wedge Y\|^2_{\mathrm{g}_s}},$$

and using the above expansions, we obtain the desired conclusion (3.17). □

Remark 3.8. The equality (3.17) can also be proved by comparing the first variations of $\sec_{\mathrm{g}_s}(X \wedge Y)$ and $\sec_{\overline{\mathrm{g}}_s}(X \wedge Y)$, as given in Proposition 3.2.

As an immediate consequence of Corollary 3.6 and Lemma 3.7, we have:

Corollary 3.9. *Let $(\overline{M}, \overline{\mathrm{g}})$ be a Riemannian manifold with $\sec_{\overline{\mathrm{g}}} \geq 0$ and a totally geodesic immersed compact submanifold $i\colon M \to \overline{M}$. Let $\overline{\mathrm{g}}_s$ be a first-order deformation of $\overline{\mathrm{g}}$, such that $\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\overline{\mathrm{g}}_s}(\sigma)\big|_{s=0} \geq 0$ for all $\sigma \in \sec^{-1}_{\overline{\mathrm{g}}}(0) \subset \mathrm{Gr}_2 T\overline{M}$. If M does not have* $\sec > 0$, *then $\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\overline{\mathrm{g}}_s}(\sigma)\big|_{s=0} = 0$ for some $\sigma \in \sec^{-1}_{\overline{\mathrm{g}}}(0)$.*

In connection with the Hopf Problem I and the Local Hopf Problem I, let us discuss deformations of a compact product manifold $(M, \mathrm{g}) = \left(M_1 \times M_2, \mathrm{g}_1 \oplus \mathrm{g}_2\right)$ such that (M_i, g_i), $i = 1, 2$, have $\sec_{\mathrm{g}_i} > 0$. From (2.19), we have that the set of flat planes $\sec^{-1}_{\mathrm{g}}(0) \subset \mathrm{Gr}_2 TM$ coincides with the set of mixed planes.[2] Since (M_i, g_i) is a compact Riemannian manifold, it admits at least one closed geodesic $\gamma_i\colon S^1 \to M_i$, see Jost [55, Thm. 7.11.4]. The product $T := \gamma_1(S^1) \times \gamma_2(S^1)$ is a totally geodesic flat torus immersed in (M, g). Since a 2-torus does not satisfy $\sec > 0$ by the Gauss-Bonnet Theorem, it follows from Corollary 3.9 that no first-order deformation g_s of

[2] At each $p \in M$, this is simply the product $S_{p_1}M_1 \times S_{p_2}M_2$ of the unit spheres $S_{p_i}M_i \subset T_{p_i}M_i$, up to orientation.

$\mathtt{g}$ can satisfy $\frac{\mathrm{d}}{\mathrm{d}s}\sec_{\mathtt{g}_s}(\sigma)\big|_{s=0} > 0$ for all $\sigma \in \sec_{\mathtt{g}}^{-1}(0)$.[3] Furthermore, it is possible to show that the first variation of sectional curvature vanishes for *all* mixed planes, as originally observed by Berger [9].

Proposition 3.10. *Let* $(M,\mathtt{g}) = \big(M_1 \times M_2, \mathtt{g}_1 \oplus \mathtt{g}_2\big)$ *be a compact product Riemannian manifold and let* $\mathtt{g}_s$ *be a first-order deformation of its product metric* $\mathtt{g}$*, such that* $\frac{\mathrm{d}}{\mathrm{d}s}\sec_{\mathtt{g}_s}(\sigma)\big|_{s=0} \geq 0$ *for all mixed planes* $\sigma \in \mathrm{Gr}_2 TM$*. Then* $\frac{\mathrm{d}}{\mathrm{d}s}\sec_{\mathtt{g}_s}(\sigma)\big|_{s=0} = 0$ *for all mixed planes* $\sigma \in \mathrm{Gr}_2 TM$*.*

Proof. Let $X, Y \in T_pM$ be such that $X \wedge Y \in \mathrm{Gr}_2(T_pM)$ is a mixed plane, and assume $X_2 = 0$ and $Y_1 = 0$. Identify $X = X_1 \in T_{p_1}M_1 \subset T_pM$ and $Y = Y_2 \in T_{p_2}M_2 \subset T_pM$. From (3.13),

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}s}\sec_{\mathtt{g}_s}(X \wedge Y)\Big|_{s=0} = \nabla_X\nabla_Y \mathtt{h}(X,Y) - \tfrac{1}{2}\nabla_X\nabla_X \mathtt{h}(Y,Y) - \tfrac{1}{2}\nabla_Y\nabla_Y \mathtt{h}(X,X).$$

Taking a trace of the above inequality on $X \in T_{p_1}M_1$, that is, applying it to a $\mathtt{g}_1$-orthonormal basis $\{e_i\}$ of $T_{p_1}M_1 \subset T_pM$, and summing over i, we obtain

$$0 \leq \mathrm{div}_{M_1}(\nabla_Y \mathtt{h})(Y) + \tfrac{1}{2}\Delta_{M_1}\mathtt{h}(Y,Y) - \tfrac{1}{2}\nabla_Y\nabla_Y \mathrm{tr}_{M_1}(\mathtt{h}), \tag{3.18}$$

where div_{M_1} and Δ_{M_1} respectively denote the divergence and Laplacian on $(M_1, \mathtt{g}_1)$, which are identified with the corresponding operators on $M_1 \times \{p_2\} \subset M$, and $\mathrm{tr}_{M_1}(\mathtt{h}) = \sum_i \mathtt{h}(e_i, e_i)$. Integrating (3.18) over the submanifold $M_1 \times \{p_2\}$, since the divergence terms on this closed manifold $(M_1, \mathtt{g}_1)$ have zero integral by the Stokes Theorem, we have

$$0 \leq -\tfrac{1}{2}\int_{M_1 \times \{p_2\}} \nabla_Y\nabla_Y \mathrm{tr}_{M_1}(\mathtt{h}) = -\tfrac{1}{2}\nabla_Y\nabla_Y \int_{M_1 \times \{p_2\}} \mathrm{tr}_{M_1}(\mathtt{h}).$$

[3]This technique implies that the same conclusion holds if $(M,\mathtt{g})$ is a compact symmetric space of rank ≥ 2 with $\sec_{\mathtt{g}} \geq 0$, since it also contains a totally geodesic flat torus, cf. Strake [94, §1.D].

Taking a trace of the above inequality on $Y \in T_{p_2}M_2$, it follows that

$$0 \leq \tfrac{1}{2}\Delta_{M_2}\left(\int_{M_1\times\{p_2\}} \mathrm{tr}_{M_1}(\mathtt{h})\right). \tag{3.19}$$

By the Maximum Principle, since $(M_2, \mathtt{g}_2)$ is a closed manifold, the inequality (3.19) is an equality, and hence so are all the preceding inequalities, concluding the proof. □

Remark 3.11. With regard to the Hopf Problem I and the Local Hopf Problem I, the above results imply that there are no first-order deformations of product metrics $\mathtt{g}$ on $S^2 \times S^2$ that could yield a metric with $\sec > 0$ via Corollary 3.6. Higher-order deformations $\mathtt{g}_s$ of such product metrics were found by Bourguignon, Deschamps and Sentenac [19], with the property that, for all mixed planes $\sigma \in \mathrm{Gr}_2\big(T(S^2 \times S^2)\big)$, the first nonzero derivative of $\sec_{\mathtt{g}_s}(\sigma)$ at $s = 0$ is positive. Although this implies that $\sec_{\mathtt{g}_s}(\sigma)$ is an increasing function on a neighborhood of $s = 0$ for each fixed mixed plane σ, this property is not sufficient to yield $\sec_{\mathtt{g}_s} > 0$ for $s > 0$ sufficiently small. In fact, there is no uniform control on all mixed planes, or, equivalently, on an open neighborhood of $\sec_{\mathtt{g}}^{-1}(0) \subset \mathrm{Gr}_2TM$, see [19, §5] and [94, p. 73]. Furthermore, such metrics $\mathtt{g}_s$ descend to $\mathbb{R}P^2 \times \mathbb{R}P^2$, which does not have $\sec > 0$ by Synge's Theorem, see Petersen [75, Thm. 26, p. 172].

Notice also that, in order to generalize (3.17) to higher-order variations, one must assume higher-order vanishing of $\mathrm{I\!I}_s$ at $s = 0$. In particular, possessing a totally geodesic submanifold that does not have $\sec > 0$ is not necessarily an obstruction for higher-order deformations to develop $\sec > 0$. This can also be deduced by direct inspection of the higher-order variations of sectional curvature, which can be found in an earlier paper of Bourguignon, Deschamps and Sentenac [18].

Let us prove yet another obstruction to first-order variations developing $\sec > 0$ via Corollary 3.6, due to Weinstein [103].

Proposition 3.12. *Let* G *be a compact Lie group that acts isometrically on a compact Riemannian manifold* (M, g) *with* $\sec_{\mathrm{g}} \geq 0$*. Assume that* M *does not admit* G*-invariant Riemannian metrics with* $\sec > 0$*. Then there are no first-order deformations of* g *that have* $\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathrm{g}_s}(\sigma)\big|_{s=0} > 0$ *for all* $\sigma \in \sec_{\mathrm{g}}^{-1}(0) \subset \mathrm{Gr}_2 TM$*.*

Proof. Let $\mathrm{g}_s = \mathrm{g} + s\,\mathrm{h}$ be a first-order deformation of g, and consider the averaging

$$\widehat{\mathrm{g}_s} := \int_{\mathsf{G}} g^*(\mathrm{g}_s)\,\mathrm{d}g,$$

where $\mathrm{d}g$ is the Haar measure on G. Clearly $\widehat{\mathrm{g}_s}$ is G-invariant for all $s \in \mathbb{R}$ such that g_s is a metric. Note that $\widehat{\mathrm{g}_0} = \mathrm{g}$, since the G-action on (M, g) is isometric, hence g is G-invariant. Thus, we have that $\widehat{\mathrm{g}_s} = \mathrm{g} + s\,\widehat{\mathrm{h}}$, where $\widehat{\mathrm{h}}$ is obtained via the averaging

$$\widehat{\mathrm{h}} := \int_{\mathsf{G}} g^*(\mathrm{h})\,\mathrm{d}g. \tag{3.20}$$

For a fixed 2-plane $\sigma \in \mathrm{Gr}_2 TM$, define

$$k^\sigma \colon \mathrm{Met}(M) \to \mathbb{R}, \quad k^\sigma(\mathrm{g}) := \sec_{\mathrm{g}}(\sigma),$$

where $\mathrm{Met}(M) \subset \vee^2 TM^*$ is the open subset formed by Riemannian on M. By (3.20),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\widehat{\mathrm{g}_s}}(\sigma)\Big|_{s=0} &= \mathrm{d}k^\sigma(\mathrm{g})\,\widehat{\mathrm{h}} = \mathrm{d}k^\sigma(\mathrm{g}) \int_{\mathsf{G}} g^*(\mathrm{h})\,\mathrm{d}g = \int_{\mathsf{G}} \mathrm{d}k^\sigma(\mathrm{g}) g^*(\mathrm{h})\,\mathrm{d}g \\ &= \int_{\mathsf{G}} g^*\big(\mathrm{d}k^\sigma(\mathrm{g})\mathrm{h}\big)\,\mathrm{d}g = \int_{\mathsf{G}} g^*\left(\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathrm{g}_s}(\sigma)\Big|_{s=0}\right)\mathrm{d}g.\end{aligned}$$

Thus, if $\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathrm{g}_s}(\sigma)\big|_{s=0} > 0$ for all $\sigma \in \sec_{\mathrm{g}}^{-1}(0)$, then also $\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\widehat{\mathrm{g}_s}}(\sigma)\big|_{s=0} > 0$ for all $\sigma \in \sec_{\mathrm{g}}^{-1}(0)$, and, by Corollary 3.6, we would have G-invariant metrics $\widehat{\mathrm{g}_s}$ on M with $\sec_{\widehat{\mathrm{g}_s}} > 0$, contradicting the assumption that M has no such metrics. □

Combining Proposition 3.12 with the celebrated result of Hsiang and Kleiner [52] and its recent strengthening by Grove and Wilking [45], we have the following:

Corollary 3.13. *Let (M^4, g) be a closed simply-connected Riemannian manifold with $\sec_{\mathsf{g}} \geq 0$, and assume (M^4, g) admits an isometric circle action. Unless M is equivariantly diffeomorphic to S^4 or $\mathbb{C}P^2$, there are no first-order deformations g_s of g that have $\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathsf{g}_s}(\sigma)\big|_{s=0} > 0$ for all $\sigma \in \sec_{\mathsf{g}}^{-1}(0) \subset \mathrm{Gr}_2 TM$.*

Proof. By the results cited above, the only closed simply-connected 4-manifolds that admit a metric with $\sec > 0$ invariant under a circle action are equivariantly diffeomorphic to S^4 or $\mathbb{C}P^2$, so the statement follows directly from Proposition 3.12.[4] □

Together with Remark 3.11, we conclude that the only metrics g on $S^2 \times S^2$ with $\sec_{\mathsf{g}} \geq 0$ that are *candidates* to have a first-order deformation that yields metrics with $\sec > 0$ on $S^2 \times S^2$ via Corollary 3.6 are not product metrics, and such that $(S^2 \times S^2, \mathsf{g})$ has finite isometry group. Although this is a generic subset of the metrics with $\sec \geq 0$ on $S^2 \times S^2$, this first-order approach to the Hopf Problem I (and the Local Hopf Problem I) seems unlikely to produce further results by itself.

Remark 3.14. Under the above assumption that $(S^2 \times S^2, \mathsf{g})$ has $\sec_{\mathsf{g}} \geq 0$ and finite isometry group, Bourguignon, Deschamps and Sentenac [18] proved that there are no *real-analytic* deformations g_s of g for which there exists $s_* > 0$ such that $\sec_{\mathsf{g}_s} > 0$ for all $0 < s < s_*$. However, we observe that this result alone does not solve the Local Hopf Problem I for such a metric g. In other words, it does not exclude that $\mathsf{g} \in \mathrm{Met}(S^2 \times S^2)$ is in the closure of the (possibly nonempty) subset of metrics on $S^2 \times S^2$ with $\sec > 0$, as it is not clear whether all points at the boundary of this subset can be reached as limits of real-analytic paths in the interior. Finally, we

[4] We should also remark that, by the same results, the only closed simply-connected 4-manifolds that admit a metric with $\sec \geq 0$ invariant under a circle action are equivariantly diffeomorphic to S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$, or $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$. Hence, Corollary 3.13 only yields obstructions to first-order deformations of metrics with $\sec \geq 0$ to $\sec > 0$ on $S^2 \times S^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$.

remark that the result in [18] uses the absence of Killing fields on $(S^2 \times S^2, \mathsf{g})$ in a crucial way, and hence does not apply to the standard product metric.

Despite the above obstructions to first-order deformations from $\sec \geq 0$ to $\sec > 0$, Corollary 3.6 was successfully used by Strake [95, Thm. 3.4] to prove the following:

Theorem 3.15. *Let (M, g) be a complete Riemannian manifold with $\sec_{\mathsf{g}} \geq 0$, and let $A \subset M$ be a connected, locally convex, compact subset with nonempty interior and nonempty boundary. Assume that the only points $p \in M$ that support flat planes are contained in the interior of A, that is, $\pi\big(\sec_{\mathsf{g}}^{-1}(0)\big) \subset \operatorname{int} A$, where $\pi \colon \mathrm{Gr}_2 TM \to M$ is the bundle projection. Then g admits a first-order deformation g_s such that $\frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathsf{g}_s}(\sigma)\big|_{s=0} > 0$ for all $\sigma \in \sec_{\mathsf{g}}^{-1}(0)$. In particular,[5] M satisfies $\sec > 0$.*

[5] Recall Corollary 3.6.

CHAPTER 4

CHEEGER DEFORMATION

A *Cheeger deformation* of a Riemannian manifold (M, g) consists of rescaling the metric g in the directions tangent to the orbits of an isometric group action. This method is inspired by the construction of *Berger spheres*,[1] in which the round metric on the sphere S^{2n+1} is rescaled by λ in the direction of the fibers of the Hopf bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$, yielding a 1-parameter family of metrics on S^{2n+1}. Recall that this is a principal S^1-bundle, so the Hopf fibers are orbits of the isometric circle action on S^{2n+1} whose orbit space is $\mathbb{C}P^n$. The terminology is due to the work of Cheeger [23], who applied this technique to construct metrics with $\sec \geq 0$ on the connected sum of any two compact rank one symmetric spaces. Cheeger deformations were systematically studied by Müter [72], and a summary of his results is provided in Ziller [111, 112]. In this section, we discuss some of these key results, following the approach in the previous references as well as Alexandrino and Bettiol [2, §6.1].

4.1 Construction of a Cheeger deformation

Let G be a compact Lie group that acts isometrically on the Riemannian manifold (M, g), and fix a bi-invariant metric Q on G. Endowing $M \times \mathsf{G}$ with the product metric $\mathsf{g} \oplus \frac{1}{t}Q$, where $t > 0$, there is a free isometric G-action on $M \times \mathsf{G}$ given by:

$$h \cdot (p, g) := (h\,p, h\,g), \quad p \in M,\ g, h \in \mathsf{G}. \tag{4.1}$$

[1] See Sections 10.2, 10.3, and 10.4 for details, in particular Proposition 10.7.

The orbit space $(M \times \mathsf{G})/\mathsf{G}$ of this action is easily seen to be diffeomorphic to M, since it is the associated bundle $M \to M \times_{\mathsf{G}} \mathsf{G} \to \{\mathrm{Id}\}$ to the trivial principal G-bundle $\mathsf{G} \to \mathsf{G} \to \{\mathrm{Id}\}$. The quotient map can be explicitly computed to be the submersion:

$$\rho\colon M \times \mathsf{G} \to M, \quad \rho(p, g) = g^{-1}p. \tag{4.2}$$

Since the G-action (4.1) is free and isometric, there is a unique Riemannian metric g_t on M such that $\rho\colon \big(M \times \mathsf{G}, \mathrm{g} \oplus \frac{1}{t}Q\big) \to (M, \mathrm{g}_t)$ is a Riemannian submersion. The family of metrics g_t is called the *Cheeger deformation* of g with respect to the G-action. We remark that, as $t \searrow 0$, the metrics g_t converge to $\mathrm{g}_0 = \mathrm{g}$, see Proposition 4.2.

The original G-action on (M, g) remains isometric on the Cheeger deformed manifold (M, g_t), $t > 0$. Indeed, the isometric G-action on $\big(M \times \mathsf{G}, \mathrm{g} \oplus \frac{1}{t}Q\big)$ given by

$$k \star (p, g) := \big(p, g\, k^{-1}\big), \quad p \in M,\ g, k \in \mathsf{G} \tag{4.3}$$

commutes with (4.1), and hence descends to an isometric G-action on the corresponding orbit space (M, g_t). As $k\,\rho(p, g) = k\, g^{-1}\, p = \rho(k \star (p, g))$, the G-action induced by (4.3) coincides with the original G-action on M.

For each $p \in M$, we denote by $\mathsf{G}_p := \{g \in \mathsf{G} : g\, p = p\}$ its isotropy group and by $\mathfrak{g}_p$ the Lie algebra of G_p, which is a subalgebra of the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{m}_p$ be the Q-orthogonal complement of $\mathfrak{g}_p$ inside $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{g}_p \oplus \mathfrak{m}_p$ is a Q-orthogonal splitting, and identify $\mathfrak{m}_p$ with the tangent space $T_p\mathsf{G}(p)$ to the G-orbit through p via *action fields*, i.e., we identify each $X \in \mathfrak{m}_p$ with

$$X^*_p = \frac{\mathrm{d}}{\mathrm{d}s} \exp(sX)\, p\Big|_{s=0} \in T_p\mathsf{G}(p). \tag{4.4}$$

This defines a g-orthogonal splitting of T_pM into *vertical* and *horizontal spaces*,

$$\mathcal{V}_p := T_p\mathsf{G}(p) = \{X^*_p : X \in \mathfrak{m}_p\} \quad \text{and} \quad \mathcal{H}_p := \{X \in T_pM : \mathsf{g}(X, \mathcal{V}_p) = 0\}, \tag{4.5}$$

and $T_pM = \mathcal{V}_p \oplus \mathcal{H}_p$ remains g_t-orthogonal for all $t > 0$. For $X \in T_pM$, we write $X = X^{\mathcal{V}} + X^{\mathcal{H}}$, where $X^{\mathcal{V}}$ and $X^{\mathcal{H}}$ are its vertical and horizontal components.

Remark 4.1. The restriction of the quotient map $\pi\colon M \to M/\mathsf{G}$ to the open and dense subset $M_{\text{princ}} \subset M$ of points that lie in principal orbits[2] is a smooth Riemannian submersion, and $\mathcal{V}_p$ and $\mathcal{H}_p$ are its vertical and horizontal spaces (2.23).

For each $t \geq 0$, there is a Q-symmetric automorphism $P_t\colon \mathfrak{m}_p \to \mathfrak{m}_p$ such that

$$Q(P_t(X), Y) = \mathsf{g}_t(X^*_p, Y^*_p), \quad X, Y \in \mathfrak{m}_p. \tag{4.6}$$

Furthermore, let $C_t\colon T_pM \to T_pM$ be the g-symmetric automorphism such that

$$\mathsf{g}(C_t(X), Y) = \mathsf{g}_t(X, Y), \quad X, Y \in T_pM. \tag{4.7}$$

In order to describe how the metrics g_t evolves compared to the original metric g, we explicitly compute the automorphisms C_t, following Müter [72, Satz 3.3] and Ziller [111, Prop. 1.1].

Proposition 4.2. *The automorphisms $P_t\colon \mathfrak{m}_p \to \mathfrak{m}_p$ and $C_t\colon T_pM \to T_pM$ for $t \geq 0$ are determined by P_0 in the following way:*

$$\begin{aligned} P_t(X) &= (P_0^{-1} + t\,\mathrm{Id})^{-1}(X) = P_0\,(\mathrm{Id} + tP_0)^{-1}(X), && X \in \mathfrak{m}_p, \\ C_t(X) &= \big((\mathrm{Id} + tP_0)^{-1}(X_{\mathfrak{m}})\big)^*_p + X^{\mathcal{H}}, && X \in T_pM, \end{aligned} \tag{4.8}$$

[2]A G-orbit $\mathsf{G}(p)$ is *principal* if the corresponding isotropy group G_p is the smallest possible (up to conjugacy), see, e.g., Alexandrino and Bettiol [2, §3.4].

where $X_{\mathfrak{m}}$ *is the unique vector in* $\mathfrak{m}_p$ *such that* $(X_{\mathfrak{m}})^*_p = X^{\mathcal{V}}$*. In other words, using the identification via action fields,* $C_t(X) = (\mathrm{Id} + tP_0)^{-1}(X^{\mathcal{V}}) + X^{\mathcal{H}}$ *for all* $X \in T_pM$*.*

Proof. The differential $\mathrm{d}\rho(p,e)\colon T_pM \oplus \mathfrak{g} \to T_pM$ of (4.2) satisfies

$$\mathrm{d}\rho(p,e)(X^*,Y) = X^*_p - Y^*_p, \quad X, Y \in \mathfrak{g}. \tag{4.9}$$

The vertical space[3] at (p,e) for the Riemannian submersion ρ is formed by the pairs $(Z^*,Z) \in T_pM \oplus \mathfrak{g}$. Thus, the vector $(X^*,Y) \in T_pM \oplus \mathfrak{g}$ is $\big(\mathsf{g} \oplus \frac{1}{t}Q\big)$-orthogonal to all pairs (Z^*,Z), and hence horizontal for ρ, if and only if for all $Z \in \mathfrak{g}$,

$$0 = \mathsf{g}(X^*,Z^*) + \tfrac{1}{t}Q(Y,Z) = Q(P_0\,X_{\mathfrak{m}},Z) + Q(\tfrac{1}{t}Y,Z) = Q\left(P_0\,X_{\mathfrak{m}} + \tfrac{1}{t}Y, Z\right),$$

that is, $Y = -tP_0\,X_{\mathfrak{m}}$. Here, for $X \in \mathfrak{g}$, we denote by $X_{\mathfrak{m}}$ the component of X in $\mathfrak{m}_p$. In particular, the horizontal lift with respect to the Riemannian submersion ρ of the action field $X^* \in T_pM$ is given by $(W^*, -tP_0\,W_{\mathfrak{m}}) \in T_pM \oplus \mathfrak{g}$ where $W \in \mathfrak{g}$ satisfies $X^* = \mathrm{d}\rho(p,e)(W^*, -tP_0\,W_{\mathfrak{m}})$. Using (4.9), this becomes $X^*_p = W^*_p + (tP_0\,W_{\mathfrak{m}})^*_p = \big((\mathrm{Id} + tP_0)W_{\mathfrak{m}}\big)^*_p$, to which $W = (\mathrm{Id} + tP_0)^{-1}X \in \mathfrak{m}$ is clearly a solution. Therefore, the horizontal lift of $X^* \in T_pM$ is given by

$$\overline{X^*} = \Big(\big((\mathrm{Id} + tP_0)^{-1}X\big)^*, -tP_0(\mathrm{Id} + tP_0)^{-1}X\Big) \in T_pM \oplus \mathfrak{g}. \tag{4.10}$$

Since $P_0(\mathrm{Id} + tP_0)^{-1} = P_0\big(P_0(P_0^{-1} + t\,\mathrm{Id})\big)^{-1} = P_0(P_0^{-1} + t\,\mathrm{Id})^{-1}P_0^{-1} = (P_0^{-1} + t\,\mathrm{Id})^{-1}$, the above horizontal lift (4.10) can be rewritten as:

$$\overline{X^*} = \Big(\big(P_0^{-1}(P_0^{-1} + t\,\mathrm{Id})^{-1}X\big)^*_p, -t(P_0^{-1} + t\,\mathrm{Id})^{-1}X\Big) \in T_pM \oplus \mathfrak{g}. \tag{4.11}$$

[3]Recall that vertical/horizontal spaces for the submersion $\rho\colon M \times \mathsf{G} \to M$ are subspaces of $T_pM \oplus \mathfrak{g}$, see (2.23), and vertical/horizontal spaces $\mathcal{V}_p$ and $\mathcal{H}_p$ in (4.5) are subspaces of T_pM.

Rewriting it in this way is convenient for the computations in the remainder of this proof, see (4.13). Analogously, the horizontal lift of a vector $V = V^{\mathcal{V}} + V^{\mathcal{H}} = (V_{\mathfrak{m}})^*_p + V^{\mathcal{H}}_p \in T_pM$, is given by

$$\overline{V} = \Big(\big(P_0^{-1}(P_0^{-1} + t\,\mathrm{Id})^{-1}V_{\mathfrak{m}}\big)^*_p + V^{\mathcal{H}}_p, -t(P_0^{-1} + t\,\mathrm{Id})^{-1}V_{\mathfrak{m}}\Big) \in T_pM \oplus \mathfrak{g}, \tag{4.12}$$

since $\overline{V}$ is horizontal for ρ and $\mathrm{d}\rho(p,e)\overline{V} = V$.

Since $\rho\colon \big(M \times \mathsf{G}, \mathsf{g} \oplus \frac{1}{t}Q\big) \to (M, \mathsf{g}_t)$ is a Riemannian submersion, the square norm of $X^* \in T_pM$ with respect to g_t equals the square norm of its horizontal lift (4.11) with respect to $\mathsf{g} \oplus \frac{1}{t}Q$. Thus, we have that for all $X \in \mathfrak{m}$,

$$\begin{aligned}
\mathsf{g}_t(X^*_p, X^*_p) &= \mathsf{g}\Big(\big(P_0^{-1}(P_0^{-1} + t\,\mathrm{Id})^{-1}X\big)^*_p, \big(P_0^{-1}(P_0^{-1} + t\,\mathrm{Id})^{-1}X\big)^*_p\Big) \\
&\quad + \tfrac{1}{t}Q\big(t(P_0^{-1} + t\,\mathrm{Id})^{-1}X, t(P_0^{-1} + t\,\mathrm{Id})^{-1}X\big) \\
&= Q\big((P_0^{-1} + t\,\mathrm{Id})^{-1}X, P_0^{-1}(P_0^{-1} + t\,\mathrm{Id})^{-1}X\big) \\
&\quad + tQ\big((P_0^{-1} + t\,\mathrm{Id})^{-1}X, (P_0^{-1} + t\,\mathrm{Id})^{-1}X\big) \\
&= Q\big((P_0^{-1} + t\,\mathrm{Id})^{-1}X, P_0^{-1}(P_0^{-1} + t\,\mathrm{Id})^{-1}X + t(P_0^{-1} + t\,\mathrm{Id})^{-1}X\big) \\
&= Q\big((P_0^{-1} + t\,\mathrm{Id})^{-1}X, (P_0^{-1} + t\,\mathrm{Id})(P_0^{-1} + t\,\mathrm{Id})^{-1}X\big) \\
&= Q\big((P_0^{-1} + t\,\mathrm{Id})^{-1}X, X\big),
\end{aligned} \tag{4.13}$$

which, according to the definition (4.6), proves that $P_t = (P_0^{-1} + t\,\mathrm{Id})^{-1}$. The formula for C_t follows immediately from the above and (4.7). $\square$

Remark 4.3. From (4.8), if P_0 has eigenvalues λ_i, then C_t has eigenvalues $\frac{1}{1+t\lambda_i}$ corresponding to the vertical directions and eigenvalues 1 in the horizontal directions. Thus, as t grows, the metric g_t *shrinks* in the direction of the orbits and remains unchanged in the orthogonal directions. Note that the speed in which the orbits shrink may vary with the orbit. As $t \nearrow +\infty$, the manifolds (M, g_t) converge, in Gromov-Hausdorff sense, to the orbit space M/G equipped with the orbital distance.

Example 4.4. In the case of the S^1-action on S^{2n+1} discussed in the beginning of this section, $\mathfrak{m}_p$ is 1-dimensional for all $p \in S^{2n+1}$ and the automorphism $P_0 \colon \mathfrak{m}_p \to \mathfrak{m}_p$ has the unique eigenvalue $\lambda_1 = 1$. Thus, denoting by $\mathsf{g}_0 = \mathsf{g}$ the round metric, we have that the Cheeger deformed metric g_t is a Berger metric on S^{2n+1} given by $\mathsf{g}_t = \frac{1}{1+t}\,\mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$, where $\mathsf{g} = \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$ is the splitting of g into vertical and horizontal parts for the Hopf bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. In this case, all orbits are shrunk at the same speed, since the action fields have constant norm. Note that (S^{2n+1}, g_t) converges in Gromov-Hausdorff sense to $\mathbb{C}P^n$ with its standard metric as $t \nearrow +\infty$. Moreover, note that the metrics $\lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$, $\lambda > 1$, are *not* Cheeger deformations of g.

4.2 Evolution of sectional curvatures

As mentioned in the previous section, we are particularly interested in deforming metrics with $\sec_{\mathsf{g}} \geq 0$. Cheeger deformations are remarkably useful in this context due to the following fundamental result.

Proposition 4.5. *Let (M, g) be a Riemannian manifold with $\sec_{\mathsf{g}} \geq 0$, and let g_t be a Cheeger deformation of g. Then $\sec_{\mathsf{g}_t} \geq 0$ for all $t \geq 0$.*

Proof. Recall that a Lie group (G, Q) with bi-invariant metric has $\sec_Q \geq 0$, see (2.17). Thus, if $\sec_{\mathsf{g}} \geq 0$, then $M \times \mathsf{G}$ with the product metric $\mathsf{g} \oplus \frac{1}{t} Q$ also has $\sec \geq 0$. Since $\rho \colon \big(M \times \mathsf{G}, \mathsf{g} \oplus \frac{1}{t} Q\big) \to (M, \mathsf{g}_t)$ is a Riemannian submersion, see (4.2), it follows from the Gray-O'Neill formula (2.28) that $\sec_{\mathsf{g}_t} \geq 0$ for all $t \geq 0$. $\square$

As observed by Müter [72], in order to study in further details how curvature evolves along a Cheeger deformation, it is very convenient to define a reparametrization of $\mathrm{Gr}_2 TM$ using the automorphism C_t defined in (4.7) and computed in (4.8). Given $\sigma = X \wedge Y \in \mathrm{Gr}_2(T_p M)$, set

$$C_t^{-1}(\sigma) := C_t^{-1}(X \wedge Y) = C_t^{-1} X \wedge C_t^{-1} Y, \quad t \geq 0.$$

The above is a 1-parameter family of bundle automorphisms of Gr_2TM, called *Cheeger reparametrization*, which considerably simplifies curvature computations, cf. the horizontal lifts (4.12) and (4.18).

The following result, due to Müter [72, Satz 3.10] (see also Ziller [111, Prop. 1.3]), completely determines how sectional curvatures evolve along a Cheeger deformation.

Proposition 4.6. *Let (M,g) be a Riemannian manifold, and let $X,Y\in T_pM$ be g-orthonormal vectors. Let g_t be a Cheeger deformation of g, and set*

$$k_C(t) := \mathsf{g}_t\big(R_t(C_t^{-1}X,\, C_t^{-1}Y)\, C_t^{-1}X,\, C_t^{-1}Y\big), \tag{4.14}$$

where R_t is the curvature tensor of g_t, so that the sectional curvature of $C_t^{-1}(X\wedge Y)$ is

$$\sec_{\mathsf{g}_t}\big(C_t^{-1}(X\wedge Y)\big) = \frac{k_C(t)}{\big\|C_t^{-1}(X\wedge Y)\big\|_{\mathsf{g}_t}^2}. \tag{4.15}$$

Then, denoting by $S_{\mathfrak{g}}$ the unit sphere in the Lie algebra $\mathfrak{g}$ with respect to the bi-invariant metric Q and using the same notation as in Proposition 4.2, we have that:

$$\begin{aligned} k_C(t) = {} & \sec_{\mathsf{g}}(X\wedge Y) + \frac{t^3}{4}\big\|[P_0\,X_{\mathfrak{m}}, P_0\,Y_{\mathfrak{m}}]\big\|_Q^2 \\ & + \frac{3t}{4}\max_{Z\in S_{\mathfrak{g}}} \frac{\Big(\mathrm{d}(Z^*)^\flat(X,Y) + t\,Q\big(\big[P_0\,X_{\mathfrak{m}}, P_0\,Y_{\mathfrak{m}}\big], Z\big)\Big)^2}{t\,\mathsf{g}(Z^*,Z^*)+1}, \end{aligned} \tag{4.16}$$

and

$$\big\|C_t^{-1}(X\wedge Y)\big\|_{\mathsf{g}_t}^2 = t^2\big\|P_0\,X_{\mathfrak{m}}\wedge P_0\,Y_{\mathfrak{m}}\big\|_Q^2 + t\big(\big\|P_0\,X_{\mathfrak{m}}\big\|_Q^2 + \big\|P_0\,Y_{\mathfrak{m}}\big\|_Q^2\big) + 1. \tag{4.17}$$

Proof. In order to explicitly compute $k_C(t)$, we use the Gray-O'Neill formula for the defining Riemannian submersion $\rho\colon \big(M\times\mathsf{G}, \mathsf{g}\oplus\frac{1}{t}Q\big)\to(M,\mathsf{g}_t)$, see (4.2). From (4.8) and (4.10), if $X \in T_pM$, the horizontal lift of $C_t^{-1}(X) = \big((\mathrm{Id} + tP_0)(X_{\mathfrak{m}})\big)^*_p + X^{\mathcal{H}}$ is

$$\overline{C_t^{-1}(X)} = \big(X, -tP_0\, X_{\mathfrak{m}}\big) \in T_pM \oplus \mathfrak{g}. \tag{4.18}$$

Recall from the proof of Proposition 4.2 that the vertical space of the Riemannian submersion ρ consists of the vectors $(Z^*, Z) \in T_pM \oplus \mathfrak{g}$, where $Z \in \mathfrak{g}$. In particular, using (2.26), we have that the square norm of the tensor A applied to horizontal lifts $\overline{C_t^{-1}(X)}$ and $\overline{C_t^{-1}(Y)}$ with respect to $\big(\mathsf{g} \oplus \frac{1}{t}Q\big)$ is[4]

$$\begin{aligned}
\Big\| A_{\overline{C_t^{-1}(X)}}\, \overline{C_t^{-1}(Y)} \Big\|^2 &= \max_{Z \in S_{\mathfrak{g}}} \frac{\big(\mathsf{g} \oplus \frac{1}{t}Q\big)\Big(A_{\overline{C_t^{-1}(X)}}\, \overline{C_t^{-1}(Y)}, (Z^*, Z)\Big)^2}{\big(\mathsf{g} \oplus \frac{1}{t}Q\big)\big((Z^*, Z), (Z^*, Z)\big)} \\
&= \max_{Z \in S_{\mathfrak{g}}} \frac{\Big(\frac{1}{2}\mathrm{d}\big((Z^*, Z)^\flat\big)\Big(\overline{C_t^{-1}(X)}, \overline{C_t^{-1}(Y)}\Big)\Big)^2}{\mathsf{g}(Z^*, Z^*) + \frac{1}{t}\, Q(Z, Z)} \\
&= \frac{1}{4} \max_{Z \in S_{\mathfrak{g}}} \frac{\Big(\mathrm{d}(Z^*)^\flat(X, Y) + \mathrm{d}(Z^\flat)\big(-tP_0\, X_{\mathfrak{m}}, -tP_0\, Y_{\mathfrak{m}}\big)\Big)^2}{\mathsf{g}(Z^*, Z^*) + \frac{1}{t}} \\
&= \frac{t}{4} \max_{Z \in S_{\mathfrak{g}}} \frac{\Big(\mathrm{d}(Z^*)^\flat(X, Y) + t\, Q\big(\big[P_0\, X_{\mathfrak{m}}, P_0\, Y_{\mathfrak{m}}\big], Z\big)\Big)^2}{t\, \mathsf{g}(Z^*, Z^*) + 1}.
\end{aligned}$$

Thus, applying the Gray-O'Neill formula, see (2.27) and (2.28), to the Riemannian submersion ρ, and using (2.16) and (4.18), one deduces the desired formula (4.16):

$$\begin{aligned}
k_C(t) &= \mathsf{g}(R(X, Y)X, Y) + \frac{1}{4t}\big\|[tP_0\, X_{\mathfrak{m}}, tP_0\, Y_{\mathfrak{m}}]\big\|_Q^2 + 3\,\Big\| A_{\overline{C_t^{-1}(X)}}\, \overline{C_t^{-1}(Y)} \Big\|^2 \\
&= \sec_{\mathsf{g}}(X \wedge Y) + \frac{t^3}{4}\big\|[P_0\, X_{\mathfrak{m}}, P_0\, Y_{\mathfrak{m}}]\big\|_Q^2 \\
&\quad + \frac{3t}{4} \max_{Z \in S_{\mathfrak{g}}} \frac{\Big(\mathrm{d}(Z^*)^\flat(X, Y) + t\, Q\big(\big[P_0\, X_{\mathfrak{m}}, P_0\, Y_{\mathfrak{m}}\big], Z\big)\Big)^2}{t\, \mathsf{g}(Z^*, Z^*) + 1}.
\end{aligned}$$

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[4]Here, we are using the elementary fact that, in a vector space V with inner product $\langle \cdot, \cdot \rangle$, one can write $\|v\|^2 = \max_{w \in S_V} \langle v, w \rangle^2$ where S_V is the unit sphere of V. The advantage of expressing this term in such way is that it allows to use (2.26), which provides a tensorial expression on $X, Y \in T_pM$, without resorting to local extensions of these vectors.

Finally, formula (4.17) follows by applying (4.6), (4.7) and (4.8),

$$\begin{aligned}\left\|C_t^{-1}(X\wedge Y)\right\|_{\mathsf{g}_t}^2 &= \mathsf{g}_t(C_t^{-1}X, C_t^{-1}X)\,\mathsf{g}_t(C_t^{-1}Y, C_t^{-1}Y) - \mathsf{g}_t(C_t^{-1}X, C_t^{-1}Y)^2\\ &= \mathsf{g}(X, C_t^{-1}X)\,\mathsf{g}(Y, C_t^{-1}Y) - \mathsf{g}(X, C_t^{-1}Y)^2\\ &= \big(1+t\left\|P_0\,X_{\mathfrak{m}}\right\|_Q^2\big)\big(1+t\left\|P_0\,Y_{\mathfrak{m}}\right\|_Q^2\big) - \big(t\,Q(P_0\,X_{\mathfrak{m}}, P_0\,Y_{\mathfrak{m}})\big)^2\\ &= t^2\left\|P_0\,X_{\mathfrak{m}}\wedge P_0\,Y_{\mathfrak{m}}\right\|_Q^2 + t\big(\left\|P_0\,X_{\mathfrak{m}}\right\|_Q^2 + \left\|P_0\,Y_{\mathfrak{m}}\right\|_Q^2\big) + 1.\end{aligned}$$

With (4.16) and (4.17), one completely determines $\sec_{\mathsf{g}_t}\big(C_t^{-1}(X\wedge Y)\big)$, see (4.15). □

Remark 4.7. Note that Proposition 4.5 also follows by direct inspection of (4.16).

Corollary 4.8. *Let* (M,g) *be a Riemannian manifold with* $\sec_{\mathsf{g}}\geq 0$*, and let* g_t *be a Cheeger deformation of* g*. If* $X, Y\in T_pM$ *are* g*-orthonormal vectors such that* $[P_0\,X_{\mathfrak{m}}, P_0\,Y_{\mathfrak{m}}]\neq 0$*, then* $\sec_{\mathsf{g}_t}\big(C_t^{-1}(X\wedge Y)\big) > 0$ *for all* $t>0$*. In particular, if* $\mathsf{G}=\mathsf{SO}(3)$ *or* $\mathsf{G}=\mathsf{SU}(2)$ *and the vertical projection*[5] *of* $X\wedge Y\in \mathrm{Gr}_2(T_pM)$ *is* 2*-dimensional, then* $\sec_{\mathsf{g}_t}\big(C_t^{-1}(X\wedge Y)\big)>0$ *for all* $t>0$*.*

Proof. If $[P_0\,X_{\mathfrak{m}}, P_0\,Y_{\mathfrak{m}}]\neq 0$, then $k_C(t)>0$ for all $t>0$ by Proposition 4.6, so $\sec_{\mathsf{g}_t}\big(C_t^{-1}(X\wedge Y)\big)>0$ for all $t>0$. Moreover, if $\mathsf{G}=\mathsf{SO}(3)$ or $\mathsf{G}=\mathsf{SU}(2)$, then $\sec_Q>0$ and hence $[P_0\,X_{\mathfrak{m}}, P_0\,Y_{\mathfrak{m}}]\neq 0$ whenever $X_{\mathfrak{m}}\neq 0$ and $Y_{\mathfrak{m}}\neq 0$, see (2.17). □

Remark 4.9. More generally, it follows from (2.17) that for any compact Lie group (G,Q), we have $[P_0\,X_{\mathfrak{m}}, P_0\,Y_{\mathfrak{m}}]\neq 0$ if and only if $\sec_Q\big(P_0\,X_{\mathfrak{m}}\wedge P_0\,Y_{\mathfrak{m}}\big)>0$.

4.3 First-order properties

In the last section of this chapter, we analyze Cheeger deformations from the first-order viewpoint of Chapter 3.

[5] By vertical projection of $X\wedge Y\in\mathrm{Gr}_2(T_pM)$ we mean the projection onto $T_p\mathsf{G}(p)\subset T_pM$.

Let (M,g) be a Riemannian manifold with $\sec_{\mathrm{g}} \geq 0$ and let g_t be a Cheeger deformation of g. By Proposition 4.5, if $\sigma \in \sec_{\mathrm{g}}^{-1}(0)$ is a flat plane, then $\sec_{\mathrm{g}_t}(\sigma) \geq 0$ for all $t \geq 0$, and an explicit formula for $\sec_{\mathrm{g}_t}\big(C_t^{-1}(\sigma)\big)$ is given in Proposition 4.6. This formula can be used to compute the first variation of $\sec_{\mathrm{g}_t}(\sigma)$, without the Cheeger reparametrization C_t, since σ has extremal curvature. More precisely, since $\sec_{\mathrm{g}} \geq 0$ and $\sec_{\mathrm{g}}(\sigma) = 0$ is a minimum, it follows that that σ is a critical point of $\sec_{\mathrm{g}} \colon \mathrm{Gr}_2 TM \to \mathbb{R}$, i.e., $\mathrm{d}(\sec_{\mathrm{g}})(\sigma) = 0$. Thus, by the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \sec_{\mathrm{g}_t}\big(C_t^{-1}(\sigma)\big)\Big|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t} \sec_{\mathrm{g}_t}(\sigma)\Big|_{t=0} + \mathrm{d}(\sec_{\mathrm{g}})(\sigma)\sigma' \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \sec_{\mathrm{g}_t}(\sigma)\Big|_{t=0},\end{aligned}$$

where σ' is the first variation of $C_t^{-1}(\sigma)$ at $t=0$, that is, $C_t^{-1}(\sigma) = \sigma + t\,\sigma' + O(t^2)$ in $\wedge^2 TM$. Furthermore, we can compute the above first variation directly from the unnormalized curvature $k_C(t)$, since $\sec_{\mathrm{g}}(\sigma) = 0$ and hence by (4.15),

$$\begin{aligned}k_C'(0) &= \frac{\mathrm{d}}{\mathrm{d}t} \big\| C_t^{-1}(\sigma) \big\|_{\mathrm{g}_t}^2 \Big|_{t=0} \sec_{\mathrm{g}}(\sigma) + \|\sigma\|_{\mathrm{g}}^2 \frac{\mathrm{d}}{\mathrm{d}t} \sec_{\mathrm{g}_t}\big(C_t^{-1}(\sigma)\big)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \sec_{\mathrm{g}_t}\big(C_t^{-1}(\sigma)\big)\Big|_{t=0}.\end{aligned}$$

Thus, altogether, we have $\frac{\mathrm{d}}{\mathrm{d}t} \sec_{\mathrm{g}_t}(\sigma)\big|_{t=0} = k_C'(0)$.

Let $X, Y \in T_pM$ be g-orthonormal vectors such that $\sigma = X \wedge Y$, and differentiate (4.16) at $t=0$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \sec_{\mathrm{g}_t}(\sigma)\Big|_{t=0} = k_C'(0) = \tfrac{3}{4} \max_{Z \in S_{\mathfrak{g}}} \mathrm{d}(Z^*)^\flat(X,Y)^2. \tag{4.19}$$

Notice that $\frac{\mathrm{d}}{\mathrm{d}t} \sec_{\mathrm{g}_t}(\sigma)\big|_{t=0} \geq 0$, and $\frac{\mathrm{d}}{\mathrm{d}t} \sec_{\mathrm{g}_t}(\sigma)\big|_{t=0} = 0$ if and only if $\mathrm{d}(Z^*)^\flat(X,Y) = 0$ for all $Z \in \mathfrak{g}$, in which case (4.16) simplifies to

$$k_C(t) = \frac{t^3}{4} \big\| [P_0\, X_{\mathfrak{m}}, P_0\, Y_{\mathfrak{m}}] \big\|_Q^2 + \frac{3t^3}{4} \max_{Z \in S_{\mathfrak{g}}} \frac{Q\big([P_0\, X_{\mathfrak{m}}, P_0\, Y_{\mathfrak{m}}], Z\big)^2}{t\,\mathrm{g}(Z^*, Z^*) + 1}. \tag{4.20}$$

In particular, if $\frac{\mathrm{d}}{\mathrm{d}t}\sec_{\mathrm{g}_t}(\sigma)\big|_{t=0}=0$, then (4.20) implies that either $\sec_{\mathrm{g}_t}\big(C_t^{-1}(\sigma)\big)>0$ for all $t>0$, or $\sec_{\mathrm{g}_t}\big(C_t^{-1}(\sigma)\big)=0$ for all $t>0$; according to $[P_0\,X_{\mathfrak{m}},P_0\,Y_{\mathfrak{m}}]\neq 0$ or $[P_0\,X_{\mathfrak{m}},P_0\,Y_{\mathfrak{m}}]=0$. Notice also that if $\frac{\mathrm{d}}{\mathrm{d}t}\sec_{\mathrm{g}_t}(\sigma)\big|_{t=0}=k_C'(0)=0$, then by (4.20),

$$\begin{aligned}
k_C''(0)&=0,\\
k_C'''(0)&=\frac{3}{2}\big\|[P_0\,X_{\mathfrak{m}},P_0\,Y_{\mathfrak{m}}]\big\|_Q^2+\frac{9}{2}\max_{Z\in S_{\mathfrak{g}}}\,Q\big([P_0\,X_{\mathfrak{m}},P_0\,Y_{\mathfrak{m}}],Z\big)^2\\
&=6\big\|[P_0\,X_{\mathfrak{m}},P_0\,Y_{\mathfrak{m}}]\big\|_Q^2,
\end{aligned}\tag{4.21}$$

cf. Müter [72, Satz 4.9] and Ziller [111, Cor. 1.4].

Remark 4.10. By the above observations, the 2-forms $\mathrm{d}(Z^*)^\flat\in\wedge^2TM$, with $Z\in\mathfrak{g}$, play a fundamental role in the study of $k_C(t)$ and $\sec_{\mathrm{g}_t}\big(C_t^{-1}(\sigma)\big)$. Note that, since Z^* is a Killing vector field, denoting by $X,Y\in TM$ local coordinate fields that extend $X,Y\in T_pM$, we have from (3.6) that

$$\begin{aligned}
\mathrm{d}(Z^*)^\flat(X,Y)&=X\big(\mathrm{g}(Z^*,Y)\big)-Y\big(\mathrm{g}(Z^*,X)\big)\\
&=\mathrm{g}(\nabla_XZ^*,Y)-\mathrm{g}(\nabla_YZ^*,X)\\
&=2\,\mathrm{g}(\nabla_XZ^*,Y).
\end{aligned}\tag{4.22}$$

In particular, if $p\in M_{\mathrm{princ}}$, see Remark 4.1, then by (2.24) and (2.26),

$$\mathrm{d}(Z^*)^\flat(X,Y)=\begin{cases}-2\,\mathrm{g}(A_XY,Z_p^*), & \text{if } X,Y\in\mathcal{H}_p,\\ -2\,\mathrm{g}(T_XY,Z_p^*)=2\mathrm{g}\big(\mathrm{I\!I}_p(X,Z_p^*),Y\big), & \text{if } X\in\mathcal{V}_p, Y\in\mathcal{H}_p,\\ Q\big([P_0\,X_{\mathfrak{m}},Y]+[X,P_0\,Y_{\mathfrak{m}}]+P_0\,[X_{\mathfrak{m}},Y_{\mathfrak{m}}],Z\big), & \text{if } X,Y\in\mathcal{V}_p,\end{cases}$$

where T and A are the tensors of the Riemannian submersion $\pi\colon M_{\mathrm{princ}}\to M_{\mathrm{princ}}/\mathsf{G}$, and $\mathrm{I\!I}_p$ is the second fundamental form of the orbit $\mathsf{G}(p)$, see Müter [72, Lemma 4.15] or Ziller [111, Prop. 1.6] for details.

As discussed in Section 3.4, the presence of totally geodesic immersed flat tori is a first-order obstruction to deforming metrics with $\sec \geq 0$ to $\sec > 0$. From the above computation (4.19) of the first variation of sectional curvature for a Cheeger deformation, one deduces the following obstruction, cf. Ziller [111, Cor. 1.5].

Proposition 4.11. *Let (M, g) be a Riemannian manifold with $\sec_{\mathsf{g}} \geq 0$, and g_t be a Cheeger deformation of g. If $\sigma \in \mathrm{Gr}_2 TM$ is tangent to a totally geodesic flat torus in (M, g) and σ contains a horizontal direction, then $\sec_{\mathsf{g}_t}\big(C_t^{-1}(\sigma)\big) = 0$ for all $t \geq 0$.*

Proof. Let $i\colon T \to M$ be the totally geodesic flat torus in (M, g), and let e_1, e_2 be a global orthonormal frame on $(T, i^*\mathsf{g})$. Define vector fields X and Y along $i\colon T \to M$ by setting $X(p) := \mathrm{d}i(p)e_1$ and $Y(p) := \mathrm{d}i(p)e_2$ for all $p \in T$. From (4.19) and Lemma 3.7, the function $k_C(t)$ defined as the unnormalized g_t-sectional curvature of $C_t^{-1}(X \wedge Y)$ satisfies

$$k_C'(0) = \frac{\mathrm{d}}{\mathrm{d}t} \sec_{\mathsf{g}_t}(X \wedge Y)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \sec_{i^*\mathsf{g}_t}(e_1 \wedge e_2)\Big|_{t=0}. \tag{4.23}$$

On the other hand, by the Gauss-Bonnet Theorem, for all $t \geq 0$,

$$\int_T \sec_{i^*\mathsf{g}_t}(e_1 \wedge e_2)\, \mathrm{vol}_{i^*\mathsf{g}_t} = 2\pi\chi(T) = 0.$$

Thus, differentiating the above at $t = 0$ and using (4.23), we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} \int_T \sec_{i^*\mathsf{g}_t}(e_1 \wedge e_2)\, \mathrm{vol}_{i^*\mathsf{g}_t}\Big|_{t=0} \\ &= \int_T \frac{\mathrm{d}}{\mathrm{d}t} \sec_{i^*\mathsf{g}_t}(e_1 \wedge e_2)\Big|_{t=0} \mathrm{vol}_{i^*\mathsf{g}} + \int_T \sec_{i^*\mathsf{g}}(e_1 \wedge e_2)\, \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}_{i^*\mathsf{g}_t}\Big|_{t=0} \\ &= \int_T \frac{\mathrm{d}}{\mathrm{d}t} \sec_{i^*\mathsf{g}_t}(e_1 \wedge e_2)\Big|_{t=0} \mathrm{vol}_{i^*\mathsf{g}} \\ &= \int_T k_C'(0)\, \mathrm{vol}_{i^*\mathsf{g}}. \end{aligned}$$

Since the above integrand $k_C'(0)$ is nonnegative by (4.19), we have that $k_C'(0) = 0$ on all of T. In particular, if $\sigma = X \wedge Y$ contains a horizontal direction, then either

$X_{\mathfrak{m}} = 0$ or $Y_{\mathfrak{m}} = 0$, hence $[P_0\, X_{\mathfrak{m}}, P_0\, Y_{\mathfrak{m}}] = 0$. Thus, it follows from (4.20) that for all $t \geq 0$ we have $k_C(t) = 0$ and hence $\sec_{\mathsf{g}_t}\big(C_t^{-1}(\sigma)\big) = 0$. □

We conclude this section computing the first variation of a Cheeger deformation g_t, that combined with the first variation formula (3.13), can be used to reobtain (4.19).

Proposition 4.12. *Let (M, g) be a Riemannian manifold and g_t be a Cheeger deformation of g with respect to a G-action on M. Let $\{Z_i\}$, $1 \leq i \leq n$, be a Q-orthonormal basis of $\mathfrak{g}$. Then $\mathsf{g}_t = \mathsf{g} + t\,\mathtt{h} + O(t^2)$, where $\mathtt{h} = -\sum_{i=1}^n (Z_i^*)^\flat \otimes (Z_i^*)^\flat \in \vee^2 TM^*$.*

Proof. From (4.8), we have that given $X \in T_pM$, the first variation of $C_t(X)$ is

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} C_t(X)\big|_{t=0} &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big((\mathrm{Id} + tP_0)^{-1}(X_{\mathfrak{m}})_p^* + X^{\mathcal{H}}\big)\big|_{t=0} \\ &= \big(\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Id} + tP_0)^{-1}(X_{\mathfrak{m}})\big|_{t=0}\big)_p^* \\ &= (-P_0\, X_{\mathfrak{m}})_p^*. \end{aligned} \tag{4.24}$$

Thus, if $X, Y \in T_pM$, we have from the above, (4.6) and (4.7) that

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\, \mathsf{g}_t(X, Y)\big|_{t=0} &= \tfrac{\mathrm{d}}{\mathrm{d}t}\, \mathsf{g}(C_t(X), Y)\big|_{t=0} \\ &= \mathsf{g}\big((-P_0\, X_{\mathfrak{m}})_p^*, Y\big) \\ &= -Q(P_0\, X_{\mathfrak{m}}, P_0\, Y_{\mathfrak{m}}) \\ &= -\textstyle\sum_{i=1}^n Q(P_0\, X_{\mathfrak{m}}, Z_i)\, Q(P_0\, Y_{\mathfrak{m}}, Z_i) \\ &= -\textstyle\sum_{i=1}^n \mathsf{g}(X, Z_i^*)\, \mathsf{g}(Y, Z_i^*) \\ &= -\big(\textstyle\sum_{i=1}^n (Z_i^*)^\flat \otimes (Z_i^*)^\flat\big)\,(X, Y), \end{aligned}$$

proving that $\mathtt{h} = -\sum_{i=1}^n (Z_i^*)^\flat \otimes (Z_i^*)^\flat$ is the first variation of g_t. □

PART II

WEAKLY POSITIVE CURVATURE

CHAPTER 5

AVERAGING SECTIONAL CURVATURES

In this chapter, we discuss two *weakly positive curvature* conditions, defined in terms of averages of pairs of sectional curvatures. Although we focus on averaging only 2 sectional curvatures, we remark that notable results on curvature positivity conditions with averages of more sectional curvatures have been obtained by Labbi [62, 63] and Wu [108], among others.

5.1 Distance between planes

Let V be a real vector space of finite dimension $\dim V = n$ endowed with an inner product. We denote by $\mathrm{Gr}_2(V)$ the Grassmannian of 2-planes in V. In this section, we compare three distance functions $\mathrm{dist}\colon \mathrm{Gr}_2(V) \times \mathrm{Gr}_2(V) \to \mathbb{R}$, that provide a notion of *aperture*, or *gap*, between 2-planes in V and induce the standard topology on $\mathrm{Gr}_2(V)$.

Let $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ and denote by S_σ and $S_{\sigma'}$ the great circles in the unit sphere $S_V := \{v \in V : \|v\| = 1\}$, obtained by intersecting it with the 2-planes σ and σ', respectively. The *spherical distance* between $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ is defined as:

$$\mathrm{dist}_S(\sigma, \sigma') := \mathrm{dist}_{\mathrm{H}}\big(S_\sigma, S_{\sigma'}\big), \tag{5.1}$$

where $\mathrm{dist}_{\mathrm{H}}$ is the Hausdorff distance between closed subsets of the sphere S_V. By the properties of the Hausdorff distance, we have

$$\operatorname{dist}_S(\sigma, \sigma') = \max \left\{ \max_{v \in S_\sigma} \operatorname{dist}(v, S_{\sigma'}),\ \max_{w \in S_{\sigma'}} \operatorname{dist}(w, S_\sigma) \right\}, \tag{5.2}$$

where the distance between a point in S_V and a great circle is measured with respect to the round metric on S_V. In particular, we have that $0 \leq \operatorname{dist}_S(\sigma, \sigma') \leq \frac{\pi}{2}$.

Remark 5.1. Suppose $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ achieve the above maximum $\operatorname{dist}_S(\sigma, \sigma') = \frac{\pi}{2}$. The linear subspace of V spanned by σ and σ' has dimension 3 or 4, according to whether the great circles $S_\sigma, S_{\sigma'} \subset S_V$ intersect or not. In the first case, there are orthonormal vectors $e_1, e_2, e_3 \in V$ such that $\sigma = e_1 \wedge e_2$ and $\sigma' = e_1 \wedge e_3$, while in the second case, there are orthonormal vectors $e_1, e_2, e_3, e_4 \in V$ such that $\sigma = e_1 \wedge e_2$ and $\sigma' = e_3 \wedge e_4$, that is, σ and σ' are orthogonal.

Another definition of distance between $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ can be given in terms of the operators of orthogonal projection onto these subspaces. Let $\Pi_\sigma \colon V \to V$ be the orthogonal projection operator onto $\sigma \subset V$, and analogously for σ'. The *chordal distance* between $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ is defined as:

$$\operatorname{dist}_C(\sigma, \sigma') := \|\Pi_\sigma - \Pi_{\sigma'}\|. \tag{5.3}$$

Since orthogonal projection operators have norm ≤ 1 and the vectors $\Pi_\sigma(\mathrm{Id} - \Pi_{\sigma'})v$ and $(\mathrm{Id} - \Pi_\sigma)\Pi_{\sigma'} v$ are orthogonal for all $v \in T_pM$, we have that

$$\begin{aligned} \|(\Pi_\sigma - \Pi_{\sigma'})v\|^2 &= \left\| \big(\Pi_\sigma(\mathrm{Id} - \Pi_{\sigma'}) - (\mathrm{Id} - \Pi_\sigma)\Pi_{\sigma'}\big) v \right\|^2 \\ &= \|\Pi_\sigma(\mathrm{Id} - \Pi_{\sigma'})v\|^2 + \|(\mathrm{Id} - \Pi_\sigma)\Pi_{\sigma'} v\|^2 \\ &\leq \|(\mathrm{Id} - \Pi_{\sigma'})v\|^2 + \|\Pi_{\sigma'} v\|^2 \\ &\leq \|v\|^2, \end{aligned} \tag{5.4}$$

which implies $0 \leq \operatorname{dist}_C(\sigma, \sigma') \leq 1$. It is proved in Akhiezer and Glazman [1, §34] that (5.3) can also be written as

$$\text{dist}_C(\sigma, \sigma') = \max \left\{ \max_{v \in S_\sigma} \inf_{w \in \sigma'} \|v - w\|, \max_{w \in S_{\sigma'}} \inf_{v \in \sigma} \|v - w\| \right\}. \tag{5.5}$$

This allows us to compare dist_S and dist_C, using (5.2) and (5.5). Notice that the first arguments are the maximum over $v \in S_\sigma$ of the distance between v and σ', either measured in the *spherical* fashion $\text{dist}(v, S_{\sigma'}) = \min\limits_{w \in S_{\sigma'}} \arccos |\langle v, w \rangle|$, i.e., taking the length of the shortest path on S_V that joins v to σ'; or in the *chordal* fashion $\|v - \Pi_{\sigma'}(v)\| = \inf\limits_{w \in \sigma'} \|v - w\|$, i.e., taking the length of the shortest straight line segment that joins v to σ'. An analogous statement holds for the second arguments, interchanging the roles of σ and σ'. This geometric interpretation is what motivates the above terminology, and it also implies

$$\text{dist}_C(\sigma, \sigma') = \sin\big(\text{dist}_S(\sigma, \sigma')\big), \tag{5.6}$$

proving that dist_S and dist_C are, in fact, equivalent in the above context.[1]

Remark 5.2. The maximum in (5.2) is achieved simultaneously by both expressions, that is,

$$\text{dist}_S(\sigma, \sigma') = \max_{v \in S_\sigma} \text{dist}(v, S_{\sigma'}) = \max_{w \in S_{\sigma'}} \text{dist}(w, S_\sigma) \tag{5.7}$$

and analogously for the maximum in (5.5),

$$\text{dist}_C(\sigma, \sigma') = \max_{v \in S_\sigma} \inf_{w \in \sigma'} \|v - w\| = \max_{w \in S_{\sigma'}} \inf_{v \in \sigma} \|v - w\|. \tag{5.8}$$

A proof of the above can be found in Morris [71, Lemma 3.2].

Remark 5.3. If $\sigma, \sigma' \in \text{Gr}_2(V)$ achieve the maximum $\text{dist}_C(\sigma, \sigma') = 1$, then it follows from (5.6) that the same conclusions as in Remark 5.1 are valid for σ and σ'.

[1]A detailed comparison of the distance functions dist_S and dist_C in more general contexts (such as Banach spaces) can be found in Berkson [11], see also Kato [57, Chap. IV §2.1].

Our last distance function in $\mathrm{Gr}_2(V)$ is obtained via the diffeomorphism

$$\mathrm{Gr}_2(V) \cong \frac{\mathsf{O}(n)}{\mathsf{O}(2)\mathsf{O}(n-2)}. \tag{5.9}$$

In order to describe the distance function induced by the Riemannian metric on (5.9) that makes it a symmetric space, we first discuss basic properties of the *principal angles* between two subspaces, see [40, 54, 73] for details.

Definition 5.4. Given $\sigma, \sigma' \in \mathrm{Gr}_2(V)$, the principal angles $0 \leq \theta_1 \leq \theta_2 \leq \frac{\pi}{2}$ between σ and σ' are respectively the smallest and the largest angle that a line in σ makes with the plane σ'. In other words,

$$\theta_1 = \arccos\left(\max_{v\in S_\sigma}\max_{w\in S_{\sigma'}}\langle v, w\rangle\right) \quad \text{and} \quad \theta_2 = \arccos\left(\min_{v\in S_\sigma}\max_{w\in S_{\sigma'}}\langle v, w\rangle\right). \tag{5.10}$$

It is easy to see that interchanging the roles of σ and σ' does not change these angles. Clearly, $\theta_1 = 0$ if and only if σ and σ' intersect nontrivially, and $\theta_1 = \theta_2 = 0$ if and only if $\sigma = \sigma'$. Furthermore, $\theta_1 = \theta_2 = \frac{\pi}{2}$ if and only if σ and σ' are orthogonal, that is, $\sigma' \subset \sigma^\perp$, where $\sigma^\perp$ is the complement of σ in V such that $\sigma \oplus \sigma^\perp = V$ is an orthogonal direct sum. More generally, if the principal angles between $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ are θ_1 and θ_2, then there are orthonormal vectors $e_1, e_2, e_3, e_4 \in V$ such that $\sigma = e_1 \wedge e_2$ and $\sigma' = (\cos\theta_1 e_1 + \sin\theta_1 e_3) \wedge (\cos\theta_2 e_2 + \sin\theta_2 e_4)$. In particular, $\langle \sigma, \sigma'\rangle = \cos\theta_1 \cos\theta_2$.

Remark 5.5. A useful alternative characterization of principal angles θ_1 and θ_2 between σ and σ' is that $\cos\theta_1$ and $\cos\theta_2$ are the singular values[2] of the operator $(\Pi_{\sigma'})|_\sigma \colon \sigma \to \sigma'$ of orthogonal projection of σ onto σ'. Direct computations show that the principal angles between $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ are $\theta_i = \arccos\sqrt{\lambda_i}$, where $\lambda_1 \geq \lambda_2 \geq 0$ are the roots of the polynomial[3] $\mathcal{P}_{\sigma,\sigma'}(\lambda) = \lambda^2 - \big(1 - \|\sigma \wedge \sigma'\|^2 + \langle \sigma, \sigma'\rangle^2\big)\lambda + \langle \sigma, \sigma'\rangle^2$.

[2]Recall that the *singular values* of an operator $A \colon V \to V$ are the eigenvalues of $\sqrt{A^*A}$.

[3]In particular, notice that $\theta_1 = \theta_2 = 0$ if and only if $\lambda_1 = \lambda_2 = 1$, which is equivalent to

The *symmetric space distance* between $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ is defined as:

$$\mathrm{dist}_{SS}(\sigma, \sigma') = \sqrt{\theta_1^2 + \theta_2^2}, \tag{5.11}$$

where θ_i are the principal angles between σ and σ'. This is the length of the shortest geodesic in (5.9) that joins σ and σ', which can be computed to be $\gamma(t) = \big(\cos(t\theta_1)e_1 + \sin(t\theta_1)e_3\big) \wedge \big(\cos(t\theta_2)e_2 + \sin(t\theta_2)e_4\big)$ in terms of the above orthonormal vectors $e_1, e_2, e_3, e_4 \in V$ such that $\gamma(0) = \sigma$ and $\gamma(1) = \sigma'$.

Remark 5.6. From (5.7) and (5.10), it follows that the spherical distance between $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ is precisely the largest principal angle between these planes, i.e., $\mathrm{dist}_S(\sigma, \sigma') = \theta_2$. In particular, by (5.6), the chordal distance is $\mathrm{dist}_C(\sigma, \sigma') = \sin\theta_2$.

Remark 5.7. From the above discussion, $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ achieve the maximum distance $\mathrm{dist}_{SS}(\sigma, \sigma') = \frac{\pi}{\sqrt{2}}$ if and only if σ and σ' are orthogonal. Thus, there is more information about planes at maximal distance with respect to dist_{SS} than planes at maximal distance with respect to dist_S or dist_C, see Remarks 5.1 and 5.3.

5.2 The family of conditions $\sec^\theta > 0$

Let M be a manifold and fix a *fiberwise* distance function dist on $\mathrm{Gr}_2 TM$, that is, a distance function on each $\mathrm{Gr}_2(T_pM)$ that varies continuously with $p \in M$. For instance, endowing M with a Riemannian metric g, one may use local orthonormal frames to construct smoothly varying linear isometries $\iota_p \colon V \to T_pM$, and define a distance function on $\mathrm{Gr}_2(T_pM)$ by setting

$$\mathrm{dist}(\sigma, \sigma') := \mathrm{dist}_{\mathrm{Gr}_2(V)}\big(\iota_p^{-1}(\sigma), \iota_p^{-1}(\sigma')\big), \quad \text{for all } \sigma, \sigma' \in \mathrm{Gr}_2(T_pM), \tag{5.12}$$

$\langle \sigma, \sigma' \rangle = 1$ and $\|\sigma \wedge \sigma'\| = 0$, that is, $\sigma = \sigma'$. Similarly, $\theta_1 = \theta_2 = \frac{\pi}{2}$ if and only if $\lambda_1 = \lambda_2 = 0$, which is equivalent to $\langle \sigma, \sigma' \rangle = 0$ and $\|\sigma \wedge \sigma'\| = 1$, that is, σ and σ' are orthogonal.

where $\mathrm{dist}_{\mathrm{Gr}_2(V)}$ is a distance function on $\mathrm{Gr}_2(V)$, e.g., one of (5.1), (5.3) or (5.11).

Given a Riemannian metric g on M and $\theta > 0$, we define:

$$\sec^\theta_\mathsf{g}\colon \mathrm{Gr}_2 TM \to \mathbb{R}, \quad \sec^\theta_\mathsf{g}(\sigma) := \min_{\substack{\sigma' \in \mathrm{Gr}_2(T_pM) \\ \mathrm{dist}(\sigma,\sigma') \geq \theta}} \tfrac{1}{2}\big(\sec_\mathsf{g}(\sigma) + \sec_\mathsf{g}(\sigma')\big). \tag{5.13}$$

Note $\sec^\theta_\mathsf{g} > 0$ means that, at every point $p \in M$, the average of sectional curvatures of any 2-planes tangent at p that are at least θ apart from each other is positive. One can intuitively think of θ as a lower bound for the *aperture*, or *gap*, between the planes considered in the averaging, see the previous section for geometric interpretations.

Clearly, if $\sec_\mathsf{g} > 0$, then $\sec^\theta_\mathsf{g} > 0$ for any $\theta > 0$. Moreover, if $\theta_1 < \theta_2$, then $\sec^{\theta_1}_\mathsf{g} > 0$ implies $\sec^{\theta_2}_\mathsf{g} > 0$. Thus, $\sec^\theta_\mathsf{g} > 0$ is a family of curvature positivity conditions parametrized by $\theta > 0$, that becomes stronger as $\theta \searrow 0$. At the limit $\theta = 0$, the condition $\sec^\theta_\mathsf{g} > 0$ is equivalent to $\sec_\mathsf{g} > 0$, since:

$$\sec_\mathsf{g}(\sigma) = \tfrac{1}{2}\big(\sec_\mathsf{g}(\sigma) + \sec_\mathsf{g}(\sigma)\big) \geq \min_{\substack{\sigma' \in \mathrm{Gr}_2(T_pM) \\ \mathrm{dist}(\sigma,\sigma') \geq 0}} \tfrac{1}{2}\big(\sec_\mathsf{g}(\sigma) + \sec_\mathsf{g}(\sigma')\big) = \sec^0_\mathsf{g}(\sigma).$$

In the setting of Section 2.3, $\sec^\theta_\mathsf{g} > 0$ are pointwise curvature conditions if dist is of the form (5.12), since $\sec^\theta_\mathsf{g} > 0$ corresponds to the open $\mathsf{O}(n)$-invariant convex cone

$$C_{\sec^\theta > 0} := \left\{ R \in S_\mathfrak{b}(\wedge^2 V) : \begin{array}{l} \langle R(\sigma), \sigma\rangle + \langle R(\sigma'), \sigma'\rangle > 0 \\ \text{for all } \sigma, \sigma' \in \mathrm{Gr}_2(V) \text{ such that } \mathrm{dist}_{\mathrm{Gr}_2(V)}(\sigma,\sigma') \geq \theta \end{array} \right\}.$$

5.3 The condition $\sec^{0+} > 0$

Using the above family of conditions $\sec^\theta > 0$ without fixing the metric g, consider the following curvature positivity condition on M regarding the limit $\theta \searrow 0$.

Definition 5.8. A manifold M satisfies $\sec^{0+} > 0$ if, for all $\theta > 0$, there exists a Riemannian metric g^θ on M with $\sec^\theta_{\mathsf{g}^\theta} > 0$, and g^θ converges to a metric g^0 as $\theta \searrow 0$.

Remark 5.9. The notion of convergence $\mathsf{g}^\theta \to \mathsf{g}^0$ in the above definition can be chosen, e.g., as convergence in the C^k-topology for some $k \geq 2$, and is only explicitly mentioned if necessary. Note that the choice of fiberwise distance function on $\mathrm{Gr}_2 TM$ is not very important in this context, since we are interested in $\theta \searrow 0$, and also note that $\sec^{0+} > 0$ is *not* a pointwise curvature condition in the sense of Section 2.3.

The interest in $\sec^{0+} > 0$ is mainly due to its relation with $\sec > 0$ and $\mathrm{Ric} > 0$:

Proposition 5.10. *If M satisfies $\sec > 0$, then it also satisfies $\sec^{0+} > 0$. If M is compact and satisfies $\sec^{0+} > 0$, then it also satisfies $\mathrm{Ric} > 0$ and $\sec \geq 0$.*

Proof. If M admits a Riemannian metric g with $\sec_{\mathsf{g}} > 0$, then setting $\mathsf{g}^\theta = \mathsf{g}$ for all $\theta > 0$ we have that M satisfies $\sec^{0+} > 0$.

Given a Riemannian metric g on a compact manifold M, we define:

$$\theta_0(\mathsf{g}) := \min_{p \in M} \min_{v \in S_pM} \min_{\substack{e_i, e_j \in S_pM, \\ \mathsf{g}(v, e_i) = 0, \\ \mathsf{g}(e_i, e_j) = \delta_{ij}}} \mathrm{dist}\big(v \wedge e_i, v \wedge e_j\big). \tag{5.14}$$

The above is clearly a positive number, that depends continuously on g. Moreover, if $\sec^\theta_{\mathsf{g}} > 0$ for some $0 < \theta \leq \theta_0(\mathsf{g})$, then $\mathrm{Ric}_{\mathsf{g}} > 0$. In fact, $\mathrm{Ric}_{\mathsf{g}}(v) > 0$ for any direction $v \in TM$, since, by (2.9) and (5.14), this is a sum of sectional curvatures whose pairwise average is positive.

If M satisfies $\sec^{0+} > 0$, set $\mathcal{G} := \{\mathsf{g}^\theta : \theta \in [0, 1]\}$, and $\theta_* := \min\{\theta_0(\mathsf{g}) : \mathsf{g} \in \mathcal{G}\}$. Since $\theta_0(\mathsf{g})$ depends continuously on g and $\mathcal{G}$ is compact, we have that $\theta_* > 0$. By the above, for any $0 < \theta \leq \theta_*$, we have $\mathrm{Ric}_{\mathsf{g}^\theta} > 0$. Therefore, M satisfies $\mathrm{Ric} > 0$. Moreover, we claim that $\sec_{\mathsf{g}^0} \geq 0$ and hence M satisfies $\sec \geq 0$. In fact, given $\sigma \in \mathrm{Gr}_2(T_pM)$, let θ_n be a sequence of positive numbers converging to 0, say $\theta_n = \frac{1}{n}$, and let $\sigma_n \in \mathrm{Gr}_2(T_pM)$ be a sequence such that $\mathrm{dist}(\sigma, \sigma_n) = \theta_n$. Then, we have:

$$0 < \sec^{\theta_n}_{\mathsf{g}^{\theta_n}}(\sigma) = \min_{\substack{\sigma' \in \mathrm{Gr}_2(T_pM) \\ \mathrm{dist}(\sigma, \sigma') \geq \theta_n}} \tfrac{1}{2}\big(\sec_{\mathsf{g}^{\theta_n}}(\sigma) + \sec_{\mathsf{g}^{\theta_n}}(\sigma')\big) \leq \tfrac{1}{2}\big(\sec_{\mathsf{g}^{\theta_n}}(\sigma) + \sec_{\mathsf{g}^{\theta_n}}(\sigma_n)\big).$$

As $n \to +\infty$, the right-hand side of the above inequality converges to $\sec_{\mathsf{g}}(\sigma)$, which is hence a nonnegative number, concluding the proof. $\square$

By the above, on compact manifolds, $\sec^{0+} > 0$ is an intermediate curvature positivity condition between $\sec > 0$ and $\mathrm{Ric} > 0$, as well as $\sec > 0$ and $\sec \geq 0$. We remark that $\sec^{0+} > 0$ is indeed *intermediate*, since there are compact manifolds M that have $\mathrm{Ric} > 0$ but do not have $\sec^{0+} > 0$; and that have $\sec^{0+} > 0$ but do not have $\sec > 0$. Namely, the connected sum $\#^k(S^n \times S^m)$, $k \in \mathbb{N}$, has $\mathrm{Ric} > 0$ by the work of Sha and Yang [89], and, for k sufficiently large, it does not have $\sec \geq 0$ by the celebrated a priori bounds on Betti numbers of Gromov [41], see also Petersen [75, Thm. 86, p. 357]. Thus, by Proposition 5.10, $\#^k(S^n \times S^m)$ does not satisfy $\sec^{0+} > 0$ for k sufficiently large. Moreover, it is proved in Corollary 6.6 that $\mathbb{R}P^2 \times \mathbb{R}P^2$ satisfies $\sec^{0+} > 0$, and it does not satisfy $\sec > 0$ by Synge's Theorem, see Petersen [75, Thm. 26, p. 172].

5.4 The condition $\sec^{\perp} > 0$

The range of $\theta > 0$ for which $\sec^{\theta}_{\mathsf{g}} > 0$ is a meaningful condition is between 0 and the diameter of the largest fiber of $\mathrm{Gr}_2 TM$, according to the chosen fiberwise distance. In the previous section, we explored the limit $\theta \searrow 0$, and, in this section, we discuss a curvature positivity condition related to the opposite limit. Differently from the previous section, the choice of fiberwise distance function is very important here. As observed in Remark 5.7, 2-planes $\sigma, \sigma' \in \mathrm{Gr}_2(V)$ are at maximal distance $\mathrm{dist}_{SS}(\sigma, \sigma') = \frac{\pi}{\sqrt{2}}$ with respect to the symmetric space distance (5.11) if and only if they are *orthogonal*, that is, $\sigma' \subset \sigma^{\perp}$, while this is not necessarily the case if σ and σ' are at maximal distance with respect to the distances (5.1) or (5.3). For this reason, given a metric g on M, we use the symmetric space distance via (5.12) to define the *biorthogonal curvature* of $\sigma \in \mathrm{Gr}_2(T_pM)$ as $\sec_{\mathsf{g}}^{\perp}(\sigma) := \sec_{\mathsf{g}}^{\theta}(\sigma)$ with $\theta = \frac{\pi}{\sqrt{2}}$, that is,

$$\sec_{\mathsf{g}}^{\perp}(\sigma) := \min_{\substack{\sigma' \in \mathrm{Gr}_2(T_pM) \\ \sigma' \subset \sigma^{\perp}}} \tfrac{1}{2}\big(\sec_{\mathsf{g}}(\sigma) + \sec_{\mathsf{g}}(\sigma')\big). \tag{5.15}$$

Note $\sec_{\mathsf{g}}^{\perp} > 0$ means that, at every point $p \in M$, the average of sectional curvatures of any 2-planes tangent at p that are orthogonal to each other is positive. Clearly, this condition is vacuous[4] if $\dim M \leq 3$, so we henceforth assume $\dim M \geq 4$.

In the setting of Section 2.3, $\sec_{\mathsf{g}}^{\perp} > 0$ is a pointwise curvature condition , corresponding to the open $\mathsf{O}(n)$-invariant convex cone

$$C_{\sec^{\perp}>0} := \left\{ R \in S_{\mathfrak{b}}(\wedge^2 V) : \begin{array}{c} \langle R(\sigma), \sigma\rangle + \langle R(\sigma'), \sigma'\rangle > 0 \\ \text{for all } \sigma, \sigma' \in \mathrm{Gr}_2(V) \text{ such that } \sigma' \subset \sigma^{\perp} \end{array} \right\}, \tag{5.16}$$

which is precisely the cone $C_{\sec^{\theta}>0}$, with $\mathrm{dist}_{\mathrm{Gr}_2(V)} = \mathrm{dist}_{SS}$ and $\theta = \frac{\pi}{\sqrt{2}}$.

Definition 5.11. A manifold M satisfies $\sec^{\perp} > 0$ if there exists a Riemannian metric g on M with $\sec_{\mathsf{g}}^{\perp} > 0$.

Since the conditions $\sec^{\theta} > 0$ become weaker as θ grows, it is natural to expect that the maximal possible θ yields a condition much weaker than $\sec > 0$. Indeed, while $\sec^{0_+} > 0$ is intermediate between $\sec > 0$ and $\mathrm{Ric} > 0$ (see Proposition 5.10), we now verify that $\sec^{\perp} > 0$ is intermediate between $\sec > 0$ and $\mathrm{scal} > 0$.

Proposition 5.12. *If* (M, g) *has* $\sec_{\mathsf{g}} > 0$*, then it also has* $\sec_{\mathsf{g}}^{\perp} > 0$*. If* M *has* $\sec_{\mathsf{g}}^{\perp} > 0$*, then it also has* $\mathrm{scal}_{\mathsf{g}} > 0$*.*

Proof. If a Riemannian metric g on M has $\sec_{\mathsf{g}} > 0$, then by definition (5.15) it clearly has $\sec_{\mathsf{g}}^{\perp} > 0$. Moreover, if the metric g has $\sec_{\mathsf{g}}^{\perp} > 0$, then it also has $\mathrm{scal}_{\mathsf{g}} > 0$,

[4]On 3-manifolds, there are no pairs of orthogonal 2-planes. However, requiring that averages of sectional curvatures of 2-planes at maximal distance (with respect to (5.11)) is positive determines an intermediate condition between $\sec > 0$ and $\mathrm{Ric} > 0$.

since, by (2.10), $\mathrm{scal}_{\mathrm{g}}$ is a sum of sectional curvatures of pairwise orthogonal 2-planes. Alternatively, note that $C_{\sec>0} \subset C_{\sec^\perp>0} \subset C_{\mathrm{scal}>0} := \{R \in S_{\mathfrak{b}}(\wedge^2 V) : \operatorname{tr} R > 0\}$. $\square$

Remark 5.13. There are manifolds that have $\sec^\perp > 0$ but do not have $\mathrm{Ric} > 0$, for example $S^{n-1} \times S^1$. The standard product metric g on $S^{n-1} \times S^1$ has $\sec^\perp_{\mathrm{g}} > 0$, since it has $\sec_{\mathrm{g}} \geq 0$ and the only 2-planes with $\sec_{\mathrm{g}}(\sigma) = 0$ are mixed planes, i.e., those of the form $\sigma = v \wedge \frac{\partial}{\partial\theta}$, where v is tangent to S^{n-1} and $\frac{\partial}{\partial\theta}$ is tangent to S^1. In particular, if one of the two sectional curvatures in the average (5.15) vanishes, then the other is positive, since it is the sectional curvature of a plane orthogonal to $\frac{\partial}{\partial\theta}$, hence tangent to S^{n-1}. On the other hand, $S^{n-1} \times S^1$ does not satisfy $\mathrm{Ric} > 0$ by the Bonnet-Myers Theorem (see Petersen [75, p. 171]), since its fundamental group $\pi_1(S^{n-1} \times S^1) \cong \mathbb{Z}$ is infinite.

In Proposition 7.11, we prove that connected sums of manifolds with $\sec^\perp > 0$ also satisfy $\sec^\perp > 0$. In particular, it follows that $M_k = \#^k(S^{n-1} \times S^1)$, $k \in \mathbb{N}$, satisfy $\sec^\perp > 0$. Since $\pi_1(M_k)$ is the free group in k generators, the class of manifolds that satisfy $\sec^\perp > 0$ is *much larger* than that of manifolds that satisfy $\mathrm{Ric} > 0$. Furthermore, in Chapters 6 and 7, we prove that also $\#^k(S^2 \times S^2)$, $k \in \mathbb{N}$, satisfy $\sec^\perp > 0$, see Remark 7.15.

Riemannian manifolds with $\sec^\perp > 0$, particularly *pinched* biorthogonal curvature, were first studied by Seaman [84–86]. The biorthogonal curvature of (M, g) is said to be *weakly $\frac{1}{4}$-pinched* if there is a nonnegative function $\delta \colon M \to \mathbb{R}_+$ such that

$$\tfrac{\delta}{4} \leq \sec^\perp_{\mathrm{g}}(\sigma) \leq \delta, \qquad \text{for all } \sigma \in \mathrm{Gr}_2 TM, \tag{5.17}$$

and *strictly $\frac{1}{4}$-pinched* if at least one of the above inequalities is strict. The following common feature with manifolds whose sectional curvature is $\frac{1}{4}$-pinched was observed by Seaman [84–86], using the Berger inequalities for the curvature tensor.

Proposition 5.14. *Let* (M, g) *be a Riemannian manifold. If* (M, g) *has weakly* $\frac{1}{4}$*-pinched biorthogonal curvature, then it has nonnegative isotropic curvature. Moreover, if* (M, g) *has strictly* $\frac{1}{4}$*-pinched biorthogonal curvature, then it has positive isotropic curvature.*

Proof. Suppose (M, g) has weakly $\frac{1}{4}$-pinched biorthogonal curvature, i.e., satisfies (5.17). Let $\sigma \in \mathrm{Gr}_2(T_pM)^{\mathbb{C}}$ be an isotropic 2-plane, and $X, Y, Z, W \in T_pM$ be orthonormal vectors such that $\sigma = (X+iY)\wedge(Z+iW)$. From the Berger inequalities for R_{g}, see Seaman [84, Prop. 2.7], [85, Prop. 1.1] or Karcher [56], it follows that

$$\left|\langle R_{\mathrm{g}}(X,Y)Z,W\rangle\right| \leq \tfrac{2}{3}(\delta - \tfrac{\delta}{4}) = \tfrac{\delta}{2}. \tag{5.18}$$

Thus, from (2.12) and (5.18), we have that

$$\begin{aligned} \sec_{\mathrm{g}}^{\mathbb{C}}(\sigma) &= 2\,\sec_{\mathrm{g}}^{\perp}(X\wedge Z) + 2\,\sec_{\mathrm{g}}^{\perp}(X\wedge W) - 2\,\langle R_{\mathrm{g}}(X,Y)Z,W\rangle \\ &\geq \delta - 2\,\langle R_{\mathrm{g}}(X,Y)Z,W\rangle \\ &\geq 0, \end{aligned} \tag{5.19}$$

hence (M, g) has nonnegative isotropic curvature. Similarly, if (M, g) has strictly $\frac{1}{4}$-pinched biorthogonal curvature, then (5.19) is strict and hence (M, g) has positive isotropic curvature. □

The following is a direct consequence of Proposition 5.14 and the recent extensions of the classical work of Micallef and Moore [67] by Brendle and Schoen [20], using the results of Böhm and Wilking [17].

Corollary 5.15. *Let* (M, g) *be a simply-connected Riemannian manifold with strictly* $\frac{1}{4}$*-pinched biorthogonal curvature. Then* M *is diffeomorphic to a sphere.*

Proposition 5.14 was also used by Seaman [84, 85] to study manifolds with weakly $\frac{1}{4}$-pinched biorthogonal curvature. Finally, the following was proved by Seaman [86, Thm. A], and independently by Micallef and Wang [68].

Theorem 5.16. *Let M be an even-dimensional closed orientable manifold such that $b_2(M) \neq 0$. Suppose M admits a Riemannian metric g with nonnegative isotropic curvature and $\sec^{\perp}_{\mathrm{g}} > 0$ at a point.*[5] *Then M is simply-connected, $b_2(M) = 1$ and (M, g) is Kähler.*

5.5 The condition $\sec^{\perp} > 0$ in dimension 4

The condition $\sec^{\perp} > 0$ is particularly interesting in the lowest possible dimension where it is meaningful, $\dim M = 4$. In this case, the orthogonal complement of any 2-plane $\sigma \in \mathrm{Gr}_2(T_pM)$ is another 2-plane $\sigma^{\perp} \in \mathrm{Gr}_2(T_pM)$. Thus, (5.15) is simply:

$$\sec^{\perp}_{\mathrm{g}}(\sigma) = \tfrac{1}{2}\big(\sec_{\mathrm{g}}(\sigma) + \sec_{\mathrm{g}}(\sigma^{\perp})\big). \tag{5.20}$$

A classification (up to homeomorphism) of closed simply-connected 4-manifolds with $\sec^{\perp} > 0$ is given in Chapter 7, see Theorem 7.1. In what follows, we recall previous results in the literature related to 4-manifolds with $\sec^{\perp} > 0$ and prove a new deformation result for 4-manifolds with $\sec \geq 0$ and $\sec^{\perp} > 0$ at a point (Proposition 5.19).

We begin by observing that, from Proposition 5.12, if a 4-manifold M satisfies $\sec^{\perp} > 0$, then M satisfies $\mathrm{scal} > 0$ and hence its Seiberg-Witten invariants vanish, see Moore [70] for details. Let (M, g) be a Riemannian 4-manifold and denote by $*\colon \wedge^2 TM \to \wedge^2 TM$ the *Hodge star operator*. Then $\sigma^{\perp} = *\sigma$ and, since $*$ is self-adjoint, the biorthogonal curvature $\sec^{\perp}$ coincides with the quadratic form

$$\sec^{\perp}_{\mathrm{g}}\colon \mathrm{Gr}_2TM \to \mathbb{R}, \quad \sec^{\perp}_{\mathrm{g}}(\sigma) = \tfrac{1}{2}\langle (R_{\mathrm{g}} + *R_{\mathrm{g}}*)(\sigma), \sigma\rangle, \tag{5.21}$$

cf. (2.7). Consider the decomposition $\wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM$ into self-dual and

[5] That is, $\sec^{\perp}_{\mathrm{g}}(\sigma) > 0$ for all 2-planes $\sigma \subset T_{p_0}M$ tangent to M at a point $p_0 \in M$.

anti-self-dual parts, i.e., into the eigenspaces of $*\colon \wedge^2 TM \to \wedge^2 TM$ with eigenvalues $+1$ and -1, respectively. The curvature operator $R_{\mathtt{g}}\colon \wedge^2 TM \to \wedge^2 TM$ decomposes as

$$R_{\mathtt{g}} = \begin{pmatrix} W^+_{\mathtt{g}} + \frac{\mathrm{scal}_{\mathtt{g}}}{12}\,\mathrm{Id} & \mathrm{Ric}^0_{\mathtt{g}} \\ \left(\mathrm{Ric}^0_{\mathtt{g}}\right)^{\mathtt{t}} & W^-_{\mathtt{g}} + \frac{\mathrm{scal}_{\mathtt{g}}}{12}\,\mathrm{Id} \end{pmatrix}, \tag{5.22}$$

where $W^{\pm}_{\mathtt{g}}$ are self-dual and anti-self-dual parts of the Weyl tensor $W_{\mathtt{g}}$, and $\mathrm{Ric}^0_{\mathtt{g}} := \mathrm{Ric}_{\mathtt{g}} - \frac{\mathrm{scal}_{\mathtt{g}}}{4}\mathtt{g}$ is the trace-free Ricci tensor, identified as $\mathrm{Ric}^0_{\mathtt{g}}\colon \wedge^2_- TM \to \wedge^2_+ TM$, see Besse [12, p. 51] for details. We recall the following result that follows from the above decomposition (5.22), due to Singer and Thorpe [91, Thm. 1.3, 1.4].

Proposition 5.17. *Let $(M^4, \mathtt{g})$ be a 4-manifold. The following are equivalent:*

(i) $(M^4, \mathtt{g})$ *is Einstein, that is,* $\mathrm{Ric}_{\mathtt{g}} = \lambda\,\mathtt{g}$*;*

(ii) *The curvature operator of* $(M^4, \mathtt{g})$ *commutes with* $*$*, that is,* $*R_{\mathtt{g}} = R_{\mathtt{g}}*$*;*

(iii) $\sec_{\mathtt{g}}(\sigma) = \sec^{\perp}_{\mathtt{g}}(\sigma)$ *for all* $\sigma \in \mathrm{Gr}_2 TM$*.*

Proof (Sketch). In order to prove that (i) and (ii) are equivalent, note that $\mathrm{Ric}_{\mathtt{g}} = \lambda\,\mathtt{g}$ if and only if $\mathrm{Ric}^0_{\mathtt{g}} = 0$, which is equivalent to $R_{\mathtt{g}}$ and $*$ commuting by (5.22). If (ii) holds, then $R_{\mathtt{g}} + *R_{\mathtt{g}}* = 2R_{\mathtt{g}}$ and hence (iii) follows from (5.21). Conversely, if (iii) holds, then $R_{\mathtt{g}} - *R_{\mathtt{g}}*$ has identically zero sectional curvature and satisfies $\mathfrak{b}(R_{\mathtt{g}} - *R_{\mathtt{g}}*) = 0$, from which $R_{\mathtt{g}} - *R_{\mathtt{g}}* = 0$ (see Lemma 8.1), hence (ii) holds. □

In particular, Proposition 5.17 shows that the problem of classifying Einstein 4-manifolds with $\sec > 0$ is precisely the same as classifying Einstein 4-manifolds with $\sec^{\perp} > 0$. We recall that both the Hopf Problem I and the Local Hopf Problem I regarding $S^2 \times S^2$ are open even when restricted to Einstein metrics.[6]

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[6]Partial results in this direction have been obtained by Gursky and LeBrun [48] and Yang [109]. The former implies that a closed oriented Einstein 4-manifold with $\sec \geq 0$ (or $\sec^{\perp} \geq 0$) and *definite intersection form* (see Section 7.1 for details) is isometric to $\mathbb{C}P^2$ with a multiple of its standard metric. The latter implies that an Einstein 4-manifold with $\sec_{\mathtt{g}} \geq 0$ (or $\sec^{\perp}_{\mathtt{g}} \geq 0$), $\mathrm{Ric}_{\mathtt{g}} = \mathtt{g}$, and *sufficiently large* sectional curvature ($\sec_{\mathtt{g}} \geq \varepsilon_0 \cong 0.1$) is isometric to either S^4, $\mathbb{R}P^4$, or $\mathbb{C}P^2$ with a multiple of its standard metric.

Note that, in terms of the decomposition $\wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM$, the operator $\frac{1}{2}(R_{\mathrm{g}} + *R_{\mathrm{g}}*)$ whose quadratic form restricted to $\mathrm{Gr}_2 TM$ is $\sec^\perp_{\mathrm{g}}$, see (5.21), reads

$$\frac{1}{2}(R_{\mathrm{g}} + *R_{\mathrm{g}}*) = \begin{pmatrix} W^+_{\mathrm{g}} + \frac{\mathrm{scal}_{\mathrm{g}}}{12}\,\mathrm{Id} & 0 \\ 0 & W^-_{\mathrm{g}} + \frac{\mathrm{scal}_{\mathrm{g}}}{12}\,\mathrm{Id} \end{pmatrix}. \tag{5.23}$$

Since $\mathfrak{b}(R_{\mathrm{g}}) = 0$, we have $0 = \langle R_{\mathrm{g}}, *\rangle = \mathrm{tr}(R_{\mathrm{g}}\, *) = \mathrm{tr}(*R_{\mathrm{g}} * *) = \langle *R_{\mathrm{g}}*, *\rangle$, so also $\mathfrak{b}(*R_{\mathrm{g}}*) = 0$, and hence $\frac{1}{2}(R_{\mathrm{g}} + *R_{\mathrm{g}}*)$ is an algebraic curvature operator, see Sections 2.3 and 8.2 for details. Furthermore, it is an *Einstein* algebraic curvature operator, since it clearly commutes with $*$.

Recently, 4-manifolds with $\sec^\perp > 0$ have been studied by Bettiol [13], Costa [25], Costa, Diógenes and Ribeiro [26], and Costa and Ribeiro [27]. A few remarks related to the Hopf Problem I are made in Costa [25], where it is asked whether $S^2 \times S^2$ satisfies $\sec^\perp > 0$. This question was answered affirmatively in Bettiol [13], see also Theorem 6.1 and Remark 6.3. Costa and Ribeiro [27, Thm. 1] proved the following:

Theorem 5.18. *Let (M, g) be a closed simply-connected 4-manifold such that*

$$\sec^\perp_{\mathrm{g}} \geq \frac{\mathrm{scal}_{\mathrm{g}}}{24} > 0. \tag{5.24}$$

Then (M, g) has nonnegative isotropic curvature, and is diffeomorphic to S^4 or $\mathbb{C}P^2$.

Proof (Sketch). Decomposing $\sigma \in \mathrm{Gr}_2 TM$ as $\sigma = \sigma_+ + \sigma_- \in \wedge^2_+ TM \oplus \wedge^2_- TM$, where $\|\sigma_\pm\|^2 = \frac{1}{2}$, cf. (6.5), we have from (5.21) and (5.23) that[7]

$$\sec^\perp_{\mathrm{g}}(\sigma) = \frac{\mathrm{scal}_{\mathrm{g}}}{12} + \langle W^+_{\mathrm{g}}(\sigma_+), \sigma_+\rangle + \langle W^-_{\mathrm{g}}(\sigma_-), \sigma_-\rangle. \tag{5.25}$$

Thus, from (5.24) and (5.25) it follows that $\frac{\mathrm{scal}_{\mathrm{g}}}{6}\,\mathrm{Id} - W_{\mathrm{g}}$ is positive-semidefinite. This is well-known to imply that (M, g) has nonnegative isotropic curvature, see

[7] In particular, note that $\sec^\perp_{\mathrm{g}} - \frac{\mathrm{scal}_{\mathrm{g}}}{12}$ is a conformally invariant quantity.

Micallef and Moore [67, p. 201]. If $(M, \mathtt{g})$ has positive isotropic curvature, then M is diffeomorphic to S^4 by [67]. Else, by the work of Seshadri [88], either $(M, \mathtt{g})$ is locally symmetric or Kähler. Using (5.24) and results of Derdziński [29] and Micallef and Wang [68], the conclusion that M is diffeomorphic to S^4 or $\mathbb{C}P^2$ follows. □

We conclude this section with a new deformation result for closed 4-manifolds with $\sec \geq 0$ and $\sec^\perp > 0$ at a point, illustrating some of the techniques discussed in Chapter 3.

Proposition 5.19. *If* $(M^4, \mathtt{g})$ *is a closed* 4*-manifold with* $\sec_{\mathtt{g}} \geq 0$ *and* $\sec^\perp_{\mathtt{g}} > 0$ *at a point* $p_0 \in M$*, then* M *satisfies* $\sec^\perp > 0$*.*

Proof. In order to prove that M admits Riemannian metrics with $\sec^\perp > 0$, we exhibit a first-order conformal deformation $\mathtt{g}_s$ of $\mathtt{g}$ such that $\sec^\perp_{\mathtt{g}_s} > 0$ for all $s > 0$ sufficiently small. In particular, the metrics with $\sec^\perp > 0$ (on all of M) can be chosen arbitrarily close to $\mathtt{g}$, and in the same conformal class.

Denote by $\pi\colon \mathrm{Gr}_2 TM \to M$ the projection of the Grassmannian bundle of 2-planes tangent to M, so that $\pi^{-1}(p) = \mathrm{Gr}_2(T_pM)$ for all $p \in M$. By continuity of $\sec^\perp_{\mathtt{g}}\colon \mathrm{Gr}_2 TM \to \mathbb{R}$ and compactness of $\pi^{-1}(p_0)$, there exist $\varepsilon > 0$ and an open neighborhood U of $p_0 \in M$, such that $\sec^\perp_{\mathtt{g}}(\sigma) \geq \varepsilon$ for all $\sigma \in \pi^{-1}(U)$. Let $\psi\colon M \to \mathbb{R}$ be a smooth function such that $\psi(x) = 1$ for all $x \in M \backslash U$ and $\int_M \psi = 0$. By standard elliptic PDE results, see, e.g., Aubin [4, Thm. 4.7], there exists a smooth function $\phi\colon M \to \mathbb{R}$ such that $\Delta\phi = \psi$. Consider the first-order conformal deformation $\mathtt{g}_s := \mathtt{g} + s\,\mathtt{h}$, with $\mathtt{h} := \phi\,\mathtt{g}$, and set $f\colon [0, S] \times \mathrm{Gr}_2 TM \to \mathbb{R}$, $f(s, \sigma) := \sec^\perp_{\mathtt{g}_s}(\sigma)$.

Since $\sec_{\mathtt{g}} \geq 0$, we have that $f(0, \sigma) \geq 0$ for all $\sigma \in \mathrm{Gr}_2 TM$. Furthermore, if $\sigma \in \mathrm{Gr}_2(T_pM)$ is such that $f(0, \sigma) = 0$, then $\sigma \in \mathrm{Gr}_2(T_pM)$ for some $p \in M \setminus U$. Let $\{e_i\}$ be a $\mathtt{g}$-orthonormal basis of T_pM, so that $\sigma = e_1 \wedge e_2$ and $\sigma^\perp = e_3 \wedge e_4$. From

(5.20) and Corollary 3.4, we have that

$$\begin{aligned} \frac{\partial f}{\partial s}(0,\sigma) &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}\big(\sec_{\mathtt{g}_s}(e_1\wedge e_2)+\sec_{\mathtt{g}_s}(e_3\wedge e_4)\big)\Big|_{s=0} \\ &= -\frac{1}{4}\sum_{i=1}^{4}\operatorname{Hess}\phi(e_i,e_i) \\ &= \tfrac{1}{4}\,(\Delta\phi)(p) \\ &= \tfrac{1}{4}\,\psi(p) \\ &= \tfrac{1}{4}. \end{aligned} \tag{5.26}$$

Therefore, by Lemma 3.5, there exists $s_*>0$ such that $f(s,\sigma)=\sec^{\perp}_{\mathtt{g}_s}(\sigma)>0$ for all $\sigma\in \mathrm{Gr}_2TM$ and $0<s<s_*$, concluding the proof. □

Remark 5.20. The above technique can also be used to prove that a closed manifold M (of any dimension) satisfies $\sec>0$ if it has a Riemannian metric $\mathtt{g}$ with *quasi-positive curvature* (that is, $\sec_{\mathtt{g}}\geq 0$ and $\sec_{\mathtt{g}}>0$ at a point $p_0\in M$), and all flat planes $\sec_{\mathtt{g}}^{-1}(0)\subset \mathrm{Gr}_2TM$ are contained in a fixed nonholonomic rank 2 distribution[8] $\mathcal{D}$ on M. Similarly to Proposition 5.19, there exist $\varepsilon>0$ and an open neighborhood U of $p_0\in M$, such that $\sec_{\mathtt{g}}(\sigma)\geq\varepsilon$ for all $\sigma\in\pi^{-1}(U)$. Let $\psi\colon M\to\mathbb{R}$ be a smooth function with $\psi(x)=1$ for all $x\in M\setminus U$ and $\int_M\psi=0$. Since $\mathcal{D}$ is nonholonomic, there exists a smooth function $\phi\colon M\to\mathbb{R}$ such that $\Delta_{\mathcal{D}}\phi=\psi$, where $\Delta_{\mathcal{D}}$ is the *sub-Laplacian* associated with the distribution $\mathcal{D}$, see Khesin and Lee [59, Prop. 2.7]. Setting $\mathtt{g}_s:=\mathtt{g}+s\,\mathtt{h}$, with $\mathtt{h}:=\phi\,\mathtt{g}$, we have by Corollary 3.4 that $\frac{\mathrm{d}}{\mathrm{d}s}\sec_{\mathtt{g}_s}(\sigma)\big|_{s=0}=\frac{1}{2}(\Delta_{\mathcal{D}}\phi)(p)=\frac{1}{2}\,\psi(p)=\frac{1}{2}$ whenever $\sigma\in\mathrm{Gr}_2(T_pM)$ satisfies $\sec_{\mathtt{g}}(\sigma)=0$, since we are assuming this implies $\sigma=\mathcal{D}_p$. Thus, by Corollary 3.6, there exists $s_*>0$ such that $\sec_{\mathtt{g}_s}>0$ for all $0<s<s_*$.

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[8]A rank 2 distribution $\mathcal{D}$ on M is a rank 2 subbundle of TM, and it is said to be *nonholonomic*, or *bracket-generating*, if local vector fields tangent to $\mathcal{D}$ and their iterated Lie brackets span TM.

CHAPTER 6

IMPROVING THE CURVATURE OF $S^2 \times S^2$

The goal of this chapter is to prove the following result, from Bettiol [13].

Theorem 6.1. *The manifold $S^2 \times S^2$ satisfies* $\sec^{0+} > 0$.

There are two main sources of motivation for the above result. The first is that it plays a crucial role in the classification of closed simply-connected 4-manifolds that satisfy $\sec^{\perp} > 0$, discussed in Chapter 7. The second is its connection with the Hopf Problem I, since, by Proposition 5.10, it implies that $S^2 \times S^2$ satisfies an intermediate curvature condition between $\sec > 0$ and $\sec \geq 0$, as well as between $\sec > 0$ and $\mathrm{Ric} > 0$. It is also related to the Local Hopf Problem I, since the metrics g^θ with $\sec^\theta_{\mathsf{g}^\theta} > 0$ that imply the $\sec^{0+} > 0$ condition can be constructed arbitrarily close to the standard product metric g_0 as $\theta \searrow 0$.

Let us give a brief outline of how Theorem 6.1 is proved, which also informs the organization of this chapter. After recalling basic properties of g_0 in Section 6.1, we proceed to the construction of the metrics g^θ with $\sec^\theta_{\mathsf{g}^\theta} > 0$, which is done in two steps. First, in Section 6.2, we perform a Cheeger deformation of g_0 with respect to the diagonal $\mathsf{SO}(3)$-action (6.6). This produces a 1-parameter family g_t of metrics with $\sec_{\mathsf{g}_t} \geq 0$ and $\sec^\theta_{\mathsf{g}_t} > 0$ on an open and dense subset (given by the complement of two submanifolds), for any $\theta > 0$ and $t > 0$, see Proposition 6.2. Second, in Section 6.3, we perform a first-order local conformal deformation of g_t supported near these submanifolds, which yields the desired metrics $\mathsf{g}_{t,s}$ that have $\sec^\theta_{\mathsf{g}_{t,s}} > 0$ if $s > 0$ is sufficiently small and $t > 0$, see Proposition 6.5. Finally, a few comments regarding this construction are made in Section 6.4.

6.1 Standard product metric

Let us briefly recall some properties of the standard product metric g_0 on $S^2 \times S^2$. The manifold $(S^2 \times S^2, \mathsf{g}_0)$ can be isometrically embedded into $\mathbb{R}^3 \oplus \mathbb{R}^3$ as the product of unit spheres, hence this is a homogeneous space (actually, a symmetric space) and the identity connected component of its isometry group is $\mathsf{SO}(3) \times \mathsf{SO}(3)$. As described in Section 2.5, given $p = (p_1, p_2) \in S^2 \times S^2$, we have that $T_p(S^2 \times S^2) = T_{p_1} S^2 \oplus T_{p_2} S^2$, and for each $X \in T_p(S^2 \times S^2)$, we write $X = (X_1, X_2)$, where $X_i \in T_{p_i} S^2$. The product metric g_0 is given by

$$\mathsf{g}_0(X, Y) = \langle X_1, Y_1 \rangle + \langle X_2, Y_2 \rangle, \tag{6.1}$$

where $X_i, Y_i \in T_{p_i} S^2 \cong \{p_i\}^\perp \subset \mathbb{R}^3$ and $\langle \cdot, \cdot \rangle$ is the standard metric in $\mathbb{R}^3$. Routine computations, see (2.18), show that the curvature operator of g_0 is the positive-semidefinite operator $R_0 \colon \wedge^2 T(S^2 \times S^2) \to \wedge^2 T(S^2 \times S^2)$ given by

$$\mathsf{g}_0(R_0(X \wedge Y), Z \wedge W) = \langle X_1 \wedge Y_1, Z_1 \wedge W_1 \rangle + \langle X_2 \wedge Y_2, Z_2 \wedge W_2 \rangle. \tag{6.2}$$

In particular, the sectional curvature of $(S^2 \times S^2, \mathsf{g}_0)$ is given by, see (2.19),

$$\sec_{\mathsf{g}_0}(X \wedge Y) = \|X_1 \wedge Y_1\|^2 + \|X_2 \wedge Y_2\|^2. \tag{6.3}$$

Thus, $\sec_{\mathsf{g}} \geq 0$ and $\sec_{\mathsf{g}}(X \wedge Y) = 0$ if and only if $X \wedge Y$ is a mixed plane, i.e., $X_1 = 0$ and $Y_2 = 0$, or $X_2 = 0$ and $Y_1 = 0$. Therefore, for each $p \in S^2 \times S^2$, the 2-torus given by the product of the unit spheres in $T_{p_i} S^2$,

$$T := \big\{ (v, 0) \wedge (0, w) \in \mathrm{Gr}_2(T_p(S^2 \times S^2)) : v \in S_{p_1} S^2, w \in S_{p_2} S^2 \big\}, \tag{6.4}$$

consists of flat planes, i.e., $\sec_{\mathsf{g}_0} \colon \mathrm{Gr}_2(T_p(S^2 \times S^2)) \to \mathbb{R}$ vanishes on the submanifold T and is positive everywhere else.

In order to give another description of the above, see Viaclovsky [101, §11], recall that as $\dim T_p(S^2 \times S^2) = 4$, we have $\mathrm{Gr}_2(T_p(S^2\times S^2)) \cong \mathrm{Gr}_2(\mathbb{R}^4) \cong S^2\times S^2$. Indeed, consider the decomposition $\wedge^2\mathbb{R}^4 = \wedge^2_+\mathbb{R}^4 \oplus \wedge^2_-\mathbb{R}^4$ into self-dual and anti-self-dual parts, i.e., into the eigenspaces of the Hodge star operator $*\colon \wedge^2\mathbb{R}^4 \to \wedge^2\mathbb{R}^4$ with eigenvalues $+1$ and -1, respectively. Since $\alpha \in \wedge^2\mathbb{R}^4$ is decomposable if and only if $\alpha\wedge\alpha = 0$, writing $\alpha = \alpha_+ + \alpha_- \in \wedge^2_+\mathbb{R}^4 \oplus \wedge^2_-\mathbb{R}^4$,

$$0 = \alpha\wedge\alpha = \langle \alpha, *\alpha\rangle \,\mathrm{vol} = \langle \alpha_+ + \alpha_-, \alpha_+ - \alpha_-\rangle = (\|\alpha_+\|^2 - \|\alpha_-\|^2)\,\mathrm{vol},$$

and $\|\alpha\|^2 = \|\alpha_+\|^2 + \|\alpha_-\|^2$. Thus, the Grassmannian of 2-planes in $\mathbb{R}^4$ is given by

$$\begin{aligned}\mathrm{Gr}_2(\mathbb{R}^4) &= \big\{\alpha\in\wedge^2\mathbb{R}^4 : \alpha\wedge\alpha = 0, \|\alpha\| = 1\big\}\\ &= \big\{(\alpha_+,\alpha_-) \in \wedge^2_+\mathbb{R}^4\oplus\wedge^2_-\mathbb{R}^4 : \|\alpha_+\|^2 = \|\alpha_-\|^2 = \tfrac{1}{2}\big\},\end{aligned} \tag{6.5}$$

which is clearly diffeomorphic to $S^2\times S^2$, since $\dim\wedge^2_\pm\mathbb{R}^4 = 3$. Under this identification, the sectional curvature function $\sec_{\mathrm{g}_0}\colon \mathrm{Gr}_2(T_p(S^2\times S^2)) \to \mathbb{R}$ is, see (6.3),

$$\sec_{\mathrm{g}_0}(\alpha) = \mathrm{g}_0\big(R^0(\alpha),\alpha\big) = \tfrac{1}{2}\langle\alpha_+,\omega_+\rangle^2 + \tfrac{1}{2}\langle\alpha_-,\omega_-\rangle^2,$$

for $\alpha = (\alpha_+,\alpha_-) \in \mathrm{Gr}_2(T_p(S^2\times S^2))$, where $\omega_\pm := \mathrm{vol}_1 \pm \mathrm{vol}_2 \in \wedge^2_\pm(T_{p_1}S^2 \oplus T_{p_2}S^2) \cong \wedge^2_\pm\mathbb{R}^4$ and vol_i are the volume forms of each sphere factor.[1] In particular, its zero locus T, see (6.4), is the subset of 2-planes $\alpha = (\alpha_+,\alpha_-)$ such that $\langle\alpha_+,\omega_+\rangle = 0$ and $\langle\alpha_-,\omega_-\rangle = 0$. In other words, $T\cong S^1\times S^1$ is the 2-torus given by the product of the equators in $\mathrm{Gr}_2(T_p(S^2\times S^2)) \cong S^2\times S^2$, see (6.5).

From the above, for all $\theta > 0$, $p\in S^2\times S^2$, and $\sigma\in\mathrm{Gr}_2(T_p(S^2\times S^2))$, we have that $\sec^\theta_{\mathrm{g}_0}(\sigma) \geq 0$ and the equality $\sec^\theta_{\mathrm{g}_0}(\sigma) = 0$ holds if and only if σ is a mixed

[1] Moreover, the curvature operator $R_0\colon \wedge^2 T(S^2\times S^2) \to \wedge^2 T(S^2\times S^2)$ given by (6.2) is the identity on $\mathrm{span}\{\omega_+,\omega_-\} = \mathrm{span}\{\mathrm{vol}_1,\mathrm{vol}_2\}$ and vanishes identically on its orthogonal complement.

plane, i.e., $\sigma \in T$. In fact, if $\sigma \in T$, then any neighborhood of σ in $\mathrm{Gr}_2(T_p(S^2 \times S^2))$ contains other mixed planes $\sigma' \in T$. Furthermore, also $\sec^{\perp}_{\mathrm{g}_0}(\sigma) \geq 0$ and $\sec^{\perp}_{\mathrm{g}_0}(\sigma) = 0$ if and only if $\sigma \in T$, since $\sigma \in T$ if and only if $\sigma^{\perp} \in T$.

Finally, note that taking a trace of (6.2), we have $\mathrm{Ric}_{\mathrm{g}_0} = \mathrm{g}_0$, that is, g_0 is an Einstein metric; and, in particular, it has $\mathrm{Ric}_{\mathrm{g}_0} > 0$.

6.2 First step

The first step to produce the desired metrics on $S^2 \times S^2$ is to perform a Cheeger deformation with respect to the diagonal $\mathsf{SO}(3)$-action, given by

$$A \cdot (p_1, p_2) = (A\,p_1, A\,p_2), \quad A \in \mathsf{SO}(3). \tag{6.6}$$

By (6.1), this is clearly an isometric action. The isotropy group of $p = (p_1, p_2)$ is trivial if $p_1 \neq \pm p_2$, since there are no linear isometries $A \in \mathsf{SO}(3)$ of $\mathbb{R}^3$ that fix two linearly independent directions. Otherwise, if $p_1 = \pm p_2$, then the isotropy group of p consists of linear isometries $A \in \mathsf{SO}(3)$ of $\mathbb{R}^3$ that fix the line spanned by p_1 and p_2, which is hence isomorphic to $\mathsf{SO}(2)$. Thus, the principal orbits of (6.6) are hypersurfaces in $S^2 \times S^2$ diffeomorphic to $\mathbb{R}P^3$ and the only nonprincipal orbits are the singular orbits given by the *diagonal* and *anti-diagonal* submanifolds

$$\pm\Delta S^2 := \big\{(p_1, \pm p_1) \in S^2 \times S^2 : p_1 \in S^2\big\},$$

which are diffeomorphic to S^2. Consider the geodesic segment given by

$$\gamma\colon \big[0, \tfrac{\pi}{2}\big] \to S^2 \times S^2, \quad \gamma(r) = \big((\cos r)e_1 + (\sin r)e_2,\, (\cos r)e_1 - (\sin r)e_2\big),$$

where $\{e_i\}$ is the standard basis in $\mathbb{R}^3$. Clearly, $\gamma(0) \in \Delta S^2$ and $\gamma(\frac{\pi}{2}) \in -\Delta S^2$, and $\gamma(r)$ intersects all orbits of (6.6). Thus, (6.6) is a *cohomogeneity one action*, whose orbit space is isometric to $\left[0, \frac{\pi}{2}\right]$. The geodesic $\gamma(r)$ is *horizontal*, i.e., intersects all orbits perpendicularly, and is hence a *section* for this *polar action*.

The evolution of $\sec^\theta$ on $S^2 \times S^2$ along the Cheeger deformation of g_0 with respect to this action (6.6) is described in the following result, which follows from Müter [72, Satz 4.26], see also Ziller [111, p. 5] and Kerin [58, Rem. 4.3].

Proposition 6.2. *Let g_t be the Cheeger deformation of g_0 with respect to (6.6). Then, $\sec^\theta_{\mathrm{g}_t}(\sigma) \geq 0$ for all $\theta > 0$, $t \geq 0$ and $\sigma \in \mathrm{Gr}_2(T(S^2 \times S^2))$. Moreover, for $t > 0$, the equality $\sec^\theta_{\mathrm{g}_t}(\sigma) = 0$ holds only if $\sigma \in \mathrm{Gr}_2(T_p(S^2 \times S^2))$ for some $p = (p_1, \pm p_1) \in \pm\Delta S^2$, and, in this case, σ is not tangent to the submanifold $\pm\Delta S^2$. In particular, $\sec^\theta_{\mathrm{g}_t} > 0$ on an open and dense subset for any $\theta > 0$ and $t > 0$.*

Proof. From Proposition 4.5, for all $t \geq 0$, we have that $\sec_{\mathrm{g}_t} \geq 0$ and hence $\sec^\theta_{\mathrm{g}_t} \geq 0$ for all $\theta > 0$ by definition (5.13).

In order to analyze $\sec^{-1}_{\mathrm{g}_t}(0)$, we use the results and notation from Chapter 4. Identify the Lie algebra of $\mathsf{SO}(3)$ with $(\mathbb{R}^3, \wedge)$, where $\wedge \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the *cross product*, via

$$\mathfrak{so}(3) \ni Z = \begin{pmatrix} 0 & -z_3 & z_2 \\ z_3 & 0 & -z_1 \\ -z_2 & z_1 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = z \in \mathbb{R}^3. \tag{6.7}$$

Considering $(\mathfrak{so}(3), Q)$ endowed with the standard bi-invariant metric, the above is an isometric identification with Euclidean space $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$. Since the Lie exponential in $\mathsf{SO}(3)$ is given by matrix exponentiation, the action field induced by $Z \in \mathfrak{so}(3)$ is:

$$Z^*_p = \left(Z^*_{p_1}, Z^*_{p_2}\right) = (Z\, p_1, Z\, p_2) = (z \wedge p_1, z \wedge p_2) \in T_p(S^2 \times S^2), \tag{6.8}$$

see (4.4). Recall that the vertical space is $\mathcal{V}_p = \left\{\left(X^*_{p_1}, X^*_{p_2}\right) : X \in \mathfrak{m}_p\right\}$, and the horizontal space $\mathcal{H}_p$ is its orthogonal complement, see (4.5). From (6.1) and (6.8), if

$x, y \in \mathbb{R}^3$ satisfy $\langle x, p_1\rangle = \langle y, p_2\rangle = 0$, then for all $z \in \mathbb{R}^3$,

$$\begin{aligned} \mathsf{g}_0\big((X^*_{p_1}, Y^*_{p_2}), Z^*_p\big) &= \langle x \wedge p_1, z \wedge p_1\rangle + \langle y \wedge p_2, z \wedge p_2\rangle \\ &= \langle p_1 \wedge (x \wedge p_1), z\rangle + \langle p_2 \wedge (y \wedge p_2), z\rangle \\ &= \big\langle \langle p_1, p_1\rangle x - \langle x, p_1\rangle p_1, z\big\rangle + \big\langle \langle p_2, p_2\rangle y - \langle y, p_2\rangle p_2, z\big\rangle \\ &= \langle x + y, z\rangle. \end{aligned} \tag{6.9}$$

Therefore, $\big(X^*_{p_1}, -X^*_{p_2}\big) \in T_p(S^2 \times S^2)$ is horizontal if $x \in \{p_1, p_2\}^\perp := \{x \in \mathbb{R}^3 : \langle x, p_1\rangle = \langle x, p_2\rangle = 0\}$. By dimensional reasons, it follows that the horizontal space is

$$\mathcal{H}_p = \big\{\big(X^*_{p_1}, -X^*_{p_2}\big) \in T_p(S^2 \times S^2) : x \in \{p_1, p_2\}^\perp\big\}.$$

Notice that $\dim \mathcal{H}_p = \dim\{p_1, p_2\}^\perp$ is either equal to 1 or 2, according respectively to $p \notin \pm\Delta S^2$ or $p \in \pm\Delta S^2$. For any $x, y \in \mathbb{R}^3$, analogously to (6.9), we have:

$$\begin{aligned} \mathsf{g}_0\big(X^*_p, Y^*_p\big) &= \mathsf{g}_0\big(\big(X^*_{p_1}, X^*_{p_2}\big), \big(Y^*_{p_1}, Y^*_{p_2}\big)\big) \\ &= \langle x \wedge p_1, y \wedge p_1\rangle + \langle x \wedge p_2, y \wedge p_2\rangle \\ &= \langle p_1 \wedge (x \wedge p_1), y\rangle + \langle p_2 \wedge (x \wedge p_2), y\rangle \\ &= \big\langle (2x - \langle x, p_1\rangle p_1 - \langle x, p_2\rangle p_2), y\big\rangle. \end{aligned}$$

By the definition (4.6) of $P_0 \colon \mathfrak{m}_p \to \mathfrak{m}_p$, we have that the above is $\langle P_0\, X, Y\rangle$, hence

$$P_0\, X = 2X - \langle X, p_1\rangle p_1 - \langle X, p_2\rangle p_2.$$

In particular, it follows that the subspace $\{p_1, p_2\}^\perp \subset \mathfrak{m}_p$ is invariant under P_0 and hence under P_t, see Proposition 4.2.

Let $\pi \colon \{p_1, p_2\}^\perp \to \{p_1, p_2\}^\perp / \sim$ be the projection onto the corresponding real projective space. Note that $\pi(\{p_1, p_2\}^\perp)$ is diffeomorphic to either $\mathbb{R}P^0 \cong \{1\}$ or

$\mathbb{R}P^1 \cong S^1$, according respectively to $p \notin \pm\Delta S^2$ or $p \in \pm\Delta S^2$. For each $x \in \{p_1, p_2\}^\perp$, define $\sigma_{\pi(x)} \in \mathrm{Gr}_2(T_p(S^2 \times S^2))$ as the mixed plane

$$\sigma_{\pi(x)} := \big(X^*_{p_1}, 0\big) \wedge \big(0, X^*_{p_2}\big) = \big(X^*_{p_1}, X^*_{p_2}\big) \wedge \big(X^*_{p_1}, -X^*_{p_2}\big). \tag{6.10}$$

Clearly, this is the unique mixed plane at p that contains the horizontal vector $\big(X^*_{p_1}, -X^*_{p_2}\big)$. Furthermore, since $\{p_1, p_2\}^\perp$ is invariant under P_t, these planes are fixed under the Cheeger reparametrization, i.e.,

$$C_t^{-1}(\sigma_{\pi(x)}) = \sigma_{\pi(x)}, \qquad \text{for all } t \geq 0. \tag{6.11}$$

Thus, by Proposition 4.11, we have that $\sec_{\mathrm{g}_t}(\sigma_{\pi(x)}) = 0$ for all $t \geq 0$, since $\sigma_{\pi(x)}$, such as any other mixed plane in $(S^2 \times S^2, \mathrm{g}_0)$, is tangent to a totally geodesic flat torus. Furthermore, by Corollary 4.8, these are the only g_0-flat planes that remain g_t-flat for $t > 0$, since all the other mixed planes have 2-dimensional vertical projection.

Therefore, for any $t > 0$, we have that $\sigma \in \mathrm{Gr}_2(T_p(S^2 \times S^2))$ satisfies $\sec_{\mathrm{g}_t}(\sigma) = 0$ if and only if $\sigma = \sigma_{\pi(x)}$ for some $x \in \{p_1, p_2\}^\perp$. The set of flat planes in $T_p(S^2 \times S^2)$,

$$\sec_{\mathrm{g}_t}^{-1}(0) = \big\{\sigma_{\pi(x)} \in \mathrm{Gr}_2(T_p(S^2 \times S^2)) : x \in \{p_1, p_2\}^\perp\big\}, \tag{6.12}$$

is parametrized by $\pi(\{p_1, p_2\}^\perp)$ and hence consists of a unique plane if $p \notin \pm\Delta S^2$, and of a circle's worth of planes if $p \in \pm\Delta S^2$. Thus, by definition (5.13), for all $\theta > 0$ and $t > 0$, a plane $\sigma \in \mathrm{Gr}_2(T_p(S^2 \times S^2))$ can only have $\sec^\theta_{\mathrm{g}_t}(\sigma) = 0$ if $p \in \pm\Delta S^2$, in which case σ is not tangent to $\pm\Delta S^2$, as it contains a horizontal vector, see (6.10). □

Remark 6.3. By the above Proposition 6.2, the manifolds $(S^2 \times S^2, \mathrm{g}_t)$ have $\sec^\perp_{\mathrm{g}_t} \geq 0$ and $\sec^\perp_{\mathrm{g}_t} > 0$ on the open and dense subset $S^2 \times S^2 \setminus \{\pm\Delta S^2\}$. Furthermore, if $p \in \pm\Delta S^2$, the set of flat planes (6.12) at p contains pairs of orthogonal planes, hence g_t does not have $\sec^\perp_{\mathrm{g}_t} > 0$ globally. However, in order to prove that $S^2 \times S^2$ satisfies

$\sec^{\perp} > 0$, one can use Proposition 5.19 to deform g_t into metrics with $\sec^{\perp} > 0$. In the next section, we perform an analogous but slightly more careful first-order deformation to obtain the stronger conclusion that $S^2 \times S^2$ satisfies $\sec^{0+} > 0$.

Remark 6.4. For $n \geq 3$, there exists a diagonal $\mathsf{SO}(n+1)$-action on $S^n \times S^n$ of cohomogeneity one, analogous to (6.6). Nevertheless, since $\mathsf{SO}(n+1)$ has $\sec_Q \geq 0$ but does not have $\sec_Q > 0$ when $n \geq 3$, the Cheeger deformation of the standard product metric on $S^n \times S^n$ fails to destroy so many flat planes, cf. Remark 4.9. As a result, this step in the construction of metrics with $\sec^{\theta} > 0$ only works if $n = 2$.

6.3 Second step

The second step in the construction of the desired metrics on $S^2 \times S^2$ is to perform a first-order local conformal deformation on a Cheeger deformed metric g_t with $t > 0$.

Proposition 6.5. *Let* g_t *be the Cheeger deformation of* g_0 *with respect to* (6.6)*, and fix* $t > 0$*. There exists a smooth function* $\phi \colon S^2 \times S^2 \to \mathbb{R}$ *supported in a neighborhood of* $\pm\Delta S^2$*, such that for each* $\theta > 0$*, there exists* $s_* > 0$*, such that* $\mathsf{g}_{t,s} := (1 + s\,\phi)\,\mathsf{g}_t$ *satisfies* $\sec^{\theta}_{\mathsf{g}_{t,s}} > 0$ *if* $0 < s < s_*$*.*

Proof. Denote by $\mathrm{dist}_{\mathsf{g}_t}$ the distance function on $(S^2 \times S^2, \mathsf{g}_t)$, and define the functions

$$\psi_{\pm} \colon S^2 \times S^2 \to \mathbb{R}, \quad \psi_{\pm}(x) := \mathrm{dist}_{\mathsf{g}_t}(x, \pm\Delta S^2)^2. \tag{6.13}$$

In a sufficiently small tubular neighborhood $D(\pm\Delta S^2)$ of $\pm\Delta S^2$, the function $\psi_{\pm}$ is smooth. Let $\chi_{\pm} \colon S^2 \times S^2 \to \mathbb{R}$ be smooth cutoff functions that vanish outside the corresponding $D(\pm\Delta S^2)$ and are equal to 1 on a smaller tubular neighborhood of $\pm\Delta S^2$. Set

$$\phi \colon S^2 \times S^2 \to \mathbb{R}, \quad \phi := -\chi_+ \psi_+ - \chi_- \psi_-. \tag{6.14}$$

By construction, ψ is a smooth function supported in a neighborhood of $\pm\Delta S^2$. At a point $p \in \Delta S^2$, the Hessian of ϕ coincides with the Hessian of ψ_+,

$$\operatorname{Hess}\phi(X,X) = \operatorname{Hess}\psi_+(X,X) = -2\,\mathtt{g}_t(X^\perp, X^\perp) = -2\,\|X^\perp\|^2_{\mathtt{g}_t}, \tag{6.15}$$

where $X^\perp \in T_p(\Delta S^2)^\perp$ is the normal component of $X \in T_p(S^2 \times S^2)$. A completely analogous formula holds for points $p \in -\Delta S^2$. Thus, setting $\mathtt{h} := \phi\,\mathtt{g}_t$, Corollary 3.4 and (6.10) imply that the first-order variation $\mathtt{g}_{t,s} = \mathtt{g}_t + s\,\mathtt{h} = (1 + s\,\phi)\,\mathtt{g}_t$ satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathtt{g}_{t,s}}(\sigma_{\pi(x)})\Big|_{s=0} &= -\tfrac{1}{2}\operatorname{Hess}\phi\Big(\big(X^*_{p_1}, X^*_{p_2}\big), \big(X^*_{p_1}, X^*_{p_2}\big)\Big) \\ &\quad - \tfrac{1}{2}\operatorname{Hess}\phi\Big(\big(X^*_{p_1}, -X^*_{p_2}\big), \big(X^*_{p_1}, -X^*_{p_2}\big)\Big) \\ &= \big\|\big(X^*_{p_1}, -X^*_{p_1}\big)\big\|^2_{\mathtt{g}_t} \\ &> 0, \end{aligned} \tag{6.16}$$

for all $\mathtt{g}_t$-flat planes $\sigma_{\pi(x)} \in \operatorname{Gr}_2(T_p(S^2 \times S^2))$ with $p \in \pm\Delta S^2$.

For any $\theta > 0$, consider the subset of $M \times \operatorname{Gr}_2(T(S^2 \times S^2)) \times \operatorname{Gr}_2(T(S^2 \times S^2))$ given by $K_\theta := \big\{(p,\sigma,\sigma') : \sigma,\sigma' \in \operatorname{Gr}_2(T_p(S^2 \times S^2)),\ \operatorname{dist}(\sigma,\sigma') \geq \theta\big\}$, and define

$$f \colon [0,S] \times K_\theta \to \mathbb{R}, \quad f\big(s,(p,\sigma,\sigma')\big) := \tfrac{1}{2}\big(\sec_{\mathtt{g}_{t,s}}(\sigma) + \sec_{\mathtt{g}_{t,s}}(\sigma')\big). \tag{6.17}$$

From Proposition 6.2, we have that $f\big(0,(p,\sigma,\sigma')\big) \geq 0$ for all $(p,\sigma,\sigma') \in K_\theta$, and $f\big(0,(p,\sigma,\sigma')\big) = 0$ if only if $p \in \pm\Delta S^2$ and σ,σ' are of the form (6.10). In particular, by (6.16), we have that $\frac{\partial f}{\partial s}\big(0,(p,\sigma,\sigma')\big) > 0$ if $f\big(0,(p,\sigma,\sigma')\big) = 0$. Thus, by Lemma 3.5, there exists $s_* > 0$ such that $f\big(s,(p,\sigma,\sigma')\big) > 0$ for all $(p,\sigma,\sigma') \in K_\theta$ and $0 < s < s_*$, which by definition (5.13) means that $\sec^\theta_{\mathtt{g}_{t,s}} > 0$ if $0 < s < s_*$. □

The above concludes the proof of Theorem 6.1, see Definition 5.8. Note that, in Proposition 6.5, as $\theta \searrow 0$, also $s_* \searrow 0$ (see Remark 6.10) and hence the metrics $\mathtt{g}_{t,s}$ with $\sec^\theta_{\mathtt{g}_{t,s}} > 0$ converge to $\mathtt{g}_t$. In particular, the metrics $\mathtt{g}^\theta$ with $\sec^\theta_{\mathtt{g}^\theta} > 0$ may be selected among $\mathtt{g}_{t,s}$ in such way that $\mathtt{g}^\theta \to \mathtt{g}_0$ as $\theta \searrow 0$.

6.4 Comments

We conclude this chapter with some comments on the above construction.

Corollary 6.6. *The manifold* $\mathbb{R}P^2 \times \mathbb{R}P^2$ *satisfies* $\sec^{0+} > 0$*.*

Proof. Consider the free isometric $\mathbb{Z}_2 \oplus \mathbb{Z}_2$-action on $(S^2 \times S^2, \mathsf{g}_0)$ given by

$$(\pm 1, \pm 1) \cdot (p_1, p_2) = (\pm p_1, \pm p_2), \tag{6.18}$$

whose orbit space is $\mathbb{R}P^2 \times \mathbb{R}P^2$. Since the actions (6.18) and (6.6) commute, this action remains isometric under the Cheeger deformation g_t of g_0. Furthermore, the cutoff functions $\chi_\pm$ in the proof of Proposition 6.5 can be chosen to be invariant under (6.18), and hence this action is also isometric on $(S^2 \times S^2, \mathsf{g}_{t,s})$ for $t > 0$ and $s > 0$. In particular, the metrics g^θ that verify $\sec^{0+} > 0$ on $S^2 \times S^2$ descend to metrics on $\mathbb{R}P^2 \times \mathbb{R}P^2$, proving that it also satisfies $\sec^{0+} > 0$. ☐

Recall that $\mathbb{R}P^2 \times \mathbb{R}P^2$ does not have $\sec > 0$ by Synge's Theorem, see Section 5.3.

Remark 6.7. An argument totally analogous to the proof of Corollary 6.6 shows that the metrics g^θ with $\sec^\theta > 0$ on $S^2 \times S^2$ can be chosen to be invariant under the circle action given by the subaction of (6.6) by the maximal torus of $\mathsf{SO}(3)$.

Corollary 6.8. *The manifolds* $S^2 \times S^2$ *and* $\mathbb{R}P^2 \times \mathbb{R}P^2$ *satisfy* $\sec^\perp > 0$*.*

Proof. Follows directly from Theorem 6.1, Corollary 6.6 and Definition 5.11. ☐

Remark 6.9. The first-order deformation in Proposition 6.5 produces metrics with $\sec^\theta > 0$ globally because the only points $p \in S^2 \times S^2$ that have pairs of planes $\sigma, \sigma' \in \mathrm{Gr}_2(T_p(S^2 \times S^2))$ with $f\big(0, (p, \sigma, \sigma')\big) = 0$ are along the submanifolds $\pm\Delta S^2$. These submanifolds admit a neighborhood with compact closure where $\frac{\partial f}{\partial s}\big(0, (p, \sigma, \sigma')\big) > 0$. Clearly, analogous techniques cannot be used to obtain $\sec > 0$ globally because *at every point* in $S^2 \times S^2$ there is at least one plane σ with $\sec_{\mathsf{g}_t}(\sigma) = 0$. Furthermore, by Corollary 3.9, see also Strake [95, Prop. 4.3], the presence of totally geodesic flat tori in $(S^2 \times S^2, \mathrm{g}_t)$ prevents first-order deformations from developing sec > 0, and the same can be inferred in this case using that, by Corollary 6.6, the metrics $\mathrm{g}_{t,s}$ descend to $\mathbb{R}P^2 \times \mathbb{R}P^2$, which does not have sec > 0.

Remark 6.10. The metrics $\mathrm{g}_{t,s}$ on $S^2 \times S^2$ do not have $\sec_{\mathrm{g}_{t,s}} \geq 0$ if $t > 0$ and $s > 0$. In fact, the existence of planes with negative curvature can be proved using the above mentioned totally geodesic flat tori in $(S^2 \times S^2, \mathrm{g}_t)$, such as

$$i\colon T \to S^2 \times S^2, \quad i(r_1, r_2) := \big((\cos r_1)e_1 + (\sin r_1)e_2,\, (\cos r_2)e_1 - (\sin r_2)e_2\big),$$

where $\{e_i\}$ is the standard basis in $\mathbb{R}^3$, with an argument analogous to the proof of Proposition 4.11. Namely, let e_1, e_2 be a global orthonormal frame on $(T, i^*\mathrm{g}_{t,s})$ and define vector fields X and Y along $i\colon T \to S^2 \times S^2$ by setting $X(r_1, r_2) := \mathrm{d}i(r_1, r_2)e_1$ and $Y(r_1, r_2) := \mathrm{d}i(r_1, r_2)e_2$ for all $(r_1, r_2) \in T$. By the Gauss-Bonnet Theorem, for all $s \geq 0$,

$$\int_T \sec_{i^*(\mathrm{g}_{t,s})}(e_1 \wedge e_2)\, \mathrm{vol}_{i^*(\mathrm{g}_{t,s})} = 2\pi\chi(T) = 0.$$

Thus, from Lemma 3.7, differentiating the above at $s = 0$ we have:

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s} \int_T \sec_{i^*(\mathrm{g}_{t,s})}(e_1 \wedge e_2)\, \mathrm{vol}_{i^*(\mathrm{g}_{t,s})}\Big|_{s=0} \\
&= \int_T \frac{\mathrm{d}}{\mathrm{d}s} \sec_{i^*(\mathrm{g}_{t,s})}(e_1 \wedge e_2)\Big|_{s=0} \mathrm{vol}_{i^*(\mathrm{g}_t)} + \int_T \sec_{i^*(\mathrm{g}_t)}(e_1 \wedge e_2)\, \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{vol}_{i^*(\mathrm{g}_{t,s})}\Big|_{s=0} \\
&= \int_T \frac{\mathrm{d}}{\mathrm{d}s} \sec_{i^*(\mathrm{g}_{t,s})}(e_1 \wedge e_2)\Big|_{s=0} \mathrm{vol}_{i^*(\mathrm{g}_t)} \\
&= \int_T \frac{\mathrm{d}}{\mathrm{d}s} \sec_{\mathrm{g}_{t,s}}(X \wedge Y)\Big|_{s=0} \mathrm{vol}_{i^*(\mathrm{g}_t)}.
\end{aligned}$$

By the construction of $\mathrm{g}_{t,s}$, the above integrand is positive on the intersection of T and $\pm\Delta S^2$, see (6.16). Since the above integral vanishes, this integrand is also negative somewhere, which (by an argument analogous to Lemma 3.5) implies that there exists planes $\sigma \in \mathrm{Gr}_2(T_p(S^2 \times S^2))$ with $\sec_{\mathrm{g}_{t,s}}(\sigma) < 0$ if $t > 0$ and $s > 0$.

With regard to extending the construction in this chapter to other 4-manifolds, we remark that similar techniques to those used for $S^2 \times S^2$ yield the following, see Bettiol [13, Prop. 5.1] and Müter [72, Satz 4.29].

Proposition 6.11. *The manifold* $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ *satisfies* $\sec^\perp > 0$*.*

Proof. The starting point is a metric g_0 on $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ with $\sec_{\mathrm{g}_0} \geq 0$, obtained as follows. Consider the round spheres $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ and $S^2 = \{(z_3, h) \in \mathbb{C} \oplus \mathbb{R} : |z_3|^2 + h^2 = 1\}$, and define the diagonal circle action on $S^3 \times S^2$ via the Hopf action on S^3 and a rotation action on S^2, i.e., given by

$$e^{i\theta} \cdot \big((z_1, z_2), (z_3, h)\big) := \big((e^{i\theta} z_1, e^{i\theta} z_2), (e^{ki\theta} z_3, h)\big), \quad e^{i\theta} \in S^1,\ k \in \mathbb{N}. \tag{6.19}$$

This is a free isometric action, whose orbit space is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ if k is an odd number.[2] Choosing, e.g., $k = 1$, we have that there exists a Riemannian metric g_0 on $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ such that the quotient map $\pi\colon S^3 \times S^2 \to \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is a Riemannian submersion, where $S^3 \times S^2$ is endowed with the standard product metric. In particular, from (2.19) and the Gray-O'Neill formula (2.28), we have that $\sec_{\mathrm{g}_0} \geq 0$. Furthermore, the flat planes $\sigma \in \sec_{\mathrm{g}_0}^{-1}(0)$ on $\big(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2, \mathrm{g}_0\big)$ are precisely the images under $\mathrm{d}\pi$ of mixed planes $\widetilde{\sigma}$ on $S^3 \times S^2$ that are orthogonal to the action field induced by (6.19). Note that at all points $\widetilde{p} = \big((z_1, z_2), (z_3, h)\big) \in S^3 \times S^2$ with $|h| < 1$, these mixed planes $\widetilde{\sigma}$ contain a vector of the form $(0, X) \in T_{\widetilde{p}}(S^3 \times S^2)$ where $X \in T_{\widetilde{p}_2} S^2$ is orthogonal to the rotation action field, i.e., X is tangent to a great circle on S^2 that passes through the North and South poles. Thus, all flat planes $\sigma \in \sec_{\mathrm{g}_0}^{-1}(0) \subset \mathrm{Gr}_2\big(T_p\big(\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2\big)\big)$, where $p = \pi(\widetilde{p})$, must intersect along a line, and hence cannot

[2] The manifold $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ can also be thought of as two copies of the normal disk bundle of the equatorial $\mathbb{C}P^1 \subset \mathbb{C}P^2$ glued together along the boundary. This decomposition lifts to $S^3 \times D^2_+ \cup S^3 \times D^2_-$, where $D^2_\pm \subset S^2$ are disks of radius $\pi/2$ around the North and South poles.

be orthogonal. Thus, $\sec^{\perp}_{\mathrm{g}_0} > 0$ on an open and dense subset of $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, cf. Proposition 6.2. The conclusion now follows from Proposition 5.19. □

Remark 6.12. It is not known whether $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ satisfies $\sec^{0+} > 0$ (or $\sec > 0$). Note that, at all points on $\big(\mathbb{C}P^2\#\overline{\mathbb{C}P}^2, \mathrm{g}_0\big)$, there are pairs of flat planes at arbitrarily small distance, hence the methods from Section 6.3 do not apply in this case.

Remark 6.13. The transitive $\mathsf{SU}(2)$-action on S^3 given by left translation induces an isometric action on $S^3 \times S^2$ that commutes with (6.19), and hence descends to an isometric $\mathsf{SU}(2)$-action on $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$. This is a cohomogeneity one action whose principal orbits are hypersurfaces diffeomorphic to S^3 and singular orbits have codimension 2 and are diffeomorphic to S^2, analogously to the $\mathsf{SO}(3)$-action (6.6). The points $p = \pi(\widetilde{p}) \in \mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ where $\widetilde{p} = \big((z_1, z_2), (z_3, h)\big) \in S^3 \times S^2$ with $|h| < 1$ are precisely those that do not belong to singular orbits. Since $\sec^{\perp}_{\mathrm{g}_0}$ is already positive at such points, we do not need to use a Cheeger deformation with respect to this $\mathsf{SU}(2)$-action. In fact, *none* of these flat planes on $\big(\mathbb{C}P^2\#\overline{\mathbb{C}P}^2, \mathrm{g}_0\big)$ gain positive curvature under this Cheeger deformation, see Müter [72, Satz 4.29]. We also observe that there are pairs of orthogonal flat planes at all points on $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ that lie on singular orbits, hence $\sec^{\perp}_{\mathrm{g}_0}$ attains zero along these submanifolds.

Remark 6.14. If k is an even number, the orbit space of the free circle action (6.19) is diffeomorphic to $S^2 \times S^2$. Furthermore, if $k = 0$, the above mentioned isometric $\mathsf{SU}(2)$-action on $S^3 \times S^2$ descends to the $\mathsf{SO}(3)$-action (6.6) on $S^2 \times S^2$, after also taking the quotient by the ineffective kernel $\mathbb{Z}_2 \subset \mathsf{SU}(2)$. Analogous cohomogeneity one actions on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ are obtained for all even and odd values of k, respectively, with principal orbits $S^3/\mathbb{Z}_k$ covered by the above principal orbits.

Remark 6.15. By Theorem 5.16, metrics with $\sec^{\perp} > 0$ on $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ do not have nonnegative isotropic curvature. Thus, by Proposition 5.14 and Theorem 5.18, they do not have $\frac{1}{4}$-pinched biorthogonal curvature or $\sec^{\perp} \geq \frac{\mathrm{scal}}{24} > 0$.

CHAPTER 7

SIMPLY-CONNECTED 4-MANIFOLDS WITH $\sec^\perp > 0$

The goal of this chapter is to prove the following classification result.

Theorem 7.1. *Let M^4 be a smoothable closed simply-connected topological 4-manifold. Up to endowing M with different smooth structures, the following are equivalent:*

(i) *M^4 satisfies $\sec^\perp > 0$;*

(ii) *M^4 satisfies $\mathrm{Ric} > 0$;*

(iii) *M^4 satisfies $\mathrm{scal} > 0$.*

The equivalence between (ii) and (iii) above was established by Sha and Yang [90]. Furthermore, the fact that (i) implies (iii) follows from Proposition 5.12. Thus, in order to prove Theorem 7.1, we must show that (iii) implies (i), for which we use a strategy similar to that of Sha and Yang [90]. The starting point is the classification of closed simply-connected 4-manifolds that satisfy $\mathrm{scal} > 0$, which is described in Section 7.2 (see Theorem 7.8), after recalling the foundational work of Donaldson and Freedman in Section 7.1. Knowing the homeomorphism types of closed simply-connected 4-manifolds on which metrics with $\sec^\perp > 0$ need to be constructed, we combine results from Chapters 5 and 6 with a recent surgery stability result of Hoelzel [51] to verify that these constructions can be carried out, in Section 7.3. Finally, a few comments are made in Section 7.4.

Although a recent paper of Costa and Ribeiro [27] claims to contain a classification of closed 4-manifolds with $\sec^\perp \geq 0$ (a result that would extend Theorem 7.1), no classification statements are provided. In fact, all results of [27] concern 4-manifolds satisfying curvature conditions more restrictive than $\sec^\perp > 0$, see e.g. Theorem 5.18.

7.1 Donaldson-Freedman classification

In this section, we state the classification of (smooth) closed simply-connected 4-manifolds, that follows from the classical works of Donaldson [30] and Freedman [38], following Donaldson and Kronheimer [31], see also Mandelbaum [66] and Scorpan [83].

If M is a closed simply-connected topological 4-manifold, by the Hurewicz Theorem and Poincaré duality, both $H_1(M,\mathbb{Z})$ and $H_3(M,\mathbb{Z})$ are trivial, so all the homological information of M is contained in $H_2(M,\mathbb{Z})$. Furthermore, by the Universal Coefficient Theorem, $H^2(M,\mathbb{Z}) \cong \mathrm{Hom}(H_2(M,\mathbb{Z}),\mathbb{Z})$ is a free abelian group. The isomorphism $H^2(M,\mathbb{Z}) \cong H_2(M,\mathbb{Z})$ given by Poincaré duality can hence be expressed as a unimodular symmetric bilinear form, called *intersection form*,

$$Q_M\colon H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \to \mathbb{Z}, \quad Q_M(\alpha,\beta) := (\alpha \cup \beta)[M], \tag{7.1}$$

where $[M] \in H_4(M,\mathbb{Z}) \cong \mathbb{Z}$ is the fundamental class of M. If S_α and S_β are embedded surfaces in general position that represent the homology classes Poincaré dual to $\alpha,\beta \in H^2(M,\mathbb{Z})$, then $Q_M(\alpha,\beta)$ is the intersection number of S_α and S_β.

Since $H^2(M,\mathbb{Z}) \cong \mathbb{Z}^r$, where $r = b_2(M)$, the intersection form Q_M is represented by a symmetric matrix $Q_M \in \mathsf{GL}(r,\mathbb{Z})$ with $\det Q_M = \pm 1$. We call r the *rank* of Q_M. Denoting by $b_2^+(M)$ and $b_2^-(M)$ the dimensions of the largest subspaces where Q_M is positive-definite and negative-definite, the *signature* of Q_M is the difference

$$\operatorname{sign} Q_M := b_2^+(M) - b_2^-(M). \tag{7.2}$$

Clearly, $b_2^+(M) + b_2^-(M) = b_2(M) = r$. We say that Q_M is *positive* or *negative* if the corresponding matrix is positive-definite or negative-definite, and we say that Q_M is *indefinite* if it is not positive nor negative. Finally, we say that Q_M is *even* if $Q_M(\alpha,\alpha) \equiv 0 \mod 2$ for all α, otherwise Q_M is called *odd*.

Denoting by $\overline{M}$ the manifold M with opposite orientation, routine arguments show that $Q_{\overline{M}} = -Q_M$. Furthermore, if M_1 and M_2 are closed simply-connected topological 4-manifolds, then the intersection form of their connected sum $M_1 \# M_2$ is $Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2}$. In addition, we say that Q_{M_1} and Q_{M_2} are *isomorphic* if there is an isomorphism $\phi \colon H^2(M_1, \mathbb{Z}) \to H^2(M_2, \mathbb{Z})$ that commutes with Q_{M_1} and Q_{M_2}. Elementary examples of intersection forms are the following:

(i) The intersection form of S^4 has rank zero;

(ii) The intersection form of $\mathbb{C}P^2$ is $Q_{\mathbb{C}P^2} = \big(1\big)$, which has rank 1, signature 1, and is positive and odd;

(iii) The intersection form of $\overline{\mathbb{C}P}^2$ is $Q_{\overline{\mathbb{C}P}^2} = \big(-1\big)$, which has rank 1, signature -1, and is negative and odd;

(iv) The intersection form of $S^2 \times S^2$ is $Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which has rank 2, signature 0, and is indefinite and even.

Remark 7.2. By the above, the intersection form of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ is $\big(1\big) \oplus \big(-1\big)$, which is conjugate to the intersection form of $S^2 \times S^2$ over a ring of coefficients that admits an inverse of 2, such as $\mathbb{Q}$ or $\mathbb{R}$. Furthermore, the cohomology rings of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ and $S^2 \times S^2$ over $\mathbb{Q}$ or $\mathbb{R}$ are isomorphic, but not over $\mathbb{Z}$, cf. Theorem 7.3.

Another important example is the intersection form Q_{E_8}, whose matrix is determined by the Dynkin diagram E_8, by labeling the vertices from 1 through 8 in the standard way and defining the (i,j)th entry as 2 if $i = j$, and as the number of edges connecting the vertices i and j if $i \neq j$. The form Q_{E_8} has rank 8, signature 8, and is positive and even. Furthermore, there is a topological 4-manifold M_{E_8} whose intersection form is Q_{E_8}, cf. Theorem 7.4. However, M_{E_8} does not admit any smooth structures, by a result of Rokhlin [79], which states that if the intersection form of a *smooth* 4-manifold M is even, then $\operatorname{sign} Q_M \equiv 0 \mod 16$.

By a result of Serre [87], two indefinite symmetric bilinear unimodular forms are isomorphic if and only if they have the same rank, signature and parity. However, there is no classification of definite symmetric bilinear unimodular forms, and the number of different examples grows astonishingly fast with the rank.

The following classical homotopy type classification was proved by Milnor [69], as a consequence of the work of Whitehead [105].

Theorem 7.3. *Two closed simply-connected topological* 4*-manifolds are homotopy equivalent if and only if their intersection forms are isomorphic. In particular, the cohomology ring of such topological* 4*-manifolds is completely determined by the intersection form.*

A major breakthrough came from Freedman [38], who proved the following:

Theorem 7.4. *For any integral symmetric unimodular form Q, there is a closed simply-connected topological* 4*-manifold whose intersection form is Q. Furthermore, if Q is even, then there is exactly one such manifold, and if Q is odd there are exactly two such manifolds (at least one of which does not admit any smooth structures).*[1]

In particular, from Theorem 7.4, two closed simply-connected smooth 4-manifolds are homeomorphic if and only if their intersection forms are isomorphic.[2] Although all the candidates to intersection forms are realized by *topological* 4-manifolds by Theorem 7.4, including the vast number of definite forms, this is not the case among *smooth* 4-manifolds due to the next major breakthrough obtained by Donaldson [30], who proved the following:

Theorem 7.5. *The only definite symmetric bilinear unimodular forms that can be realized as intersection forms of a smooth* 4*-manifold are $\oplus^m(1)$ and $\oplus^m(-1)$.*

[1] The *Kirby-Siebenmann obstruction* $\mathrm{ks}(M) \in H^4(M, \mathbb{Z}_2) \cong \mathbb{Z}_2$ is an obstruction to the existence of a smooth structure on M, which vanishes if Q_M is even. More generally, Freedman's theorem states that if $\mathrm{ks} \in \mathbb{Z}_2$ and Q is an integral symmetric unimodular form such that $\mathrm{ks} \equiv \frac{1}{8} \operatorname{sign} Q \mod 2$, then there exists a closed simply-connected 4-manifold M, with $Q_M = Q$ and $\mathrm{ks}(M) = \mathrm{ks}$. Moreover, two closed simply-connected 4-manifolds are homeomorphic if and only if their intersection forms are isomorphic and their Kirby-Siebenmann invariants are equal.

[2] Furthermore, these manifolds become *diffeomorphic* after taking a sufficiently large number of connected sums with $S^2 \times S^2$, by the sum stabilization results of Wall, see Scorpan [83, p. 155].

Combining the above results, we arrive to the following classification:

Theorem 7.6. *Let M^4 be a smoothable closed simply-connected topological* 4*-manifold. Then M^4 is homeomorphic to S^4, $\#^m \mathbb{C}P^2 \#^n \overline{\mathbb{C}P}^2$, or $\#^{\pm m} M_{E_8} \#^n (S^2 \times S^2)$.*[3]

It is important to stress that, in the above result, not all of $\#^{\pm m} M_{E_8} \#^n (S^2 \times S^2)$ admit smooth structures. Furthermore, for later purposes, note that:

(i) The intersection form of $\#^m \mathbb{C}P^2 \#^n \overline{\mathbb{C}P}^2$ has rank $m+n$, signature $m-n$ and is odd;

(ii) The intersection form of $\#^{\pm m} M_{E_8} \#^n (S^2 \times S^2)$ has rank $8m+n$, signature $\pm 8m$ and is even.

Finally, we remark that the parity of the intersection form Q_M is related to the existence of a *spin structure* on M. Namely, if M is simply-connected, then M admits a spin structure if and only if Q_M is even. Thus, in Theorem 7.6, the spin manifolds are S^4 and $\#^{\pm m} M_{E_8} \#^n (S^2 \times S^2)$, while the nonspin manifolds are $\#^m \mathbb{C}P^2 \#^n \overline{\mathbb{C}P}^2$. Furthermore, the sentence following Theorem 7.4 can be rephrase as two closed simply-connected smooth 4-manifolds are homeomorphic if and only if they have the same Euler characteristic, their intersection forms have the same signature, and they are both spin or nonspin.

7.2 Positive scalar curvature

The question of which closed manifolds satisfy $\mathrm{scal} > 0$ is also a central problem in Riemannian geometry, similarly to the question of which closed manifolds satisfy $\sec > 0$ mentioned in the Introduction, see the surveys [80, 93] for details. There are, however, remarkable differences between what is known regarding each of these problems, and also in the nature of the techniques used in their study. In this section, we very briefly state well-known obstructions to $\mathrm{scal} > 0$ that, in conjunction with the

[3]Here, the connected sum of $-m$ copies of M_{E_8} means the connected sum of m copies of $\overline{M}_{E_8}$.

Donaldson-Freedman classification discussed in Section 7.1, yield the classification of closed simply-connected 4-manifolds that satisfy $\operatorname{scal} > 0$, see Theorem 7.8.

As an instance of the so-called *Bochner technique*, Lichnerowicz [65] proved that spin manifolds with $\operatorname{scal} > 0$ do not admit nontrivial harmonic spinors.[4] By the celebrated Atiyah-Singer Index Theorem, in dimensions multiple of 4, this corresponds to the vanishing of an invariant called the $\widehat{A}$-*genus*, yielding the following:

Theorem 7.7. *If M is a closed spin manifold of dimension $4k$ that satisfies $\operatorname{scal} > 0$, then $\widehat{A}(M) = 0$.*

By the Hirzebruch Signature Theorem, the $\widehat{A}$-genus of a 4-manifold is given by:

$$\widehat{A}(M) = -\tfrac{1}{8} \operatorname{sign} Q_M.$$

In particular, a closed spin 4-manifold whose intersection form has nonzero signature does not satisfy $\operatorname{scal} > 0$ by Theorem 7.7. Thus, from Theorem 7.6, we conclude:

Theorem 7.8. *A closed simply-connected 4-manifold that satisfies $\operatorname{scal} > 0$ is homeomorphic to either S^4, $\#^m\, \mathbb{C}P^2 \#^n\, \overline{\mathbb{C}P}^2$, or $\#^n(S^2 \times S^2)$.*

Conversely, all the above topological 4-manifolds are known to admit smooth structures with metrics with $\operatorname{scal} > 0$, since the connected sum of 4-manifolds that satisfy $\operatorname{scal} > 0$ also satisfies $\operatorname{scal} > 0$ (see Section 7.3) and the standard metrics on S^4, $\mathbb{C}P^2$, and $S^2 \times S^2$ have $\operatorname{scal} > 0$. Nevertheless, some of the above topological 4-manifolds also admit exotic smooth structures without any metrics of $\operatorname{scal} > 0$. For example, the *Barlow surface* is a complex surface homeomorphic (but not diffeomorphic) to $\mathbb{C}P^2 \#^8\, \overline{\mathbb{C}P}^2$, with nonvanishing Seiberg-Witten invariant, which hence does not carry metrics with $\operatorname{scal} > 0$, see [22, 74, 80] for details and further examples.

[4] A *harmonic spinor* is a section of the spinor bundle whose Laplacian vanishes, see [64, §2.8].

Remark 7.9. Any 4-manifold $M^4 = S^4 \#^m \mathbb{C}P^2 \#^n \overline{\mathbb{C}P}^2 \#^p (S^2 \times S^2)$ is homeomorphic to one of the 4-manifolds in Theorem 7.8. This follows easily from Theorem 7.4 and the fact that there are orientation preserving diffeomorphisms between $\overline{S^2 \times S^2}$ and $S^2 \times S^2$, and between $\mathbb{C}P^2 \#(S^2 \times S^2)$ and $\#^2 \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, see Scorpan [83, p. 151].

7.3 Surgery stability of $\sec^\perp > 0$

In this section, we discuss the surgery results that allow to construct metrics with $\sec^\perp > 0$ on all the manifolds listed in Theorem 7.8, which yields the proof of Theorem 7.1. Gromov and Lawson [42], and independently Schoen and Yau [81], pioneered surgery techniques that imply stability of $\mathrm{scal} > 0$ under surgery of codimension ≥ 3. This means that if M satisfies $\mathrm{scal} > 0$, then any manifold M' obtained from M by surgery of codimension ≥ 3 also satisfies $\mathrm{scal} > 0$. This result was recently extended to more general pointwise curvature conditions by Hoelzel [51, Thm. B], as follows.

Theorem 7.10. *Let $C \subset S_\mathfrak{b}(\wedge^2 \mathbb{R}^n)$ be a curvature condition given by open $\mathsf{O}(n)$-invariant convex cone, such that the curvature operator of the standard product metric on $S^{n-k-1} \times \mathbb{R}^{k+1}$ belongs to C for some $k \in \{0, \dots, n-3\}$. Suppose (M_1, g_1) and (M_2, g_2) are n-dimensional Riemannian manifolds that satisfy C, and let $N_1 \subset M_1$ and $N_2 \subset M_2$ be closed l-dimensional submanifolds, with $0 \leq l \leq k$. If there is an isomorphism $\Phi\colon TN_1^\perp \to TN_2^\perp$ between the normal bundles of N_1 and N_2, then*

$$M_1 \#_\Phi M_2 := \big(M_1 \setminus D(N_1)\big) \sqcup_\Phi \big(M_2 \setminus D(N_2)\big) \tag{7.3}$$

admits a Riemannian metric that satisfies C, where $D(N_i)$ are tubular neighborhoods of N_i and the gluing is given by the diffeomorphism $D(N_1) \cong D(N_2)$ induced by Φ.

We use this result with $k = 0$, in which case N_1 and N_2 are points, and (7.3) is the connected sum $M_1 \# M_2$, to prove that $\sec^\perp > 0$ is stable under connected sums:

Proposition 7.11. *If M_1 and M_2 are n-dimensional manifolds that satisfy* $\sec^\perp > 0$*, then also $M_1 \# M_2$ satisfies* $\sec^\perp > 0$.

Proof. Recall that $\sec^\perp > 0$ is a pointwise curvature condition that corresponds to the open $\mathsf{O}(n)$-invariant convex cone $C_{\sec^\perp>0}$ given by (5.16). Furthermore, the curvature operator R of the standard product metric on $S^{n-1} \times \mathbb{R}$ satisfies, see (2.18),

$$\langle R(X \wedge Y), X \wedge Y \rangle = \|X_1 \wedge Y_1\|^2, \tag{7.4}$$

where $X = (X_1, X_2) \in T_{p_1} S^{n-1} \oplus \mathbb{R}$. Thus, $\langle R(\sigma), \sigma \rangle \geq 0$ for all $\sigma \in \mathrm{Gr}_2(\mathbb{R}^n)$ and $\langle R(\sigma), \sigma \rangle = 0$ if and only if σ is a mixed plane on $T_p(S^{n-1} \times \mathbb{R})$, i.e., $\sigma = (X_1, 0) \wedge (0, 1)$ for some unit vector $X_1 \in T_{p_1} S^{n-1}$. In particular, whenever $\langle R(\sigma), \sigma \rangle = 0$, we have $\langle R(\sigma'), \sigma' \rangle = \|\sigma'\|^2 = 1$ for all $\sigma' \subset \sigma^\perp$, cf. Remark 5.13. Therefore, $R \in C_{\sec^\perp>0}$ and hence the conclusion follows from Theorem 7.10. □

This concludes the proof of Theorem 7.1, since it follows that (iii) implies (i). Indeed, if M is a closed simply-connected 4-manifold that satisfies $\mathrm{scal} > 0$, then it is homeomorphic to S^4, $\#^m \mathbb{C}P^2 \#^n \overline{\mathbb{C}P}^2$, or $\#^n (S^2 \times S^2)$, by Theorem 7.8. Clearly, S^4 and $\mathbb{C}P^2$ satisfy $\sec^\perp > 0$, since they satisfy $\sec > 0$. Furthermore, by Theorem 6.1 (see also Remark 6.3), $S^2 \times S^2$ satisfies $\sec^\perp > 0$. Thus, by Proposition 7.11, connected sums of S^4, $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$ and $S^2 \times S^2$ satisfy $\sec^\perp > 0$; in particular, all closed simply-connected 4-manifolds that satisfy $\mathrm{scal} > 0$ also satisfy $\sec^\perp > 0$.

Remark 7.12. The surgery stability result in Theorem 7.10 can be used more generally to show that $\sec^\theta > 0$ is stable under connected sums if $\theta > \frac{\pi}{2}$ and the distance is induced via (5.12) by the symmetric space distance dist_{SS} given by (5.11). Recall that the corresponding cone $C_{\sec^\theta>0}$ is an open $\mathsf{O}(n)$-invariant convex cone. Furthermore, if $\langle R(\sigma), \sigma \rangle = 0$ and $\mathrm{dist}_{SS}(\sigma, \sigma') > \frac{\pi}{2}$, then $\theta_1 > 0$ and hence, by (5.10), the plane σ' does not contain the direction spanned by $(0, 1) \in T_{p_1} S^{n-1} \oplus \mathbb{R}$, so $\langle R(\sigma'), \sigma' \rangle > 0$. Thus, $R \in C_{\sec^\theta>0}$ and Theorem 7.10 applies. Note, however, that the same does not hold if the distance is induced via (5.12) by any of the other distances (5.1) or (5.3), since, in this case, there are pairs of mixed planes for $S^{n-1} \times \mathbb{R}$ at all distances, cf. Remarks 5.1 and 5.3.

Remark 7.13. Theorem 7.10 cannot be used to generalize Proposition 7.11 to surgeries along submanifolds of lower codimension. Indeed, for any $k \geq 1$, the curvature operator of $S^{n-k-1} \times \mathbb{R}^{k+1}$ does not belong to $C_{\sec^\perp > 0}$, since, when $k \geq 1$, there are pairs of mixed planes for $S^{n-k-1} \times \mathbb{R}^{k+1}$ that are orthogonal.[5]

7.4 Comments

We conclude this chapter with some remarks on 4-manifolds with $\sec^\perp > 0$, complementing Section 5.5 after the proof of Theorem 7.1.

Remark 7.14. Since the above classification of closed simply-connected 4-manifolds with $\sec^\perp > 0$ was only obtained up to *homeomorphisms*, it is natural to wonder if this can be improved to *diffeomorphisms*. The first difficulties in achieving this originate from the (rather serious) difficulties in strengthening Theorem 7.8 to a result that detects not only homeomorphism type but also diffeomorphism type. The only currently known invariants that distinguish 4-manifolds that are homeomorphic but not diffeomorphic are the Donaldson invariants (see [31]) and Seiberg-Witten invariants (see [70]). These invariants vanish on smooth 4-manifolds that satisfy $\mathrm{scal} > 0$, preventing the distinction of diffeomorphism types. Further difficulties arise from the fact that the surgery construction via Proposition 7.11 only yields metrics with $\sec^\perp > 0$ on the manifolds listed in Theorem 7.8 with their *standard* smooth structure, and new constructions would be necessary to produce metrics with $\sec^\perp > 0$ on such manifolds endowed with exotic smooth structures.

[5]The main result of Hoelzel [51, Thm. B] actually has slightly weaker hypotheses than Theorem 7.10. Namely, instead of requiring that the curvature operator of $S^{n-k-1} \times \mathbb{R}^{k+1}$ belongs to C, it suffices to have that C *satisfies an inner cone condition* with respect to the curvature operator of $S^{n-k-1} \times \mathbb{R}^{k+1}$. However, it is not clear whether $C_{\sec^\perp > 0}$ satisfies this hypothesis when $k \geq 1$.

Remark 7.15. It follows from Theorem 6.1 (see also Remark 6.3) and Proposition 7.11 that, for all $n \in \mathbb{N}$, the manifold $\#^n(S^2 \times S^2)$ satisfies $\sec^\perp > 0$. In particular, there is no upper bound on the total Betti number for closed 4-manifolds with $\sec^\perp > 0$. Furthermore, the above provide examples of closed simply-connected 4-manifolds that satisfy $\sec^\perp > 0$ but do not satisfy $\sec \geq 0$, by the a priori bounds on Betti numbers of Gromov [41], see also Petersen [75, Thm. 86, p. 357].

Remark 7.16. Although Theorem 7.1 only deals with simply-connected 4-manifolds, many of the above techniques can be used in the non-simply-connected case. Recall from Corollary 6.6 that $\mathbb{R}P^2 \times \mathbb{R}P^2$ satisfies $\sec^\perp > 0$, and from Remark 5.13 that also $S^3 \times S^1$ and $(S^3 \times \mathbb{R})/\Gamma$ satisfy $\sec^\perp > 0$, where Γ is a discrete cocompact group. By Proposition 7.11, connected sums of such manifolds satisfy $\sec^\perp > 0$, providing many examples of non-simply-connected 4-manifolds with $\sec^\perp > 0$. In particular, we can e.g. construct 4-manifolds M with $\sec^\perp > 0$ whose fundamental group is

$$\pi_1(M) \cong *^m \, \mathbb{Z} *^{n_1} (\mathbb{Z}_{p_1} \oplus \mathbb{Z}) \cdots *^{n_k} (\mathbb{Z}_{p_k} \oplus \mathbb{Z}) *^r (\mathbb{Z}_2 \oplus \mathbb{Z}_2),$$

where $*$ denotes the free product and $p_1, \dots, p_k$ are prime, by taking the connected sum $M = \#^m(S^3 \times S^1)\#^{n_1}(S^3/\mathbb{Z}_{p_1} \times S^1) \dots \#^{n_k}(S^3/\mathbb{Z}_{p_k} \times S^1)\#^r(\mathbb{R}P^2 \times \mathbb{R}P^2)$.

As observed above, although $\sec^\perp > 0$ is stable under connected sums, it is not clear whether $\sec^\perp > 0$ is stable under surgeries of lower codimension (see Remark 7.13). However, note that if $\sec^\perp > 0$ on 4-manifolds were also invariant under surgeries of codimension 3, then it would follow that all finitely presented groups could be realized as the fundamental group of a 4-manifold with $\sec^\perp > 0$. Namely, a finitely presented group with n generators and r relations is the fundamental group of the 4-manifold obtained from $\#^n(S^3 \times S^1)$ after r surgeries of codimension 3, where tubular neighborhoods $S^1 \times D^3$ of r loops representing the relations are replaced by $S^2 \times D^2$. This is the exact same construction that yields that any finitely presented group is realized as the fundamental group of a 4-manifold with scal > 0.

Remark 7.17. Although the proof of Theorem 7.10 is constructive, and yields metrics with $\sec^\perp > 0$ on the connected sums of S^4, $\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$ and $S^2 \times S^2$, the only *explicit* metrics known (besides the trivial cases on S^4 and $\mathbb{C}P^2$) are those constructed in Chapter 6 on $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$. Furthermore, analogously to Remark 6.15, none of these 4-manifolds with $\sec^\perp > 0$ (except for S^4 and $\mathbb{C}P^2$) can have nonnegative isotropic curvature, $\frac{1}{4}$-pinched biorthogonal curvature, or $\sec^\perp \geq \frac{\text{scal}}{24} > 0$.

Remark 7.18. The surgery stability criterion (Theorem 7.10) has an equivariant version [51, §5] that, under the same conditions, allows to endow $M_1 \#_\Phi M_2$ with a G-*invariant* metric satisfying C, provided M_i have G-invariant metrics satisfying C such that N_i are fixed by the G-action and Φ is G-equivariant. This result may be used to endow the manifolds $\#^m \mathbb{C}P^2 \#^n \overline{\mathbb{C}P}^2$ and $\#^n (S^2 \times S^2)$ with metrics that are invariant under a circle action and satisfy either scal > 0 or $\sec^\perp > 0$. Namely, consider the circle action on $\mathbb{C}P^2$ given as a subaction of the transitive $\mathsf{SU}(3)$-action by the maximal torus of $\mathsf{SU}(2) \subset \mathsf{SU}(3)$, and the circle action on $S^2 \times S^2$ given as a subaction of the $\mathsf{SO}(3)$-action (6.6) by the maximal torus of $\mathsf{SO}(3)$. These actions have respectively 3 and 4 fixed points.[6] The standard metrics on $\mathbb{C}P^2$ and $S^2 \times S^2$ are invariant under these actions, and so are the metrics with $\sec^\perp > 0$ on the latter, constructed in Chapter 6, see Remark 6.7. Performing surgeries (connected sums) using these fixed points as N_i, the above equivariant version of the surgery stability criterion implies that $\#^m \mathbb{C}P^2 \#^n \overline{\mathbb{C}P}^2$ and $\#^n (S^2 \times S^2)$ carry invariant metrics with scal > 0 or $\sec^\perp > 0$. Although it does not follow from the original construction of Sha and Yang [90] that these manifolds carry invariant metrics with Ric > 0, this was recently proved[7] by Bazaĭkin and Matvienko [6]. Clearly, also the round metric

[6] Recall that the Euler characteristic of the fixed point set of a torus action on a closed manifold is the same as the Euler characteristic of the manifold. Thus, the above number of fixed points agrees with the fact that $\chi(\mathbb{C}P^2) = 3$ and $\chi(S^2 \times S^2) = 4$.

[7] In fact, it is shown in [6] that these manifolds carry metrics with Ric > 0 invariant under a

on S^4 satisfies all of these curvature conditions and is invariant under a circle action. Thus, in all items in Theorem 7.1, the metric satisfying that curvature condition may be chosen invariant under a circle action.

2-torus action.

PART III

STRONGLY POSITIVE CURVATURE

CHAPTER 8

STRONGLY POSITIVE CURVATURE

In this chapter, we study a curvature positivity condition called *strongly positive curvature*, which stems from the work of Thorpe [96, 97] on algebraic properties of curvature operators. This condition has been implicitly studied by other authors, including Zoltek [113], Puttmann [77], Dearricott [28], and Grove, Verdiani and Ziller [44], who coined the term. However, a systematic study of this condition was only recently initiated by Bettiol and Mendes [14, 15]. Although this chapter is based on the latter references, it contains several examples, auxiliary results, and proofs that were omitted in these papers.

8.1 Modified curvature operators

Let V be an n-dimensional real vector space, endowed with an inner product. Recall from Section 2.3 that the space of symmetric linear operators on $\wedge^2 V$ splits as the orthogonal direct sum

$$S(\wedge^2 V) = S_{\mathfrak{b}}(\wedge^2 V) \oplus \wedge^4 V,$$

where $S_{\mathfrak{b}}(\wedge^2 V) = \ker \mathfrak{b}$ is the space of algebraic curvature operators, given by the kernel of the Bianchi map $\mathfrak{b}\colon S(\wedge^2 V) \to \wedge^4 V$. Consider the Grassmannian of 2-planes

$$\begin{aligned} \mathrm{Gr}_2(V) &= \big\{\sigma \subset V : \dim \sigma = 2\big\} \\ &= \big\{\sigma \in \wedge^2 V : \sigma \wedge \sigma = 0 \text{ and } \|\sigma\|^2 = 1\big\}, \end{aligned} \tag{8.1}$$

which (as before) we identify with its double covering, the Grassmannian of oriented 2-planes. Note that $\sigma \in \mathrm{Gr}_2(V)$ if and only if there exist orthonormal vectors $X, Y \in V$ such that $\sigma = X \wedge Y$. The sectional curvature function of an operator $R \in S(\wedge^2 V)$ is defined as the quadratic form associated to R restricted to $\mathrm{Gr}_2(V)$,

$$\sec_R \colon \mathrm{Gr}_2(V) \to \mathbb{R}, \quad \sec_R(\sigma) := \langle R(\sigma), \sigma \rangle, \tag{8.2}$$

cf. (2.7). The observation that $\sec_R$ only depends on the component of R in $S_{\mathfrak{b}}(\wedge^2 V)$ is at the foundations of the theory of strongly positive curvature.

Lemma 8.1. *Let $R_1, R_2 \in S(\wedge^2 V)$. Then $\sec_{R_1} = \sec_{R_2}$ if and only if $R_1 - R_2 \in \wedge^4 V$.*

Proof. Since $\sec_{R_1} - \sec_{R_2} = \sec_{R_1 - R_2}$, see (8.2), it suffices to show that $R \in S(\wedge^2 V)$ satisfies $\sec_R = 0$ if and only if $R \in \wedge^4 V$. Suppose $\sec_R = 0$, i.e., for all $X, Y \in V$,

$$\langle R(X \wedge Y), X \wedge Y \rangle = 0. \tag{8.3}$$

Replacing X by $X + Z$ in (8.3), we have

$$\begin{aligned} 0 &= \big\langle R\big((X+Z) \wedge Y\big), (X+Z) \wedge Y \big\rangle \\ &= \langle R(X \wedge Y), X \wedge Y \rangle + 2 \langle R(X \wedge Y), Z \wedge Y \rangle + \langle R(Z \wedge Y), Z \wedge Y \rangle \\ &= 2 \langle R(X \wedge Y), Z \wedge Y \rangle. \end{aligned}$$

Furthermore, since the above vanishes, replacing Y with $Y + W$, we have

$$\begin{aligned} 0 &= \big\langle R\big(X \wedge (Y+W)\big), Z \wedge (Y+W) \big\rangle \\ &= \langle R(X \wedge Y), Z \wedge Y \rangle + \langle R(X \wedge Y), Z \wedge W \rangle \\ &\quad + \langle R(X \wedge W), Z \wedge Y \rangle + \langle R(X \wedge W), Z \wedge W \rangle \\ &= \langle R(X \wedge Y), Z \wedge W \rangle + \langle R(X \wedge W), Z \wedge Y \rangle. \end{aligned} \tag{8.4}$$

Since $R\colon \wedge^2 V \to \wedge^2 V$ is symmetric, it follows from (8.4) that

$$\langle R(X \wedge Y), Z \wedge W \rangle = \langle R(Y \wedge Z), X \wedge W \rangle.$$

Thus, $\langle R(X \wedge Y), Z \wedge W\rangle$ is invariant under cyclic permutations of X, Y, Z, and hence $R = \mathfrak{b}(R) \in \wedge^4 V$.

Conversely, if $R \in \wedge^4 V$, then (2.13) and (8.1) imply that, for all $\sigma \in \mathrm{Gr}_2(V)$,

$$\sec_R(\sigma) = \langle R(\sigma), \sigma \rangle = \langle R, \sigma \wedge \sigma \rangle = 0,$$

concluding the proof. □

By Lemma 8.1, if $R \in S_{\mathfrak{b}}(\wedge^2 V)$ is an algebraic curvature operator, then for all $\omega \in \wedge^4 V$ the sectional curvature functions $\sec_R$ and $\sec_{R+\omega}$ coincide, since

$$\sec_{R+\omega}(\sigma) = \langle (R+\omega)(\sigma), \sigma \rangle = \langle R(\sigma), \sigma \rangle + \langle \omega(\sigma), \sigma \rangle = \sec_R(\sigma). \tag{8.5}$$

The operator $R + \omega$ is called a *modified curvature operator*.

Definition 8.2. An operator $R \in S(\wedge^2 V)$ has *strongly positive curvature* if there exists $\omega \in \wedge^4 V$ such that $(R + \omega)\colon \wedge^2 V \to \wedge^2 V$ is positive-definite. Similarly, $R \in S(\wedge^2 V)$ has *strongly nonnegative curvature* if there exists $\omega \in \wedge^4 V$ such that $R + \omega$ is positive-semidefinite.

Definition 8.3. A Riemannian manifold (M, g) has *strongly positive curvature* if, for all $p \in M$, the curvature operator $R_{\mathrm{g}}\colon \wedge^2 T_pM \to \wedge^2 T_pM$ has strongly positive curvature; and similarly for *strongly nonnegative curvature*.

In other words, strongly positive curvature is a pointwise curvature condition in the sense of Section 2.3, corresponding to the open $\mathsf{O}(n)$-invariant convex cone

$$C_{\text{str. pos.}} := \left\{ R \in S_{\mathfrak{b}}(\wedge^2 V) : \begin{array}{c} \text{there exists } \omega \in \wedge^4 V, \text{ such that} \\ \langle (R+\omega)(\alpha), \alpha \rangle > 0 \text{ for all } \alpha \in \wedge^2 V, \alpha \neq 0 \end{array} \right\}, \quad (8.6)$$

and analogously for strongly nonnegative curvature.

Proposition 8.4. *If* (M, g) *has positive-definite curvature operator, then it has strongly positive curvature. If* (M, g) *has strongly positive curvature, then it has* $\sec_{\mathrm{g}} > 0$.

Proof. If the curvature operator R_{g} is positive-definite, then it has strongly positive curvature, using, e.g., $\omega = 0$. From (8.5), if R_{g} has strongly positive curvature, then $\sec_{\mathrm{g}} > 0$. Alternatively, note that $C_{R>0} \subset C_{\text{str. pos.}} \subset C_{\sec>0}$. □

By Proposition 8.4, strongly positive curvature is an intermediate curvature condition between $R > 0$ and $\sec > 0$, and, analogously, strongly nonnegative curvature is an intermediate curvature condition between $R \geq 0$ and $\sec \geq 0$. In dimensions ≤ 3, these conditions are clearly equivalent.[1] Remarkably, strongly positive curvature and $\sec > 0$ remain equivalent in dimension 4, see Proposition 8.9, and analogously for nonnegative curvature.

Remark 8.5. From the above definition, if (M, g) has strongly positive curvature, then the assignment $M \ni p \mapsto \omega_p \in \wedge^4 T_pM$ such that $R + \omega$ is positive-definite may, in principle, fail to be smooth. However, since (8.6) is open, a standard perturbation argument shows that there exists a smooth 4-form $\widetilde{\omega} \in \wedge^4 TM$ such that $R + \widetilde{\omega}$ is positive-definite, see also Remark 8.7. The same argument does not work for strongly nonnegative curvature, as the corresponding cone is not open in $S_{\mathfrak{b}}(\wedge^2 V)$, see Remark 8.11 and Bettiol and Mendes [15, §6.4] for details.

[1]Recall that in dimensions ≤ 3, $\sec_{\mathrm{g}} > 0$ if and only if $R_{\mathrm{g}} > 0$, see Besse [12, 1.119].

Regarding the set of 4-forms that can be used to modify a given algebraic curvature operator to make it positive-definite, we have the following elementary result.

Proposition 8.6. *Given an operator $R \in S(\wedge^2 V)$ with strongly positive curvature, the set $\Omega_R := \{\omega \in \wedge^4 V : R + \omega > 0\}$ is open, bounded and convex.*

Proof. Openness of Ω_R is evident from the definition. For any $\omega \in \wedge^4 V$, the induced operator $\omega \colon \wedge^2 V \to \wedge^2 V$ via (2.13) clearly satisfies $\operatorname{tr} \omega = 0$, and hence has eigenvalues of both signs. Thus, if $\omega \in \Omega_R$, then $\lambda\omega \notin \Omega_R$ if $|\lambda|$ is sufficiently large, proving that Ω_R is bounded. Finally, convexity of Ω_R follows from the convexity of the subset of positive-definite operators. □

Remark 8.7. Since Ω_R is bounded and convex, it has a center of mass $\omega_R \in \Omega_R$. If (M, g) has strongly positive curvature, then the center of mass ω_R of $\Omega_R \subset \wedge^4 T_p M$ can be used to construct a *smooth* 4-form $\omega_R \in \wedge^4 TM$ such that $R + \omega_R$ is positive-definite.

We conclude this section proving that algebraic curvature operators can be realized as the curvature operator of a Riemannian manifold at one given point.

Proposition 8.8. *Let $R \in S_{\mathfrak{b}}(\wedge^2 V)$ be an algebraic curvature operator on V, with $\dim V = n$. Then there exists a smooth n-dimensional submanifold $M \subset V \times \mathbb{R}^k$, $k \leq \frac{1}{2}n(n-1)$, whose curvature operator at $(0,0) \in M$ is given by R.*

Proof. Since $\mathfrak{b}(R) = 0$, it follows from [53, p. 102], see also [92, p. 422], that there exist symmetric linear operators $H_j \colon V \to V$, $1 \leq j \leq k \leq \frac{1}{2}n(n-1)$, such that $R = -\sum_{j=1}^{k}(H_j \wedge H_j)$, where $(H_j \wedge H_j)(X \wedge Y) := H_j X \wedge H_j Y$. In particular,

$$\langle R(X \wedge Y), Z \wedge W \rangle = \sum_{j=1}^{k} \langle H_j X, W \rangle \langle H_j Y, Z \rangle - \langle H_j X, Z \rangle \langle H_j Y, W \rangle. \tag{8.7}$$

For each $1 \leq j \leq k$, consider the quadratic functions $f_j \colon V \times \mathbb{R} \to \mathbb{R}$, given by

$f_j(v, y_j) := y_j + \frac{1}{2}\langle H_j v, v\rangle$, and set

$$f\colon V \times \mathbb{R}^k \to \mathbb{R}^k, \quad f(v, y_1, \dots, y_k) := \big(f_1(v, y_1), \dots, f_k(v, y_k)\big).$$

Then $f(0,0) = 0$ and $0 \in \mathbb{R}^k$ is a regular value of f, hence $M := f^{-1}(0) \subset V \times \mathbb{R}^k$ is a smooth submanifold of codimension k. We have $T_{(0,0)}M = V$, and the second fundamental form $\mathrm{I\!I}\colon V \times V \to \mathbb{R}^k$ of M at this point is

$$\mathrm{I\!I}(v, w) = -\sum_{j=1}^{k} \big\langle \nabla_v(\mathrm{grad}\ f_j), w\big\rangle \mathrm{grad}\ f_j = -\sum_{j=1}^{k} \big\langle H_j v, w\big\rangle \mathrm{grad}\ f_j.$$

Thus, (8.7) is the Gauss equation of $M \subset V \times \mathbb{R}^k$, cf. (2.21), proving that its curvature operator at $(0,0)$, with the induced metric, is exactly R. □

8.2 Modified curvature operators in dimension 4

Modified curvature operators are particularly interesting in the lowest meaningful dimension, $\dim V = 4$. In this case, there is an isometry $\wedge^4 V \cong \mathbb{R}$ given by the Hodge star operator, since any $\omega \in \wedge^4 V$ is a multiple of the volume form of V. In particular, any $\omega \in \wedge^4 V$ determines an operator $\omega\colon \wedge^2 V \to \wedge^2 V$ via (2.13) which is a multiple of $*\colon \wedge^2 V \to \wedge^2 V$. Furthermore, strongly positive curvature and $\sec > 0$ are equivalent in dimension 4. This was originally proved by Thorpe [97]; however, a much simpler proof was communicated to us by Püttmann (see also [77]), as follows.

Proposition 8.9. *An operator $R \in S(\wedge^2 V)$ with $\dim V \leq 4$ has strongly positive curvature if and only if $\sec_R > 0$.*

Proof. The only nontrivial implication is that if $\dim V = 4$ and $\sec_R > 0$, then R has strongly positive curvature. For each modified curvature operator $(R+\omega) \in S(\wedge^2 V)$, denote by $\lambda_1(R+\omega) := \min \operatorname{Spec}(R+\omega)$ its smallest eigenvalue. Consider

$$\lambda := \sup_{\omega\in\wedge^4 V} \lambda_1(R+\omega). \tag{8.8}$$

Since the endomorphisms $\omega\colon \wedge^2 V \to \wedge^2 V$ are traceless by (2.13), the above supremum λ is achieved at some $\omega_{\max} \in \wedge^4 V$. Denote by E_λ the subspace of $\wedge^2 V$ formed by the eigenvectors of $R+\omega_{\max}$ with eigenvalue λ. If there exists $\sigma \in E_\lambda \cap \operatorname{Gr}_2(V)$, then $\lambda = \sec_R(\sigma) > 0$ and hence $R+\omega_{\max}$ is positive-definite. Otherwise, the map

$$q\colon \wedge^2 V \to \wedge^4 V \cong \mathbb{R}, \quad q(\alpha) := \alpha\wedge\alpha,$$

satisfies $q(\alpha) \neq 0$ for all nonzero $\alpha \in E_\lambda$. We claim that the image $q(E_\lambda \setminus \{0\}) \subset \mathbb{R}\setminus\{0\}$ is contained in a half-line, say $\mathbb{R}_+ := \{x > 0\}$. Indeed, this is clear if $\dim E_\lambda = 1$, and if $\dim E_\lambda \geq 2$, then $E_\lambda\setminus\{0\}$ is connected and hence so is $q(E_\lambda\setminus\{0\})$. Therefore, for any nonzero $\alpha \in E_\lambda$, we can construct a new modified curvature operator $R+\omega_{\max}+\alpha\wedge\alpha$ that satisfies

$$\big\langle (R+\omega_{\max}+\alpha\wedge\alpha)(\beta),\beta\big\rangle = \big\langle (R+\omega_{\max})(\beta),\beta\big\rangle + q(\alpha)\|\beta\|^2 > \lambda\|\beta\|^2,$$

for all $\beta\in\wedge^2 V$, contradicting the maximality (8.8) of λ. □

A completely analogous statement to Proposition 8.9 holds for nonnegative curvature, following the same proof. In addition, by Definition 8.3, we have:

Corollary 8.10. *A Riemannian manifold* (M,g) *with* $\dim M \leq 4$ *has strongly positive curvature if and only if* $\sec_{\mathrm{g}} > 0$*, and analogously for nonnegative curvature.*

Remark 8.11*.* In the case $\dim V = 4$, it follows from Proposition 8.9 and Thorpe [96, Thm. 2.1] that if $R \in S_{\mathfrak{b}}(\wedge^2 V)$ has strongly nonnegative curvature and there exists $\sigma\in\operatorname{Gr}_2(V)$ such that $\sec_R(\sigma) = 0$, then there exists a *unique* $\omega\in\wedge^4 V$ such that $R+\omega$ is positive-semidefinite, which is given by $\omega_{\max}$. In particular, if (M,g) is a 4-manifold with strongly nonnegative curvature, then any ω such that $R+\omega$ is positive-semidefinite is completely determined on the subset $\mathcal{Z} := \pi\big(\sec_{\mathrm{g}}^{-1}(0)\big)$, where $\pi\colon \mathrm{Gr}_2TM \to M$ is the bundle projection. In connection to Remark 8.5, this implies that smoothness of a 4-form $\omega = f\,\mathrm{vol}_{\mathrm{g}} \in \wedge^4 TM$ such that $R+\omega$ is positive-semidefinite is the same as smoothness of $f\colon \mathcal{Z} \to \mathbb{R}$. On the other hand, if $\dim V \geq 5$ and $R \in S_{\mathfrak{b}}(\wedge^2 V)$ has strong nonnegative curvature (but does not have strongly positive curvature), then the uniqueness of $\omega \in \wedge^4 V$ such that $R+\omega$ is positive-semidefinite may fail. For instance, the curvature operator of $S^4 \times S^1$ can be modified with any sufficiently small multiple of the volume form of S^4, remaining positive-semidefinite.

Recall that curvature operators of 4-manifolds decompose as (5.22). In terms of this decomposition, Proposition 8.9 yields the following statement regarding $\sec > 0$.

Corollary 8.12. *Let (M,g) be a Riemannian manifold with $\dim M = 4$. Then $\sec_{\mathrm{g}} > 0$ if and only if there exists a function $f\colon M \to \mathbb{R}$ such that $R_{\mathrm{g}} + f\,* > 0$, i.e.,*

$$\begin{pmatrix} W_{\mathrm{g}}^{+} + \left(\frac{\mathrm{scal}_{\mathrm{g}}}{12} + f\right)\mathrm{Id} & \mathrm{Ric}_{\mathrm{g}}^{0} \\ \left(\mathrm{Ric}_{\mathrm{g}}^{0}\right)^{\mathrm{t}} & W_{\mathrm{g}}^{-} + \left(\frac{\mathrm{scal}_{\mathrm{g}}}{12} - f\right)\mathrm{Id} \end{pmatrix} \tag{8.9}$$

is a positive-definite operator on $\wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM$.

Analogously to Corollary 8.12, a similar characterization of $\sec^{\perp} > 0$ also follows from Proposition 8.9 and (5.23).

Corollary 8.13. *Let (M,g) be a Riemannian manifold with $\dim M = 4$. Then $\sec_{\mathrm{g}}^{\perp} > 0$ if and only if there exists a function $f\colon M \to \mathbb{R}$ such that $\frac{1}{2}(R_{\mathrm{g}} + *R_{\mathrm{g}}*) + f* > 0$, i.e.,*

$$\begin{pmatrix} W_{\mathrm{g}}^{+} + \left(\frac{\mathrm{scal}_{\mathrm{g}}}{12} + f\right)\mathrm{Id} & 0 \\ 0 & W_{\mathrm{g}}^{-} + \left(\frac{\mathrm{scal}_{\mathrm{g}}}{12} - f\right)\mathrm{Id} \end{pmatrix} \tag{8.10}$$

is a positive-definite operator on $\wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM$.

We remark that the existence of f such that (8.9) is positive-definite forces a certain constraint on W_{g}, $\mathrm{scal}_{\mathrm{g}}$ and $\mathrm{Ric}^0_{\mathrm{g}}$, which, in the Einstein case[2] (8.10), is precisely the inequality described in Gursky and LeBrun [48, Lemma 1], cf. Theorem 5.18.

We conclude this section by describing the known examples of 4-manifolds with $\sec > 0$ and $\sec \geq 0$ endowed with standard metrics, under the light of strongly positive and nonnegative curvature. The corresponding modified curvature operators are written according to the decomposition $\wedge^2 TM = \wedge^2_+ TM \oplus \wedge^2_- TM$, i.e., as in (8.9).

The curvature operator of the round sphere S^4 is the identity, hence

$$R_{S^4} + f* = \begin{pmatrix} (1+f)\ \mathrm{Id} & 0 \\ 0 & (1-f)\ \mathrm{Id} \end{pmatrix}. \tag{8.11}$$

This modified curvature operator is positive-definite if and only if $-1 < f < 1$. In particular, at each $p \in S^4$, the set $\Omega_R \subset \wedge^4 T_p S^4 \cong \mathbb{R}$ of Proposition 8.6 is $\Omega_R = (-1, 1)$.

The modified curvature operator of $\mathbb{C}P^2$ is given by

$$R_{\mathbb{C}P^2} + f* = \begin{pmatrix} \mathrm{diag}(6+f, f, f) & 0 \\ 0 & (2-f)\ \mathrm{Id} \end{pmatrix}. \tag{8.12}$$

This operator is positive-definite if and only if $0 < f < 2$. In particular, at each $p \in \mathbb{C}P^2$, the set $\Omega_R \subset \wedge^4 T_p \mathbb{C}P^2 \cong \mathbb{R}$ of Proposition 8.6 is $\Omega_R = (0, 2)$, see Remark 8.21.

The curvature operator of the standard product metric on $S^2 \times S^2$ is described in Section 6.1, from which it follows that

$$R_{S^2 \times S^2} + f* = \begin{pmatrix} \mathrm{diag}(1+f, f, f) & 0 \\ 0 & \mathrm{diag}(1-f, -f, -f) \end{pmatrix}. \tag{8.13}$$

[2]Recall Proposition 5.17.

This operator is positive-semidefinite if and only if $f = 0$, cf. Remark 8.11.

Note that R_{S^4}, $R_{\mathbb{C}P^2}$ and $R_{S^2\times S^2}$ are Einstein, i.e., have $\mathrm{Ric}^0 = 0$, and are homogeneous, i.e., do not depend on the point $p \in M$.

Finally, we describe the curvature operator of $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, with the metric g_0 induced by the standard product metric under the submersion $\pi\colon S^3\times S^2 \to \mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, see Proposition 6.11. Differently from the previous examples, this metric g_0 is not homogeneous, hence the curvature operator is not constant. However, $\big(\mathbb{C}P^2\#\overline{\mathbb{C}P}^2, \mathsf{g}_0\big)$ admits an $\mathsf{SU}(2)$-action of cohomogeneity one (see Remark 6.13), so its curvature operator depends on only one parameter $h \in [-1, 1]$, that corresponds to the height on the second factor of $S^3 \times S^2$. More precisely, if $\widetilde{p} = \big((z_1, z_2), (z_3, h_0)\big) \in S^3 \times S^2$ and $p = \pi(\widetilde{p}) \in \mathbb{C}P^2\#\overline{\mathbb{C}P}^2$, then $h(p) = h_0 \in [-1, 1]$. With this notation,

$$R_{\mathbb{C}P^2\#\overline{\mathbb{C}P}^2} + f * = \begin{pmatrix} R_{11} + f\ \mathrm{Id} & R_{12} \\ \big(R_{12}\big)^{\mathsf{t}} & R_{22} - f\ \mathrm{Id} \end{pmatrix}, \tag{8.14}$$

where the blocks R_{11}, R_{12} and R_{22} are given by:

$$R_{11} = \mathrm{diag}\left(\tfrac{(1-h^2)\sqrt{2-h^2}-2h}{2(2-h^2)^{3/2}}, \tfrac{(1-h^2)\sqrt{2-h^2}-2h}{2(2-h^2)^{3/2}}, \tfrac{2h^4-7h^2+12+4h\sqrt{2-h^2}}{2(2-h^2)^2}\right),$$

$$R_{12} = \begin{pmatrix} 0 & 0 & \frac{1-h^2}{2(2-h^2)} \\ 0 & \frac{1-h^2}{2(2-h^2)} & 0 \\ \frac{7h^2-8}{2(2-h^2)^2} & 0 & 0 \end{pmatrix},$$

$$R_{22} = \mathrm{diag}\left(\tfrac{2h^4-7h^2+12-4h\sqrt{2-h^2}}{2(2-h^2)^2}, \tfrac{(1-h^2)\sqrt{2-h^2}+2h}{2(2-h^2)^{3/2}}, \tfrac{(1-h^2)\sqrt{2-h^2}+2h}{2(2-h^2)^{3/2}}\right).$$

It is not difficult to check that this operator is positive-semidefinite if and only if $f(h) = h(2-h^2)^{-3/2}$, cf. Remark 8.11.

8.3 Constructions regarding strongly positive curvature

In this section, we discuss the relation between strongly positive curvature and group actions, immersions, submersions and Cheeger deformations, following Bettiol and Mendes [15, §2]. We remark that Propositions 8.14 and 8.15 were also observed by Püttmann [77]. Although the results in this section are only stated for manifolds with strongly positive curvature, the analogous statements for strongly nonnegative curvature hold, and their proof is straightforward from the positive curvature case.

As the set of 4-forms that modify a curvature operator to become positive-definite is bounded and convex (see Proposition 8.6), a routine averaging technique yields:

Proposition 8.14. *Let (M, g) be a Riemannian manifold with strongly positive curvature, on which a compact Lie group G acts isometrically. Then there exists a G-invariant 4-form $\overline{\omega} \in \wedge^4 TM$ such that $R_{\mathsf{g}} + \overline{\omega}$ is positive-definite.*

Proof. Let $\omega \in \wedge^4 TM$ be such that $R_{\mathsf{g}} + \omega$ is positive-definite and let $\overline{\omega} := \int_{\mathsf{G}} g^*\omega \, \mathrm{d}g$ be the result of averaging it with the G-action. Since the G-action is isometric, R_{g} is G-invariant and hence $R_{\mathsf{g}} + \overline{\omega} = R_{\mathsf{g}} + \int_{\mathsf{G}} g^*\omega \, \mathrm{d}g = \int_{\mathsf{G}} g^*(R_{\mathsf{g}} + \omega) \, \mathrm{d}g$, which is positive-definite by convexity of the set of positive-definite operators. □

We now study the behavior of modified curvature operators under immersions and submersions, extending well-known results for $\sec > 0$ to strongly positive curvature.

Proposition 8.15. *Let $i \colon (M, \mathsf{g}) \to (\overline{M}, \overline{\mathsf{g}})$ be a totally geodesic immersion. If $(\overline{M}, \overline{\mathsf{g}})$ has strongly positive curvature, then also (M, g) has strongly positive curvature.*

Proof. Given $p \in M$, set $V = T_pM$ and $\overline{V} = T_{i(p)}\overline{M}$. For any $X \in V$, we write $\overline{X} = \mathrm{d}i(p)X \in \overline{V}$. Since $i \colon M \to \overline{M}$ is totally geodesic, from the Gauss formula (2.21), we have that $\langle R(X \wedge Y), Z \wedge W \rangle = \langle \overline{R}(\overline{X} \wedge \overline{Y}), \overline{Z} \wedge \overline{W} \rangle$. Thus, if there exists $\overline{\omega} \in \wedge^4 \overline{V}$ such that $\overline{R} + \overline{\omega}$ is positive-definite, then its restriction $\omega = (\mathrm{d}i(p))^* \overline{\omega} \in \wedge^4 V$ is such that $R + \omega$ is positive-definite. □

Theorem 8.16. *Let* $\pi\colon (\overline{M}, \overline{\mathrm{g}}) \to (M, \mathrm{g})$ *be a Riemannian submersion. If* $(\overline{M}, \overline{\mathrm{g}})$ *has strongly positive curvature, then also* (M, g) *has strongly positive curvature.*

Proof. Given that strongly positive curvature is a pointwise condition, choose $p \in M$ and $\overline{p} \in \overline{M}$ such that $\pi(\overline{p}) = p$, and set $V = T_pM$ and $\overline{V} = T_{\overline{p}}\overline{M}$. For any $X \in V$, we denote by $\overline{X} \in \overline{V}$ its horizontal lift and consider the inclusion map $i\colon V \hookrightarrow \overline{V}$, $i(X) = \overline{X}$, through which we identify V with a subspace of $\overline{V}$, that we call horizontal. We denote by $V^{\mathcal{V}}$ the orthogonal complement of this subspace, that we call vertical.

The tensor A of the submersion satisfies $A_XY = \frac{1}{2}[\overline{X}, \overline{Y}]^{\mathcal{V}}$ for all $X, Y \in V$, where $\overline{X}, \overline{Y}$ are horizontal lifts of local extensions of X, Y, see (2.25). Thus, A induces a skew-symmetric map $A\colon V \times V \to V^{\mathcal{V}}$, which can be interpreted as $A\colon \wedge^2 V \to V^{\mathcal{V}}$. Set $\alpha := A^*A \in S(\wedge^2 V)$, i.e., for all $X, Y, Z, W \in V$,

$$\langle \alpha(X \wedge Y), Z \wedge W\rangle = \langle A_XY, A_ZW\rangle. \tag{8.15}$$

Clearly, $\alpha\colon \wedge^2 V \to \wedge^2 V$ is a positive-semidefinite operator, whose rank is $\leq \dim V^{\mathcal{V}}$. From the Gray-O'Neill formula (2.27),

$$\begin{aligned}
\langle R(X \wedge Y), Z \wedge W\rangle &= \langle \overline{R}(\overline{X} \wedge \overline{Y}), \overline{Z} \wedge \overline{W}\rangle + 2\langle A_{\overline{X}}\overline{Y}, A_{\overline{Z}}\overline{W}\rangle \\
&\quad - \langle A_{\overline{Y}}\overline{Z}, A_{\overline{X}}\overline{W}\rangle + \langle A_{\overline{X}}\overline{Z}, A_{\overline{Y}}\overline{W}\rangle \\
&= \langle \overline{R}(\overline{X} \wedge \overline{Y}), \overline{Z} \wedge \overline{W}\rangle + 3\langle \alpha(\overline{X} \wedge \overline{Y}), \overline{Z} \wedge \overline{W}\rangle \\
&\quad - \langle \alpha(\overline{Y} \wedge \overline{Z}), \overline{X} \wedge \overline{W}\rangle - \langle \alpha(\overline{Z} \wedge \overline{X}), \overline{Y} \wedge \overline{W}\rangle \\
&\quad - \langle \alpha(\overline{X} \wedge \overline{Y}), \overline{Z} \wedge \overline{W}\rangle \\
&= \langle \overline{R}(\overline{X} \wedge \overline{Y}), \overline{Z} \wedge \overline{W}\rangle + 3\langle \alpha(\overline{X} \wedge \overline{Y}), \overline{Z} \wedge \overline{W}\rangle \\
&\quad - 3\mathfrak{b}(\alpha)(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}).
\end{aligned}$$

Thus, if there exists $\overline{\omega} \in \wedge^4\overline{V}$ such that $\overline{R} + \overline{\omega}$ is positive-definite, it follows that $R + \omega$ becomes positive-definite by setting $\omega = i^*\overline{\omega} + 3\mathfrak{b}(\alpha) \in \wedge^4 V$. □

Remark 8.17. The above way of rewriting the Gray-O'Neill formula for curvature operators as

$$\begin{aligned} \langle R(X\wedge Y), Z\wedge W\rangle &= \langle \overline{R}(\overline{X}\wedge\overline{Y}), \overline{Z}\wedge\overline{W}\rangle + 3\langle \alpha(\overline{X}\wedge\overline{Y}), \overline{Z}\wedge\overline{W}\rangle \\ &\quad - 3\mathfrak{b}(\alpha)(\overline{X},\overline{Y},\overline{Z},\overline{W}), \end{aligned} \tag{8.16}$$

where α is given by (8.15), seems more natural than its conventional presentation (2.27). Notice that formula (8.16) can also be deduced from Lemma 8.1 and the standard Gray-O'Neill formula (2.28), since these imply that R and $i^*\overline{R} + 3\,\alpha$ must differ by an element of $\wedge^4 V$, namely $\mathfrak{b}\big(R - i^*\overline{R} - 3\alpha\big) = -3\mathfrak{b}(\alpha)$.

Recall from (2.16) that the curvature operator of a compact Lie group (G, Q) is positive-semidefinite, hence (G, Q) has strongly nonnegative curvature. It follows easily from (2.18) that products of manifolds with strongly nonnegative curvature also have strongly nonnegative curvature. Thus, if (M, g) has strongly nonnegative curvature and an isometric action of a compact Lie group G, then the product manifold $\big(M\times\mathsf{G}, \mathsf{g}\oplus\frac{1}{t}Q\big)$ and hence the Cheeger deformation (M, g_t) have strongly nonnegative curvature for all $t \geq 0$, by (the nonnegative curvature version of) Theorem 8.16, cf. Proposition 4.5. Furthermore, just like Cheeger deformations preserve $\sec > 0$, they also preserve strongly positive curvature, as follows.

Proposition 8.18. *If (M, g) has strongly positive curvature and an isometric action of a compact Lie group G, then also the corresponding Cheeger deformation (M, g_t) has strongly positive curvature for all $t \geq 0$.*

Proof. In this proof, we use the same notation as in Chapter 4. Applying (8.16) to the Riemannian submersion $\rho\colon \big(M\times\mathsf{G}, \mathsf{g}\oplus\frac{1}{t}Q\big) \to (M, \mathsf{g}_t)$, we have that the curvature operator $R_t\colon \wedge^2 T_pM \to \wedge^2 T_pM$ of (M, g_t) is given by

$$\big\langle R_t\big(C_t^{-1}(X\wedge Y)\big), C_t^{-1}(Z\wedge W)\big\rangle_t = \langle R(X\wedge Y), Z\wedge W\rangle$$

$$+\frac{t^3}{4}Q\big([P_0X_{\mathfrak{m}}, P_0Y_{\mathfrak{m}}], [P_0Z_{\mathfrak{m}}, P_0W_{\mathfrak{m}}]\big)$$
$$+3\left\langle \alpha\left(\overline{C_t^{-1}X}\wedge\overline{C_t^{-1}Y}\right), \overline{C_t^{-1}Z}\wedge\overline{C_t^{-1}W}\right\rangle$$
$$-3\mathfrak{b}(\alpha)\left(\overline{C_t^{-1}X}, \overline{C_t^{-1}Y}, \overline{C_t^{-1}Z}, \overline{C_t^{-1}W}\right),$$

where $\alpha = A^*A$ is the positive-semidefinite operator given by (8.15). Similarly, the second term is a multiple of the quadratic form associated to the positive-semidefinite operator L^*L, where $L(X\wedge Y) = [P_0X_{\mathfrak{m}}, P_0Y_{\mathfrak{m}}]$. Notice that the above formula recovers the formula for sectional curvatures in Proposition 4.6.

If there exists ω such that $R+\omega$ is positive-definite, then setting ω_t so that

$$\omega_t\left(\overline{C_t^{-1}X}, \overline{C_t^{-1}Y}, \overline{C_t^{-1}Z}, \overline{C_t^{-1}W}\right) = \omega(X,Y,Z,W) + 3\mathfrak{b}(\alpha)\left(\overline{C_t^{-1}X}, \overline{C_t^{-1}Y}, \overline{C_t^{-1}Z}, \overline{C_t^{-1}W}\right),$$

it follows that $R_t+\omega_t$ is positive-definite for all $t\geq 0$, concluding the proof. □

Via the above formula for the curvature operator of (M, g_t), other results from Chapter 4 regarding evolution of sectional curvatures along Cheeger deformations can be transplanted to the realm of strongly positive (and nonnegative) curvature.

8.4 Homogeneous spaces

Let (G, Q) be a compact Lie group endowed with a bi-invariant metric. Let H be a closed subgroup, and denote by $\mathfrak{h}$ the corresponding Lie subalgebra of $\mathfrak{g}$. Define $\mathfrak{m}$ as the subspace such that $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ is a Q-orthogonal direct sum. Recall that the tangent space to the homogeneous space G/H at the identity class $(e\mathsf{H})\in\mathsf{G}/\mathsf{H}$ is identified with $\mathfrak{m}$, and the isotropy representation of H on $T_{(e\mathsf{H})}\mathsf{G}/\mathsf{H}\cong\mathfrak{m}$ corresponds to the adjoint representation $\mathrm{Ad}\colon\mathsf{H}\to\mathsf{SO}(\mathfrak{m})$. In particular, G-invariant metrics on G/H are in 1-to-1 correspondence with $\mathrm{Ad}(\mathsf{H})$-invariant inner products on $\mathfrak{m}$. Any such inner product $\langle\cdot,\cdot\rangle$ is determined by the Q-symmetric H-equivariant automorphism

$P\colon \mathfrak{m} \to \mathfrak{m}$ such that $\langle X, Y\rangle = Q(PX, Y)$.

The G-invariant metric on G/H induced by the inner product $Q|_{\mathfrak{m}}$, i.e., corresponding to $P = \mathrm{Id}$, is called a *normal homogeneous metric.* Consider the Riemannian submersion $\pi\colon (\mathsf{G}, Q) \to (\mathsf{G}/\mathsf{H}, Q_{\mathfrak{m}})$ given by the quotient map. As the horizontal lift of $X \in \mathfrak{m}$ is simply its inclusion $X \in \mathfrak{g}$ and the vertical projection $X^{\mathcal{V}}$ is given by its component $X_{\mathfrak{h}}$ in $\mathfrak{h}$, the corresponding operator $\alpha_{\mathsf{G}/\mathsf{H}} \in S(\wedge^2\mathfrak{m})$ defined in (8.15) can be computed using (2.25) as:

$$\big\langle \alpha_{\mathsf{G}/\mathsf{H}}(X \wedge Y), Z \wedge W\big\rangle = \tfrac{1}{4}Q\big([X, Y]_{\mathfrak{h}}, [Z, W]_{\mathfrak{h}}\big). \tag{8.17}$$

Thus, the curvature operator of the normal homogeneous space $(\mathsf{G}/\mathsf{H}, Q|_{\mathfrak{m}})$ is

$$\begin{aligned}\big\langle R_{\mathsf{G}/\mathsf{H}}(X \wedge Y), Z \wedge W\big\rangle = \tfrac{1}{4}Q([X, Y], [Z, W]) + \tfrac{3}{4}Q\big([X, Y]_{\mathfrak{h}}, [Z, W]_{\mathfrak{h}}\big)\\ - 3\mathfrak{b}(\alpha_{\mathsf{G}/\mathsf{H}})(X, Y, Z, W).\end{aligned} \tag{8.18}$$

As observed above, (G, Q) has strongly nonnegative curvature, and hence $(\mathsf{G}/\mathsf{H}, Q|_{\mathfrak{m}})$ also has strongly nonnegative curvature. Furthermore, it is clear from (8.18) that the operator $R_{\mathsf{G}/\mathsf{H}}\colon \wedge^2 \mathfrak{m} \to \wedge^2\mathfrak{m}$ can be modified with the 4-form $3\mathfrak{b}(\alpha_{\mathsf{G}/\mathsf{H}}) \in \wedge^4\mathfrak{m}$ to become positive-semidefinite.

More generally, if $P\colon \mathfrak{m} \to \mathfrak{m}$ is not the identity map, the curvature operator of the corresponding G-invariant metric on G/H can be computed in terms of the bilinear forms $B_{\pm}$ given by

$$B_{\pm}(X, Y) := \tfrac{1}{2}\big([X, PY] \mp [PX, Y]\big),$$

using a formalism due to Püttmann [77, Lemma 3.6]. Similarly to (8.15), define the positive-semidefinite operator $\beta \in S(\wedge^2\mathfrak{m})$, by

$$\big\langle \beta(X \wedge Y), Z \wedge W\big\rangle := \tfrac{1}{4}Q\big([X, Y]_{\mathfrak{m}}, [Z, W]_{\mathfrak{m}}\big), \tag{8.19}$$

where $X_{\mathfrak{m}}$ denotes the component of $X \in \mathfrak{g}$ in $\mathfrak{m}$. Rearranging the formula for the curvature operator R of G/H in terms of the Bianchi map (analogously to the proof of Theorem 8.16), one obtains the following expression:

$$\begin{aligned}\langle R(X \wedge Y), Z \wedge W\rangle = \tfrac{1}{2}\Big(&Q(B_-(X,Y),[Z,W]) + Q([X,Y],B_-(Z,W))\Big)\\ &+ Q(B_+(X,W),P^{-1}B_+(Y,Z))\\ &- Q(B_+(X,Z),P^{-1}B_+(Y,W))\\ &- 3\langle\beta(X\wedge Y),Z\wedge W\rangle + 3\mathfrak{b}(\beta)(X,Y,Z,W).\end{aligned} \tag{8.20}$$

Note that in the normal homogeneous case discussed above, $B_+(X,Y) = 0$ and $B_-(X,Y) = [X,Y]$, so formula (8.20) simplifies to $R_{\mathsf{G}/\mathsf{H}} = 4R_{\mathsf{G}} - 3\beta + 3\mathfrak{b}(\beta)$, where R_{G} is the curvature operator (2.16) of (G, Q). As $R_{\mathsf{G}} = \alpha + \beta$ by (8.17) and (8.19), it follows that this agrees with $R_{\mathsf{G}/\mathsf{H}} = R_{\mathsf{G}} + 3\alpha - 3\mathfrak{b}(\alpha)$, which is the formula in (8.18). Finally, we remark that the curvature operator (8.20) of a general G-invariant metric on G/H might fail to have strongly nonnegative curvature.

We conclude this section with the observation that the moduli space of G-invariant metrics with strongly nonnegative curvature on a homogeneous space G/H is path-connected (actually, in some sense, star-shaped). The corresponding statement regarding $\sec \geq 0$ was proved by Schwachhöfer and Tapp [82, Prop 1.1], as an application of Cheeger deformations, and our proof is based on the same method.

Theorem 8.19. *The moduli space of* G*-invariant metrics on a compact homogeneous space* G/H *with strongly nonnegative curvature is path-connected.*

Proof. Let $\mathsf{g}_* := Q|_{\mathfrak{m}}$ be a normal homogeneous metric. Given an invariant metric g on G/H with strongly nonnegative curvature, let g_t be the corresponding Cheeger deformation with respect to the left-translation G-action on G/H. Consider the path of metrics $(1+t)\,\mathsf{g}_t$ obtained by rescaling g_t. From (4.6) and (4.8), we have

$$\begin{aligned}(1+t)\,\mathsf{g}_t(X,Y) &= Q\big((1+t)P_t(X_\mathfrak{m}),Y_\mathfrak{m}\big)\\ &= Q\big((1+t)P_0\,(\mathrm{Id}+tP_0)^{-1}X_\mathfrak{m},Y_\mathfrak{m}\big)\\ &= Q\left(P_0\Big(\tfrac{1}{1+t}\,\mathrm{Id}+\tfrac{t}{1+t}P_0\Big)^{-1}X_\mathfrak{m},Y_\mathfrak{m}\right).\end{aligned}$$

As $t\nearrow+\infty$, the above clearly converges to $Q(X_\mathfrak{m},Y_\mathfrak{m})$. Thus, $(1+t)\,\mathsf{g}_t$ converges to g_* as $t\nearrow+\infty$. Evidently, $(1+t)\mathsf{g}_t$ converges to g as $t\searrow 0$. By the nonnegative curvature version of Proposition 8.18, each metric $(1+t)\,\mathsf{g}_t$ has strongly nonnegative curvature. Therefore, any two G-invariant metrics on G/H with strongly nonnegative curvature can be joined by a path of G-invariant metrics with the same property, passing through a normal homogeneous metric. □

8.5 Compact Rank One Symmetric Spaces

In this section, we discuss metrics with strongly positive (and nonnegative) curvature on the Compact Rank One Symmetric Spaces (CROSS). Recall that these are the homogeneous spaces G/K, where (G,K) is a symmetric pair of rank one, i.e.:

(i) Spheres $S^n=\mathsf{SO}(n+1)/\mathsf{SO}(n)$;

(ii) Complex projective spaces $\mathbb{C}P^n=\mathsf{SU}(n+1)/\mathsf{S}(\mathsf{U}(n)\mathsf{U}(1))$;

(iii) Quaternionic projective spaces $\mathbb{H}P^n=\mathsf{Sp}(n+1)/\mathsf{Sp}(n)$;

(iv) Cayley plane $\mathbb{C}\mathrm{a}P^2=\mathsf{F}_4/\mathsf{Spin}(9)$.

The standard metrics on the above manifolds are normal homogeneous metrics,[3] and hence have strongly nonnegative curvature (see Section 8.4). We now analyze in which cases the curvature operator of the above spaces can be further modified to

[3]Furthermore, the normal homogeneous metric in each of the above is the *unique* G-invariant metric, since the corresponding isotropy representation $\mathrm{Ad}(\mathsf{K})$ is irreducible.

become positive-definite. Throughout this section, S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{C}\mathrm{a}P^2$ are always assumed to be endowed with their standard metric.

Theorem 8.20. *The spaces S^n, $\mathbb{C}P^n$ and $\mathbb{H}P^n$ have strongly positive curvature.*

Proof. The curvature operator of S^n is the identity map $\mathrm{Id}\colon \wedge^2 TS^n \to \wedge^2 TS^n$, which is clearly positive-definite, hence S^n has strongly positive curvature. The curvature operators of $\mathbb{C}P^n$ and $\mathbb{H}P^n$ are positive-semidefinite, but have nontrivial kernel. However, since the Hopf bundles

$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n \quad \text{and} \quad S^3 \longrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n$$

are Riemannian submersions, Theorem 8.16 implies that $\mathbb{C}P^n$ and $\mathbb{H}P^n$ also have strongly positive curvature. □

Remark 8.21. In the case of $\mathbb{C}P^n$, the operator α defined in (8.15) can be computed to be $\omega_{\mathrm{FS}} \otimes \omega_{\mathrm{FS}}$, where ω_{FS} is the standard Kähler form. In particular, the 4-form $3\mathfrak{b}(\alpha) = \frac{1}{2}\omega_{\mathrm{FS}} \wedge \omega_{\mathrm{FS}}$ modifies the curvature operator of $\mathbb{C}P^n$ to become positive-definite. In the case $n = 2$ computed explicitly in (8.12), this 4-form corresponds precisely to the volume form of $\mathbb{C}P^2$, which is the center of mass $1 \in \Omega_R$.

An analogous statement holds for $\mathbb{H}P^n$, in terms of its hyper-Kähler structure.

Notice that the above argument does not apply to $\mathbb{C}\mathrm{a}P^2$, given that there are no submersions from round spheres to the Cayley plane.[4] The following result (combined with Theorem 8.16) provides an alternative proof of this fact.

Theorem 8.22. *The space $\mathbb{C}\mathrm{a}P^2$ does not have strongly positive curvature.*

Proof. Assume by contradiction that $\mathbb{C}\mathrm{a}P^2$ has strongly positive curvature. Then, by Proposition 8.14, there exists an F_4-invariant $\omega \in \wedge^4 T\mathbb{C}\mathrm{a}P^2$ such that $R + \omega$ is

[4]Even more, for topological reasons, there are no fiber bundles $\pi\colon S^n \to \mathbb{C}\mathrm{a}P^2$, see Browder [21].

positive-definite. Notice that $\omega \neq 0$, since R is positive-semidefinite but has nontrivial kernel. Since $\mathbb{C}aP^2 = \mathsf{F}_4/\mathsf{Spin}(9)$ is a compact symmetric space, ω is F_4-invariant if and only if it is harmonic, see e.g. Helgason [50, p. 227]. By Hodge theory, this implies that $[\omega] \in H^4(\mathbb{C}aP^2, \mathbb{R})$ is a nontrivial cohomology class, contradicting the fact that $b_4(\mathbb{C}aP^2, \mathbb{R}) = 0$. □

The first examples of algebraic curvature operators $R \in S_{\mathfrak{b}}(\wedge^2 V)$ with $\dim V \geq 5$, that have $\sec_R > 0$ but do not have strongly positive curvature were found by Zoltek [113]. As proved in Proposition 8.8, every algebraic curvature operator can be realized as the curvature operator of a Riemannian manifold *at one point*. Nevertheless, to our knowledge, no closed manifolds with $\sec > 0$ were known not to have strongly positive curvature, and, by Theorem 8.22, the Cayley plane $\mathbb{C}aP^2$ is one such example. Other examples can be found on homogeneous spaces such as W^{24}, B^{13}, S^{4n+3}, and S^{15}, see Theorems 9.8 and 9.11, and Remarks 9.12, 10.12 and 10.15; the two latter remarkably have $\sec > 0$ and do not have strongly *nonnegative* curvature.

We stress that the above does not imply that the manifold $\mathbb{C}aP^2$ does not carry *any* metrics with strongly positive curvature. In fact, this is a topic of current investigation by the author.

CHAPTER 9

HOMOGENEOUS CLASSIFICATION

The goal of this chapter is to provide the classification of simply-connected homogeneous spaces with strongly positive curvature, which follows from Bettiol and Mendes [14, 15]. More precisely, we prove the following result:

Theorem 9.1. *All simply-connected homogeneous spaces with* $\sec > 0$ *admit a homogeneous metric with strongly positive curvature, except for the Cayley plane* $\mathbb{C}\mathrm{a}P^2$.

The simply-connected homogeneous spaces that admit an invariant metric with $\sec > 0$ were classified in even dimensions by Wallach [102] and in odd dimensions by Bérard-Bergery [7], see also Aloff and Wallach [3], Berger [8], and Wilking and Ziller [107]. Apart from the CROSS described in Section 8.5, other examples appear in dimensions 6, 7, 12, 13 and 24. More explicitly, the complete list of simply-connected closed manifolds to admit a homogeneous metric with $\sec > 0$ is the following:

(i) Compact Rank One Symmetric Spaces: S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{C}\mathrm{a}P^2$;

(ii) Wallach flag manifolds: $W^6 = \mathsf{SU}(3)/\mathsf{T}^2$, $W^{12} = \mathsf{Sp}(3)/\mathsf{Sp}(1)\mathsf{Sp}(1)\mathsf{Sp}(1)$ and $W^{24} = \mathrm{F}_4/\mathsf{Spin}(8)$;

(iii) Aloff-Wallach spaces: $W^7_{k,\ell} = \mathsf{SU}(3)/\mathsf{S}^1_{k,\ell}$, $\gcd(k,\ell) = 1$, $k\ell(k+\ell) \neq 0$;

(iv) Aloff-Wallach space: $W^7_{1,1} = \mathsf{SU}(3)\mathsf{SO}(3)/\mathsf{U}(2)$;

(v) Berger spaces: $B^7 = \mathsf{SO}(5)/\mathsf{SO}(3)$ and $B^{13} = \mathsf{SU}(5)/\mathsf{Sp}(2)\cdot\mathsf{S}^1$.

For details on the construction of these spaces, see Sections 9.3, 9.4, 9.5, 9.6, 9.7, 9.8, and 10.1, as well as Ziller [112]. In order to prove Theorem 9.1, we show that all the above homogeneous spaces, except for $\mathbb{C}\mathrm{a}P^2$, have strongly positive curvature.

The case of CROSS follows from the results in Section 8.5. Namely, by Theorem 8.20, S^n, $\mathbb{C}P^n$ and $\mathbb{H}P^n$ endowed with their standard homogeneous metrics have strongly positive curvature. Furthermore, up to rescaling, $\mathbb{C}\mathrm{a}P^2 = \mathsf{F}_4/\mathsf{Spin}(9)$ admits a unique F_4-invariant metric (the normal homogeneous metric) because the corresponding isotropy representation is irreducible. Thus, by Theorem 8.22, there are no homogeneous metrics with strongly positive curvature on $\mathbb{C}\mathrm{a}P^2$.

Regarding the remaining spaces, with two exceptions, the construction of homogeneous metrics with strongly positive curvature uses the fact that they are the total space of a homogeneous fibration. In other words, the above spaces G/H are such that there exists an intermediate Lie group $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ and a homogeneous bundle

$$\mathsf{K}/\mathsf{H} \longrightarrow \mathsf{G}/\mathsf{H} \stackrel{\pi}{\longrightarrow} \mathsf{G}/\mathsf{K}, \quad \pi(g\mathsf{H}) = g\mathsf{K}. \tag{9.1}$$

A unified approach to construct homogeneous metrics with sec > 0 on G/H as above was discovered by Wallach [102, §7], see also Eschenburg [37] and Ziller [112, Prop. 4.3]. This approach, that we call Wallach's Theorem (see Theorem 9.2), gives sufficient conditions on (9.1) to imply that G/H has homogeneous metrics with sec > 0. Along Sections 9.1 and 9.2, we strengthen it to handle strongly positive curvature.

Denote by $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ the Lie algebras of $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$. With respect to the bi-invariant metric Q on G, consider the Q-orthogonal splittings

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad \text{and} \quad \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p}, \quad [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}, \tag{9.2}$$

so that there are natural identifications of the tangent spaces

$$\mathfrak{m} \cong T_{(e\mathsf{K})}\mathsf{G}/\mathsf{K}, \quad \mathfrak{p} \cong T_{(e\mathsf{H})}\mathsf{K}/\mathsf{H}, \quad \text{and} \quad \mathfrak{m} \oplus \mathfrak{p} \cong T_{(e\mathsf{H})}\mathsf{G}/\mathsf{H}.$$

With the above, we also identify Ad-invariant inner products on $\mathfrak{m}$ with the induced G-

invariant metrics on G/K, Ad-invariant elements of $\wedge^k \mathfrak{m}$ with the induced G-invariant forms in $\wedge^k T(\mathsf{G}/\mathsf{K})$, and analogously for the homogeneous spaces K/H and G/H. Consider the homogeneous metrics on G/H given by

$$\mathsf{g}_t := t\, Q|_{\mathfrak{p}} \oplus Q|_{\mathfrak{m}}, \quad t > 0, \tag{9.3}$$

so that g_1 is a normal homogeneous metric, and g_t is obtained by rescaling it by t in the vertical direction for (9.1). With this setup, Wallach's Theorem reads as follows:

Theorem 9.2. *Suppose that the homogeneous fibration* (9.1) *satisfies:*

(i) *the base* $(\mathsf{G}/\mathsf{K}, Q|_{\mathfrak{m}})$ *is a CROSS and* $(\mathfrak{g}, \mathfrak{k})$ *is a symmetric pair;*[1]

(ii) *the fiber* $(\mathsf{K}/\mathsf{H}, Q|_{\mathfrak{p}})$ *has* $\sec > 0$*;*

(iii) *the bundle is* fat.[2]

Then $(\mathsf{G}/\mathsf{H}, \mathsf{g}_t)$ *has* $\sec > 0$ *for all* $0 < t < 1$.

In order to strengthen the hypotheses above to yield that $(\mathsf{G}/\mathsf{H}, \mathsf{g}_t)$ has strongly positive curvature for all $0 < t < 1$, we introduce the notion of *strongly fat* homogeneous bundles in Section 9.1. The strengthened version of Wallach's Theorem (Theorem 9.5) is established in Section 9.2. Along Sections 9.3, 9.4, 9.6 and 9.8 we verify that the above listed homogeneous spaces with $\sec > 0$ satisfy its hypotheses, with two exceptions: the Berger space B^7 and the Wallach flag manifold W^{24}.

The Berger space B^7 does not have the structure (9.1) of a homogeneous bundle, and it is verified to have strongly positive curvature in Section 9.7 by direct inspection.

The Wallach flag manifold W^{24} admits the structure (9.1) of a homogeneous bundle, with base space given by the Cayley plane $\mathbb{C}aP^2$ with its standard metric. Endowing W^{24} with any of the metrics g_t defined in (9.3), the bundle projection

[1]Recall that $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair if and only if $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$.

[2]See Definition 9.3.

$\pi\colon (W^{24}, \mathrm{g}_t) \to \mathbb{C}\mathrm{a}P^2$ is a Riemannian submersion. Thus, by Theorems 8.16 and 8.22, the manifold (W^{24}, g_t) does not have strongly positive curvature. A direct construction of homogeneous metrics with strongly positive curvature on W^{24} is carried out in Section 9.5 using different methods, which were developed in Bettiol and Mendes [14].

9.1 Strong fatness

As introduced by Weinstein [104], a bundle is *fat* if all planes spanned by a vertical and a horizontal vector have positive curvature, see also Ziller [110]. In the above setup of homogeneous bundles, this property can be stated as follows, cf. (8.18).

Definition 9.3. The homogeneous bundle (9.1) is called *fat* if, for all $X \in \mathfrak{m}$ and $Y \in \mathfrak{p}$, we have that $\|[X,Y]\|^2 = 0$ implies $X = 0$ or $Y = 0$.

In order to establish the appropriate generalization to strong fatness, recall that

$$\begin{aligned} \wedge^2(\mathfrak{m} \oplus \mathfrak{p}) &= \wedge^2\mathfrak{m} \oplus \wedge^2\mathfrak{p} \oplus (\mathfrak{m} \otimes \mathfrak{p}), \\ \wedge^4(\mathfrak{m} \oplus \mathfrak{p}) &= \wedge^4\mathfrak{m} \oplus \wedge^4\mathfrak{p} \oplus (\wedge^3\mathfrak{m} \otimes \mathfrak{p}) \oplus (\wedge^2\mathfrak{m} \otimes \wedge^2\mathfrak{p}) \oplus (\mathfrak{m} \otimes \wedge^3\mathfrak{p}). \end{aligned} \tag{9.4}$$

Consider the linear map L given on decomposable elements of $\mathfrak{m} \otimes \mathfrak{p}$ by

$$L\colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m}, \quad L(X \wedge Y) := [X, Y], \tag{9.5}$$

and extended by linearity to the entire $\mathfrak{m} \otimes \mathfrak{p}$. This linear map induces the operator

$$F\colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m} \otimes \mathfrak{p}, \quad F := L^*L, \tag{9.6}$$

which is clearly positive-semidefinite and has nontrivial kernel[3] equal to $\ker L$. Given

[3] Unless $\dim \mathfrak{p} = 1$, in which case F can have trivial kernel, and hence be positive-definite.

that F is a self-adjoint linear operator on a subspace of $\wedge^2(\mathfrak{m}\oplus\mathfrak{p})$, we can to add to it a 4-form $\tau \in \wedge^2\mathfrak{m}\otimes\wedge^2\mathfrak{p} \subset \wedge^4(\mathfrak{m}\oplus\mathfrak{p})$, by using (2.13). This allows to state the appropriate strengthening of fatness to study strongly positive curvature as follows:

Definition 9.4. The homogeneous bundle (9.1) is called *strongly fat* if there exists $\tau \in \wedge^2\mathfrak{m}\otimes\wedge^2\mathfrak{p} \subset \wedge^4(\mathfrak{m}\oplus\mathfrak{p})$, such that the operator $(F+\tau)\colon \mathfrak{m}\otimes\mathfrak{p} \to \mathfrak{m}\otimes\mathfrak{p}$ is positive-definite.

Since decomposable elements of $\mathfrak{m}\otimes\mathfrak{p}$ are of the form $X\wedge Y$, where $X\in\mathfrak{m}$ and $Y\in\mathfrak{p}$, and $\big\langle (F+\tau)(X\wedge Y), X\wedge Y\big\rangle = \|L(X\wedge Y)\|^2 = \|[X,Y]\|^2$, strong fatness clearly implies fatness.

9.2 Strong Wallach Theorem

The goal of this section is to prove the following strengthening of Wallach's Theorem (Theorem 9.2), from Bettiol and Mendes [15, Thm. 4.2].

Theorem 9.5. *Suppose that the homogeneous fibration* (9.1) *satisfies:*

(i) *the base* $(\mathsf{G}/\mathsf{K}, Q|_{\mathfrak{m}})$ *is a CROSS different from* $\mathbb{C}\mathrm{a}P^2$ *and* $(\mathfrak{g},\mathfrak{k})$ *is a symmetric pair;*

(ii) *the fiber* $(\mathsf{K}/\mathsf{H}, Q|_{\mathfrak{p}})$ *has constant positive curvature, and either* $(\mathfrak{k},\mathfrak{h})$ *is a symmetric pair or* $\dim \mathsf{K}/\mathsf{H} \leq 3$*;*

(iii) *the bundle is strongly fat.*

Then $(\mathsf{G}/\mathsf{H}, \mathrm{g}_t)$ *has strongly positive curvature for all* $0<t<1$.

In the proof of this result, as well as in the remainder of this chapter, we make repeated use of the following fact that follows easily from Lemma 3.5.

Lemma 9.6. *Let V be a finite-dimensional real vector space, endowed with an inner product. Let A and B be symmetric operators on V, such that A is positive-semidefinite and $B\colon \ker A \to \ker A$ is positive-definite. Then there exists $s_*>0$ such that $A+sB$ is positive-definite for all $0<s<s_*$.*

Proof. Let $S_V = \{x \in V : \|x\| = 1\}$ be the unit sphere in V, and set

$$f\colon [0,1] \times S_V \to \mathbb{R}, \quad f(s,x) := \langle (A+sB)x, x\rangle.$$

Since A is positive-semidefinite, $f(0,x) \geq 0$. Furthermore, since $B\colon \ker A \to \ker A$ is positive-definite, $\frac{\partial f}{\partial s}(0,x)\big|_{s=0} = \langle Bx, x\rangle > 0$ for all x such that $f(0,x) = 0$. The result now follows from Lemma 3.5. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 9.5. The starting point to show that $(\mathsf{G}/\mathsf{H}, \mathrm{g}_t)$ has strongly positive curvature is to compute its curvature operator. This is done with the Riemannian submersions

$$\big(\mathsf{G}\times\mathsf{K}, Q \oplus \tfrac{1}{s}Q|_{\mathfrak{k}}\big) \xrightarrow{\pi_1} (\mathsf{G}, Q_t) \xrightarrow{\pi_2} (\mathsf{G}/\mathsf{H}, \mathrm{g}_t), \tag{9.7}$$

where π_2 is the quotient map and π_1 is of the form (4.2), corresponding to the fact that $Q_t := t\,Q|_{\mathfrak{k}} \oplus Q|_{\mathfrak{m}}$, $t < 1$, is the result of a Cheeger deformation of (G, Q) with respect to the K-action by left multiplication. From (4.8), it is easy to see that $t = \frac{1}{1+s}$, where s is the parameter of the Cheeger deformation. In particular, large values of $s = \frac{1-t}{t} > 0$ correspond to small values of $0 < t < 1$.

It follows from (4.12) that the horizontal lift of $X \in \mathfrak{g}$ with respect to the Riemannian submersion π_1 is $\overline{X} = \big(X_{\mathfrak{m}} + \frac{1}{1+s}X_{\mathfrak{k}}, -\frac{s}{1+s}X_{\mathfrak{k}}\big) = \big(X_{\mathfrak{m}} + tX_{\mathfrak{k}}, (t-1)X_{\mathfrak{k}}\big) \in \mathfrak{g}\oplus\mathfrak{k}$, where $X_{\mathfrak{m}}$ and $X_{\mathfrak{k}}$ respectively denote the components of $X \in \mathfrak{g}$ in $\mathfrak{m}$ and $\mathfrak{k}$. Thus, by (2.18) and (2.16), the curvature operator $R_{\mathsf{G}\times\mathsf{K}}$ of $\big(\mathsf{G}\times\mathsf{K}, Q\oplus\frac{1}{s}Q|_{\mathfrak{k}}\big)$ satisfies

$$\begin{aligned}
\langle R_{\mathsf{G}\times\mathsf{K}}(\overline{X}\wedge\overline{Y}), \overline{Z}\wedge\overline{W}\rangle &= \tfrac{1}{4}Q\big([X_{\mathfrak{m}} + tX_{\mathfrak{k}}, Y_{\mathfrak{m}} + tY_{\mathfrak{k}}], [Z_{\mathfrak{m}} + tZ_{\mathfrak{k}}, W_{\mathfrak{m}} + tW_{\mathfrak{k}}]\big) \\
&\quad + \tfrac{1}{4s}Q\big([(t-1)X_{\mathfrak{k}}, (t-1)Y_{\mathfrak{k}}], [(t-1)Z_{\mathfrak{k}}, (t-1)W_{\mathfrak{k}}]\big) \\
&= \tfrac{1}{4}Q\big([X_{\mathfrak{m}} + tX_{\mathfrak{k}}, Y_{\mathfrak{m}} + tY_{\mathfrak{k}}], [Z_{\mathfrak{m}} + tZ_{\mathfrak{k}}, W_{\mathfrak{m}} + tW_{\mathfrak{k}}]\big) \\
&\quad + \tfrac{t(1-t)^3}{4}Q\big([X_{\mathfrak{k}}, Y_{\mathfrak{k}}], [Z_{\mathfrak{k}}, W_{\mathfrak{k}}]\big).
\end{aligned}$$

Denote by α_1 and α_2 the positive-semidefinite operators on $\wedge^2(\mathfrak{m}\oplus\mathfrak{p})$ induced as in (8.15) by the tensors A of the Riemannian submersions π_1 and π_2 respectively. The tensor A_1 of π_1 can be computed using (2.25) and (9.2) as follows:

$$\begin{aligned}(A_1)_{\overline{X}}\overline{Y} &= \tfrac{1}{2}\big[\overline{X},\overline{Y}\big]^{\mathcal{V}}\\
&= \tfrac{1}{2}\Big(\big[X_{\mathfrak{m}}+tX_{\mathfrak{k}},Y_{\mathfrak{m}}+tY_{\mathfrak{k}}\big],\big[(1-t)X_{\mathfrak{k}},(1-t)Y_{\mathfrak{k}}\big]\Big)^{\mathcal{V}}\\
&= \tfrac{1}{2}\Big((1-t)\big[X_{\mathfrak{m}}+tX_{\mathfrak{k}},Y_{\mathfrak{m}}+tY_{\mathfrak{k}}\big]_{\mathfrak{k}}+t\big[(1-t)X_{\mathfrak{k}},(1-t)Y_{\mathfrak{k}}\big],\\
&\qquad (1-t)\big[X_{\mathfrak{m}}+tX_{\mathfrak{k}},Y_{\mathfrak{m}}+tY_{\mathfrak{k}}\big]_{\mathfrak{k}}+t\big[(1-t)X_{\mathfrak{k}},(1-t)Y_{\mathfrak{k}}\big]\Big)\\
&= \tfrac{1}{2}\Big((1-t)[X_{\mathfrak{m}},Y_{\mathfrak{m}}]+t(1-t)[X_{\mathfrak{k}},Y_{\mathfrak{k}}],(1-t)[X_{\mathfrak{m}},Y_{\mathfrak{m}}]+t(1-t)[X_{\mathfrak{k}},Y_{\mathfrak{k}}]\Big)\\
&= \tfrac{(1-t)}{2}\Big([X_{\mathfrak{m}},Y_{\mathfrak{m}}]+t[X_{\mathfrak{k}},Y_{\mathfrak{k}}],[X_{\mathfrak{m}},Y_{\mathfrak{m}}]+t[X_{\mathfrak{k}},Y_{\mathfrak{k}}]\Big).\end{aligned}$$

Thus, the operator α_1 is given by the following expression, where $X,Y,Z,W\in\mathfrak{m}\oplus\mathfrak{p}$,

$$\begin{aligned}\langle\alpha_1(\overline{X}\wedge\overline{Y}),\overline{Z}\wedge\overline{W}\rangle &= \langle A_{\overline{X}}\overline{Y},A_{\overline{Z}}\overline{W}\rangle\\
&= \big(1+\tfrac{1}{s}\big)\tfrac{(1-t)^2}{4}Q\big([X_{\mathfrak{m}},Y_{\mathfrak{m}}]+t[X_{\mathfrak{p}},Y_{\mathfrak{p}}],[Z_{\mathfrak{m}},W_{\mathfrak{m}}]+t[Z_{\mathfrak{p}},W_{\mathfrak{p}}]\big)\\
&= \tfrac{1-t}{4}Q\big([X_{\mathfrak{m}},Y_{\mathfrak{m}}]+t[X_{\mathfrak{p}},Y_{\mathfrak{p}}],[Z_{\mathfrak{m}},W_{\mathfrak{m}}]+t[Z_{\mathfrak{p}},W_{\mathfrak{p}}]\big).\end{aligned}$$

The tensor A_2 of π_2 can also be computed using (2.25), resulting

$$(A_2)_XY=\tfrac{1}{2}[X,Y]_{\mathfrak{h}},$$

where $X_{\mathfrak{h}}$ denotes the component of X in $\mathfrak{h}$. Thus, the operator α_2 is given by the following expression, where $X,Y,Z,W\in\mathfrak{m}\oplus\mathfrak{p}$, cf. (8.17),

$$\langle\alpha_2(X\wedge Y),Z\wedge W\rangle=\tfrac{t}{4}Q\big([X,Y]_{\mathfrak{h}},[Z,W]_{\mathfrak{h}}\big).$$

Thus, applying twice formula (8.16) with the setup (9.7), we obtain that the curvature operator $R_t\colon \wedge^2(\mathfrak{m}\oplus\mathfrak{p}) \to \wedge^2(\mathfrak{m}\oplus\mathfrak{p})$ of $(\mathsf{G}/\mathsf{H}, \mathrm{g}_t)$ is given by

$$\begin{aligned}
\langle R_t(X\wedge Y), Z\wedge W\rangle_t &= \langle R_{\mathsf{G}\times\mathsf{K}}(\overline{X}\wedge\overline{Y}), \overline{Z}\wedge\overline{W}\rangle \\
&\quad + 3\langle \alpha_1(\overline{X}\wedge\overline{Y}), \overline{Z}\wedge\overline{W}\rangle + 3\langle\alpha_2(X\wedge Y), Z\wedge W\rangle \\
&\quad - 3\mathfrak{b}(\alpha_1)(\overline{X},\overline{Y},\overline{Z},\overline{W}) - 3\mathfrak{b}(\alpha_2)(X,Y,Z,W) \\
&= \tfrac{1}{4}Q\big([X_\mathfrak{m}+tX_\mathfrak{p}, Y_\mathfrak{m}+tY_\mathfrak{p}], [Z_\mathfrak{m}+tZ_\mathfrak{p}, W_\mathfrak{m}+tW_\mathfrak{p}]\big) \qquad (9.8)\\
&\quad + \tfrac{t(1-t)^3}{4}Q\big([X_\mathfrak{p}, Y_\mathfrak{p}], [Z_\mathfrak{p}, W_\mathfrak{p}]\big) \\
&\quad + \tfrac{3(1-t)}{4}Q\big([X_\mathfrak{m}, Y_\mathfrak{m}] + t[X_\mathfrak{p}, Y_\mathfrak{p}], [Z_\mathfrak{m}, W_\mathfrak{m}] + t[Z_\mathfrak{p}, W_\mathfrak{p}]\big) \\
&\quad + \tfrac{3t}{4}Q\big([X,Y]_\mathfrak{h}, [Z,W]_\mathfrak{h}\big) \\
&\quad - 3\mathfrak{b}(\alpha_1)(\overline{X},\overline{Y},\overline{Z},\overline{W}) - 3\mathfrak{b}(\alpha_2)(X,Y,Z,W).
\end{aligned}$$

Note that in the limit $t\to 1$, the above coincides with the formula (8.18) for the curvature operator of the normal homogeneous space $(\mathsf{G}/\mathsf{H}, \mathrm{g}_1)$. Fix $0<t<1$ and consider the positive-semidefinite operator given by

$$\widehat{R} := R_t + 3\mathfrak{b}(\alpha_1) + 3\mathfrak{b}(\alpha_2)\colon \wedge^2(\mathfrak{m}\oplus\mathfrak{p}) \to \wedge^2(\mathfrak{m}\oplus\mathfrak{p}).$$

Then $(\mathsf{G}/\mathsf{H}, \mathrm{g}_t)$ has strongly positive curvature if and only if there exists $\omega\in\wedge^4(\mathfrak{m}\oplus\mathfrak{p})$ such that $\widehat{R}+\omega$ is positive-definite.

Using the natural decomposition (9.4), we can write $\widehat{R}$ in blocks as follows:

$$\begin{array}{c} \\ \wedge^2\mathfrak{m} \\ \wedge^2\mathfrak{p} \\ \mathfrak{m}\otimes\mathfrak{p} \end{array}
\begin{array}{c} \begin{array}{ccc} \wedge^2\mathfrak{m} & \wedge^2\mathfrak{p} & \mathfrak{m}\otimes\mathfrak{p} \end{array} \\
\begin{pmatrix} \widehat{R}_{11} & \widehat{R}_{12} & 0 \\ \widehat{R}_{12}^{\mathrm{t}} & \widehat{R}_{22} & 0 \\ 0 & 0 & \widehat{R}_{33} \end{pmatrix} \end{array} \qquad (9.9)$$

The zeros above are obtained directly from (9.2) and (9.8), using that $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{k}$, since G/K is a CROSS. Moreover, the symmetric positive-semidefinite operators

$$\widehat{R}_{11}\colon \wedge^2\mathfrak{m}\to\wedge^2\mathfrak{m},\quad \widehat{R}_{22}\colon \wedge^2\mathfrak{p}\to\wedge^2\mathfrak{p},\quad \widehat{R}_{33}\colon \mathfrak{m}\otimes\mathfrak{p}\to\mathfrak{m}\otimes\mathfrak{p}$$

can be explicitly computed from (9.8) as follows:

$$\begin{aligned}
\big\langle \widehat{R}_{11}(X\wedge Y), Z\wedge W\big\rangle_t &= (1-\tfrac{3t}{4})Q\big([X,Y],[Z,W]\big)+\tfrac{3t}{4}Q\big([X,Y]_{\mathfrak{h}},[Z,W]_{\mathfrak{h}}\big),\\
\big\langle \widehat{R}_{22}(X\wedge Y), Z\wedge W\big\rangle_t &= \tfrac{t}{4}Q\big([X,Y],[Z,W]\big)+\tfrac{3t}{4}Q\big([X,Y]_{\mathfrak{h}},[Z,W]_{\mathfrak{h}}\big),\\
\big\langle \widehat{R}_{33}(X\wedge Y), Z\wedge W\big\rangle_t &= \tfrac{t^2}{4}Q\big([X,Y],[Z,W]\big).
\end{aligned}\tag{9.10}$$

We now use the hypotheses on G/K and K/H to relate the their curvature operators, given by (8.17) and (8.18), with the above. Since $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{k}$, we have that $\alpha_{\mathsf{G}/\mathsf{K}}$ is a multiple of $R_{\mathsf{G}/\mathsf{K}}$ and hence $\mathfrak{b}(\alpha_{\mathsf{G}/\mathsf{K}})=0$. Analogously, if K/H is locally isometric to a symmetric space, then $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{h}$ and hence $\alpha_{\mathsf{K}/\mathsf{H}}$ is a multiple of $R_{\mathsf{K}/\mathsf{H}}$, so $\mathfrak{b}(\alpha_{\mathsf{K}/\mathsf{H}})=0$. Else, $\dim\mathsf{K}/\mathsf{H}\leq 3$ and hence $\mathfrak{b}(\alpha_{\mathsf{K}/\mathsf{H}})\in\wedge^4\mathfrak{p}=\{0\}$.

In either case, $\mathfrak{b}(\alpha_{\mathsf{G}/\mathsf{K}})=0$ and $\mathfrak{b}(\alpha_{\mathsf{K}/\mathsf{H}})=0$. Consequently, we get the following:

$$\begin{aligned}
\big\langle \widehat{R}_{11}(X\wedge Y), Z\wedge W\big\rangle_t &= (1-\tfrac{3t}{4})\big\langle R_{\mathsf{G}/\mathsf{K}}(X\wedge Y), Z\wedge W\big\rangle+\tfrac{3t}{4}Q\big([X,Y]_{\mathfrak{h}},[Z,W]_{\mathfrak{h}}\big),\\
\big\langle \widehat{R}_{22}(X\wedge Y), Z\wedge W\big\rangle_t &= t\,\big\langle R_{\mathsf{K}/\mathsf{H}}(X\wedge Y), Z\wedge W\big\rangle,\\
\big\langle \widehat{R}_{33}(X\wedge Y), Z\wedge W\big\rangle_t &= \tfrac{t^2}{4}\big\langle F(X\wedge Y), Z\wedge W\big\rangle,
\end{aligned}\tag{9.11}$$

where F is related to strong fatness, and given by (9.6).

Let us first analyze the restriction of $\widehat{R}$ to $\wedge^2\mathfrak{m}\oplus\wedge^2\mathfrak{p}$, i.e., the upper 2×2 block of (9.9). Since K/H has constant positive curvature, we have that $\widehat{R}_{22}=R_{\mathsf{K}/\mathsf{H}}$ is positive-definite. Thus, the kernel of the positive-semidefinite operator

$$\widehat{R}\colon \wedge^2\mathfrak{m}\oplus\wedge^2\mathfrak{p}\to\wedge^2\mathfrak{m}\oplus\wedge^2\mathfrak{p}\tag{9.12}$$

must be contained in $\wedge^2\mathfrak{m}$. It follows from Theorem 8.20 that, since G/K is a CROSS different from $\mathbb{C}\mathrm{a}P^2$, there exists $\eta \in \wedge^4\mathfrak{m}$ such that $R_{\mathsf{G}/\mathsf{K}} + \eta$, and hence $\widehat{R}_{11} + \eta$, is positive-definite. In particular, we have that

$$\begin{array}{c} \\ \wedge^2\mathfrak{m} \\ \wedge^2\mathfrak{p} \\ \mathfrak{m}\otimes\mathfrak{p} \end{array} \begin{array}{c} \begin{array}{ccc} \wedge^2\mathfrak{m} & \wedge^2\mathfrak{p} & \mathfrak{m}\otimes\mathfrak{p} \end{array} \\ \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

is positive-definite on the kernel of (9.12). Thus, Lemma 9.6 implies that there exists an $\varepsilon_1 > 0$ such that $\widehat{R} + \varepsilon_1\eta$ is positive-definite on $\wedge^2\mathfrak{m} \oplus \wedge^2\mathfrak{p}$.

Finally, we analyze the positive-semidefinite operator

$$\big(\widehat{R} + \varepsilon_1\eta\big)\colon \ \wedge^2 (\mathfrak{m}\oplus\mathfrak{p}) \to \wedge^2(\mathfrak{m}\oplus\mathfrak{p}). \tag{9.13}$$

Since its restriction to $\wedge^2\mathfrak{m}\oplus\wedge^2\mathfrak{p}$ is positive-definite, its kernel must lie in $\mathfrak{m}\otimes\mathfrak{p}$. The restriction of (9.13) to this subspace coincides with $\widehat{R}_{33} = \frac{t^2}{4}F$. By strong fatness, there exists $\tau \in \wedge^2\mathfrak{m}\otimes\wedge^2\mathfrak{p}$ such that $F+\tau$ is positive-definite on $\mathfrak{m}\otimes\mathfrak{p}$. In particular,

$$\begin{array}{c} \\ \wedge^2\mathfrak{m} \\ \wedge^2\mathfrak{p} \\ \mathfrak{m}\otimes\mathfrak{p} \end{array} \begin{array}{c} \begin{array}{ccc} \wedge^2\mathfrak{m} & \wedge^2\mathfrak{p} & \mathfrak{m}\otimes\mathfrak{p} \end{array} \\ \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & \tau \end{pmatrix} \end{array}$$

is positive-definite on the kernel of (9.13). Thus, Lemma 9.6 implies that there exists an $\varepsilon_2 > 0$ such that $\widehat{R} + \varepsilon_1\eta + \varepsilon_2\tau$ is positive-definite on $\wedge^2(\mathfrak{m}\oplus\mathfrak{p})$. In other words, $\omega = \varepsilon_1\eta + \varepsilon_2\tau \in \wedge^4(\mathfrak{m}\oplus\mathfrak{p})$ is such that the modified curvature operator $\widehat{R}+\omega$ is positive-definite, hence $(\mathsf{G}/\mathsf{H}, \mathrm{g}_t)$ has strongly positive curvature. □

9.3 Wallach flag manifold W^6

Let the Lie groups $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ be given by $\mathsf{T}^2 \subset \mathsf{U}(2) \subset \mathsf{SU}(3)$, where:

$$\mathsf{U}(2) = \left\{\operatorname{diag}\left(A, \det \overline{A}\right) \in \mathsf{SU}(3) : A \in \mathsf{U}(2)\right\}$$
$$\mathsf{T}^2 = \left\{\operatorname{diag}(z_1, z_2, \overline{z}_1\overline{z}_2) : z_j \in \mathsf{S}^1\right\}$$

The corresponding homogeneous bundle (9.1) is

$$\mathbb{C}P^1 \longrightarrow W^6 \longrightarrow \mathbb{C}P^2.$$

The base $\mathbb{C}P^2$ is a CROSS, the fiber $\mathbb{C}P^1 \cong S^2\left(\frac{1}{2}\right)$ has constant positive curvature and dimension ≤ 3, and both $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{k}, \mathfrak{h})$ are symmetric pairs.

The only other condition needed to apply Theorem 9.5 is strong fatness. Up to rescaling, the bi-invariant metric on $\mathfrak{g} = \mathfrak{su}(3)$ is given by

$$Q(X, Y) = -\tfrac{1}{2} \operatorname{Re} \operatorname{tr}(XY). \tag{9.14}$$

The Q-orthogonal complements defined in (9.2) are $\mathfrak{m} \cong \mathbb{C}^2$ and $\mathfrak{p} \cong \mathbb{C}$, and can be identified as the subspaces $\mathfrak{m} = \operatorname{span}\{\mathbb{1}_1, \mathrm{I}_1, \mathbb{1}_2, \mathrm{I}_2\}$ and $\mathfrak{p} = \operatorname{span}\{\mathbb{1}_3, \mathrm{I}_3\}$ of $\mathfrak{su}(3)$, where $\mathbb{1}_r, \mathrm{I}_r$, $1 \leq r \leq 3$, are the Q-orthonormal matrices given by:

$$\mathbb{1}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \mathbb{1}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathbb{1}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\mathrm{I}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \quad \mathrm{I}_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \mathrm{I}_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9.15}$$

The Lie bracket operator $L\colon \mathfrak{m}\otimes\mathfrak{p}\to\mathfrak{m}$ defined in (9.5) is determined by:

$[\cdot,\cdot]$	$\mathbb{1}_3$	I_3
$\mathbb{1}_1$	$\mathbb{1}_2$	$-\mathrm{I}_2$
I_1	$-\mathrm{I}_2$	$-\mathbb{1}_2$
$\mathbb{1}_2$	$-\mathbb{1}_1$	I_1
I_2	I_1	$\mathbb{1}_1$

Thus, by (9.6), $\ker F = \ker L$ is spanned by the following 4 vectors of $\mathfrak{m}\otimes\mathfrak{p}$:

$$\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3,\quad -\mathbb{1}_1\wedge\mathrm{I}_3+\mathrm{I}_1\wedge\mathbb{1}_3,\quad \mathbb{1}_2\wedge\mathbb{1}_3+\mathrm{I}_2\wedge\mathrm{I}_3,\quad -\mathbb{1}_2\wedge\mathrm{I}_3+\mathrm{I}_2\wedge\mathbb{1}_3.$$

Consider the symmetric operator induced by the H-invariant 4-form $\tau\in\wedge^2\mathfrak{m}\otimes\wedge^2\mathfrak{p}$,

$$\tau = -(\mathbb{1}_1\wedge\mathrm{I}_1+\mathbb{1}_2\wedge\mathrm{I}_2)\otimes(\mathbb{1}_3\wedge\mathrm{I}_3).$$

The restriction $\tau\colon \ker F\to\ker F$ is the identity operator,[4] hence positive-definite. From Lemma 9.6, there exists $\varepsilon>0$ such that $(F+\varepsilon\tau)\colon \mathfrak{m}\otimes\mathfrak{p}\to\mathfrak{m}\otimes\mathfrak{p}$ is positive-definite, proving strong fatness. Therefore, by Theorem 9.5, the homogeneous space (W^6,g_t) has strongly positive curvature for all $0<t<1$.

[4]This fact follows from representation theoretic arguments, since $\tau\in\wedge^2\mathfrak{m}\otimes\wedge^2\mathfrak{p}$ being H-invariant is equivalent to $\tau\colon\mathfrak{m}\otimes\mathfrak{p}\to\mathfrak{m}\otimes\mathfrak{p}$ being H-equivariant, see (2.13). This can also be easily verified using the above list of vectors that span $\ker F$. Note that such vectors are orthogonal, but have square length 2, e.g., $\|\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3\|^2=\|\mathbb{1}_1\wedge\mathbb{1}_3\|^2+\|\mathrm{I}_1\wedge\mathrm{I}_3\|^2=2$. Thus, we have:

$$\begin{aligned}
&\langle\tau(\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3),\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3\rangle = 2\langle\tau,\mathbb{1}_1\wedge\mathbb{1}_3\wedge\mathrm{I}_1\wedge\mathrm{I}_3\rangle = 2 = \|\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3\|^2,\\
&\langle\tau(\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3),-\mathbb{1}_1\wedge\mathrm{I}_3+\mathrm{I}_1\wedge\mathbb{1}_3\rangle = \langle\tau,0\rangle = 0,\\
&\langle\tau(\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3),\mathbb{1}_2\wedge\mathbb{1}_3+\mathrm{I}_2\wedge\mathrm{I}_3\rangle = \langle\tau,\mathbb{1}_1\wedge\mathbb{1}_3\wedge\mathrm{I}_2\wedge\mathrm{I}_3+\mathrm{I}_1\wedge\mathrm{I}_3\wedge\mathbb{1}_2\wedge\mathbb{1}_3\rangle = 0,\\
&\langle\tau(\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3),-\mathbb{1}_2\wedge\mathrm{I}_3+\mathrm{I}_2\wedge\mathbb{1}_3\rangle = \langle\tau,-\mathbb{1}_1\wedge\mathbb{1}_3\wedge\mathbb{1}_2\wedge\mathrm{I}_3-\mathrm{I}_1\wedge\mathrm{I}_3\wedge\mathrm{I}_2\wedge\mathbb{1}_3\rangle = 0,
\end{aligned}$$

where the last two inner products can be easily seen to vanish since τ is the sum of decomposable elements with only 2 distinct indices each, while the decomposable elements on the other slot have 3 distinct indices each. Therefore, $\tau(\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3)=\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3$, and analogously for the other vectors listed above that span $\ker F$.

9.4 Wallach flag manifold W^{12}

Let $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ be given by $\mathsf{Sp}(1)\mathsf{Sp}(1)\mathsf{Sp}(1) \subset \mathsf{Sp}(2)\mathsf{Sp}(1) \subset \mathsf{Sp}(3)$, where:

$$\mathsf{Sp}(2)\mathsf{Sp}(1) = \big\{\operatorname{diag}(A, q) : A \in \mathsf{Sp}(2), q \in \mathsf{Sp}(1)\big\}$$

$$\mathsf{Sp}(1)\mathsf{Sp}(1)\mathsf{Sp}(1) = \big\{\operatorname{diag}(q_1, q_2, q_3) : q_j \in \mathsf{Sp}(1)\big\}$$

The corresponding homogeneous bundle (9.1) is

$$\mathbb{H}P^1 \longrightarrow W^{12} \longrightarrow \mathbb{H}P^2.$$

The base $\mathbb{H}P^2$ is a CROSS, the fiber $\mathbb{H}P^1 \cong S^4\big(\tfrac{1}{2}\big)$ has constant positive curvature, and both $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{k}, \mathfrak{h})$ are symmetric pairs.

The only other condition needed to apply Theorem 9.5 is strong fatness. Up to rescaling, the bi-invariant metric on $\mathfrak{g} = \mathfrak{sp}(3)$ is given by the same formula as (9.14). The Q-orthogonal complements defined in (9.2) are $\mathfrak{m} \cong \mathbb{H}^2$ and $\mathfrak{p} \cong \mathbb{H}$, and can be identified as the subspaces $\mathfrak{m} = \operatorname{span}\{\mathbb{1}_1, \mathrm{I}_1, \mathrm{J}_1, \mathrm{K}_1, \mathbb{1}_2, \mathrm{I}_2, \mathrm{J}_2, \mathrm{K}_2\}$ and $\mathfrak{p} = \operatorname{span}\{\mathbb{1}_3, \mathrm{I}_3, \mathrm{J}_3, \mathrm{K}_3\}$ of $\mathfrak{sp}(3)$, where the matrices $\mathbb{1}_r, \mathrm{I}_r$, $1 \leq r \leq 3$, are defined in (9.15) and the remaining matrices $\mathrm{J}_r, \mathrm{K}_r$, $1 \leq r \leq 3$, are defined analogously using the other imaginary quaternion units j and k, that is:

$$\begin{aligned}
\mathrm{J}_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & j \\ 0 & j & 0 \end{pmatrix} \qquad \mathrm{J}_2 = \begin{pmatrix} 0 & 0 & j \\ 0 & 0 & 0 \\ j & 0 & 0 \end{pmatrix} \qquad \mathrm{J}_3 = \begin{pmatrix} 0 & j & 0 \\ j & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\mathrm{K}_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k \\ 0 & k & 0 \end{pmatrix} \qquad \mathrm{K}_2 = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix} \qquad \mathrm{K}_3 = \begin{pmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{9.16}$$

The Lie bracket operator $L\colon \mathfrak{m}\otimes\mathfrak{p}\to\mathfrak{m}$ defined in (9.5) is determined by:

$[\cdot,\cdot]$	$\mathbb{1}_3$	I_3	J_3	K_3
$\mathbb{1}_1$	$\mathbb{1}_2$	$-\mathrm{I}_2$	$-\mathrm{J}_2$	$-\mathrm{K}_2$
I_1	$-\mathrm{I}_2$	$-\mathbb{1}_2$	K_2	$-\mathrm{J}_2$
J_1	$-\mathrm{J}_2$	$-\mathrm{K}_2$	$-\mathbb{1}_2$	I_2
K_1	$-\mathrm{K}_2$	J_2	$-\mathrm{I}_2$	$-\mathbb{1}_2$
$\mathbb{1}_2$	$-\mathbb{1}_1$	I_1	J_1	K_1
I_2	I_1	$\mathbb{1}_1$	K_1	$-\mathrm{J}_1$
J_2	J_1	$-\mathrm{K}_1$	$\mathbb{1}_1$	I_1
K_2	K_1	J_1	$-\mathrm{I}_1$	$\mathbb{1}_1$

Thus, by (9.6), $\ker F = \ker L$ is spanned by the following 24 vectors of $\mathfrak{m}\otimes\mathfrak{p}$:

$$\begin{array}{ccc}
\mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{K}_1\wedge\mathrm{K}_3, & -\mathbb{1}_1\wedge\mathrm{I}_3+\mathrm{K}_1\wedge\mathrm{J}_3, & \mathbb{1}_1\wedge\mathrm{J}_3+\mathrm{K}_1\wedge\mathrm{I}_3,\\
-\mathbb{1}_1\wedge\mathrm{K}_3+\mathrm{K}_1\wedge\mathbb{1}_3, & \mathbb{1}_1\wedge\mathrm{I}_3+\mathrm{J}_1\wedge\mathrm{K}_3, & \mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{J}_1\wedge\mathrm{J}_3,\\
-\mathbb{1}_1\wedge\mathrm{K}_3+\mathrm{J}_1\wedge\mathrm{I}_3, & -\mathbb{1}_1\wedge\mathrm{J}_3+\mathrm{J}_1\wedge\mathbb{1}_3, & -\mathbb{1}_1\wedge\mathrm{J}_3+\mathrm{I}_1\wedge\mathrm{K}_3,\\
\mathbb{1}_1\wedge\mathrm{K}_3+\mathrm{I}_1\wedge\mathrm{J}_3, & \mathbb{1}_1\wedge\mathbb{1}_3+\mathrm{I}_1\wedge\mathrm{I}_3, & -\mathbb{1}_1\wedge\mathrm{I}_3+\mathrm{I}_1\wedge\mathbb{1}_3,\\
\mathbb{1}_2\wedge\mathbb{1}_3+\mathrm{K}_2\wedge\mathrm{K}_3, & \mathbb{1}_2\wedge\mathrm{I}_3+\mathrm{K}_2\wedge\mathrm{J}_3, & -\mathbb{1}_2\wedge\mathrm{J}_3+\mathrm{K}_2\wedge\mathrm{I}_3,\\
-\mathbb{1}_2\wedge\mathrm{K}_3+\mathrm{K}_2\wedge\mathbb{1}_3, & -\mathbb{1}_2\wedge\mathrm{I}_3+\mathrm{J}_2\wedge\mathrm{K}_3, & \mathbb{1}_2\wedge\mathbb{1}_3+\mathrm{J}_2\wedge\mathrm{J}_3,\\
\mathbb{1}_2\wedge\mathrm{K}_3+\mathrm{J}_2\wedge\mathrm{I}_3, & -\mathbb{1}_2\wedge\mathrm{J}_3+\mathrm{J}_2\wedge\mathbb{1}_3, & \mathbb{1}_2\wedge\mathrm{J}_3+\mathrm{I}_2\wedge\mathrm{K}_3,\\
-\mathbb{1}_2\wedge\mathrm{K}_3+\mathrm{I}_2\wedge\mathrm{J}_3, & \mathbb{1}_2\wedge\mathbb{1}_3+\mathrm{I}_2\wedge\mathrm{I}_3, & -\mathbb{1}_2\wedge\mathrm{I}_3+\mathrm{I}_2\wedge\mathbb{1}_3.
\end{array}$$

Consider the symmetric operator induced by the H-invariant 4-form $\tau\in\wedge^2\mathfrak{m}\otimes\wedge^2\mathfrak{p}$,

$$\begin{aligned}
\tau &= (\mathbb{1}_1\wedge\mathrm{I}_1+\mathrm{J}_1\wedge\mathrm{K}_1)\otimes(\mathrm{J}_3\wedge\mathrm{K}_3-\mathbb{1}_3\wedge\mathrm{I}_3)\\
&\quad+(\mathrm{I}_1\wedge\mathrm{K}_1-\mathbb{1}_1\wedge\mathrm{J}_1)\otimes(\mathbb{1}_3\wedge\mathrm{J}_3+\mathrm{I}_3\wedge\mathrm{K}_3)
\end{aligned}$$

$$+ (\mathbb{1}_1 \wedge \mathrm{K}_1 + \mathrm{I}_1 \wedge \mathrm{J}_1) \otimes (\mathrm{I}_3 \wedge \mathrm{J}_3 - \mathbb{1}_3 \wedge \mathrm{K}_3)$$
$$- (\mathbb{1}_2 \wedge \mathrm{I}_2 - \mathrm{J}_2 \wedge \mathrm{K}_2) \otimes (\mathbb{1}_3 \wedge \mathrm{I}_3 + \mathrm{J}_3 \wedge \mathrm{K}_3)$$
$$- (\mathbb{1}_2 \wedge \mathrm{J}_2 + \mathrm{I}_2 \wedge \mathrm{K}_2) \otimes (\mathbb{1}_3 \wedge \mathrm{J}_3 - \mathrm{I}_3 \wedge \mathrm{K}_3)$$
$$- (\mathbb{1}_2 \wedge \mathrm{K}_2 - \mathrm{I}_2 \wedge \mathrm{J}_2) \otimes (\mathbb{1}_3 \wedge \mathrm{K}_3 + \mathrm{I}_3 \wedge \mathrm{J}_3).$$

The restriction $\tau\colon \ker F \to \ker F$ is the identity operator, hence positive-definite. From Lemma 9.6, there exists $\varepsilon > 0$ such that $(F + \varepsilon\tau)\colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m} \otimes \mathfrak{p}$ is positive-definite, proving strong fatness. Therefore, by Theorem 9.5, the homogeneous space (W^{12}, g_t) has strongly positive curvature for all $0 < t < 1$.

9.5 Wallach flag manifold W^{24}

Let the Lie groups $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ be given by $\mathsf{Spin}(8) \subset \mathsf{Spin}(9) \subset \mathsf{F}_4$. In order to explain these inclusions, recall that the exceptional Lie group F_4 can be realized as the automorphism group of the exceptional Jordan algebra

$$\mathfrak{h}_3(\mathbb{C}\mathrm{a}) = \big\{ H \in \mathrm{Mat}_{3\times 3}(\mathbb{C}\mathrm{a}) : H^* = H \big\},$$

of Hermitian 3×3 octonionic matrices. As described in Baez [5, §3.4], this algebra can also be constructed using the scalar, vector and spinor representations of $\mathsf{Spin}(9)$, which is hence a subgroup of F_4, see also Harvey [49, Thm. 14.99]. The Lie algebra $\mathfrak{f}_4$ is the algebra of derivations of $\mathfrak{h}_3(\mathbb{C}\mathrm{a})$; in particular, $\mathfrak{f}_4$ contains a copy of the algebra $\mathfrak{g}_2$ of derivations of $\mathbb{C}\mathrm{a}$, which is a subalgebra of $\mathfrak{so}(7) \cong \mathfrak{so}(\mathrm{Im}(\mathbb{C}\mathrm{a}))$, and hence also a subalgebra of the Lie algebra $\mathfrak{so}(8) \cong \mathfrak{so}(\mathbb{C}\mathrm{a})$ of H. More precisely, there is a splitting

$$\mathfrak{f}_4 = \mathfrak{sa}_3(\mathbb{C}\mathrm{a}) \oplus \mathfrak{g}_2, \tag{9.17}$$

where $\mathfrak{sa}_3(\mathbb{C}\mathrm{a}) := \big\{A \in \mathrm{Mat}_{3\times 3}(\mathbb{C}\mathrm{a}) : A^* = -A, \mathrm{tr}(A) = 0\big\}$. Up to rescaling, the bi-invariant metric on $\mathfrak{f}_4$ is given by

$$Q(X,Y) = \tfrac{1}{2}\,\mathrm{Re}(\mathrm{tr}(X_1 Y_1^*)) + \tfrac{1}{8}\,\mathrm{tr}(X_2 Y_2^*),$$

where $X_1 \in \mathfrak{sa}_3(\mathbb{C}\mathrm{a})$ and $X_2 \in \mathfrak{g}_2 \subset \mathfrak{so}(7)$ denote the components of $X \in \mathfrak{f}_4$, according to (9.17). For more details on the above, including formulas for the Lie bracket on $\mathfrak{f}_4$, see Baez [5, §4.2].

The homogeneous bundle (9.1) determined by the above Lie groups $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ is

$$\mathbb{C}\mathrm{a}P^1 \longrightarrow W^{24} \longrightarrow \mathbb{C}\mathrm{a}P^2. \tag{9.18}$$

The base $\mathbb{C}\mathrm{a}P^2$ is a CROSS, the fiber $\mathbb{C}\mathrm{a}P^1 \cong S^8\big(\frac{1}{2}\big)$ has constant positive curvature, and $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair.[5] However, since the base is $\mathbb{C}\mathrm{a}P^2$, Theorem 9.5 cannot be applied. In fact, the above projection becomes a Riemannian submersion if W^{24} is endowed with a metric of the form g_t defined in (9.3). From Theorems 8.16 and 8.22, it follows that (W^{24}, g_t) does not have strongly positive curvature.

Nevertheless, since (9.7) is a Riemannian submersion and $\big(\mathsf{G} \times \mathsf{K}, Q \oplus \frac{1}{s} Q|_{\mathfrak{k}}\big)$ is the product of Riemannian manifolds with strongly nonnegative curvature, it follows that (W^{24}, g_t) has strongly nonnegative curvature for all $0 < t < 1$. In what follows, we prove that arbitrarily small perturbations of g_t in the space of F_4-invariant metrics (with normalized volume) have strongly positive curvature. Furthermore, we actually compute the moduli spaces of F_4-invariant metrics on W^{24} with strongly positive and nonnegative curvature in Theorems 9.9 and 9.11, see also Section 10.1.

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a Q-orthogonal splitting. The isotropy representation of $\mathsf{Spin}(8)$ on $\mathfrak{m}$ has 3 irreducible factors $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$, each isomorphic to $\mathbb{C}\mathrm{a}$ as a real

[5] We remark that $(\mathfrak{k}, \mathfrak{h})$ is *not* a symmetric pair in this case, although $\mathsf{K}/\mathsf{H} \cong S^8\big(\frac{1}{2}\big)$.

vector space. More precisely, $\mathfrak{m} \cong \mathbb{C}\mathrm{a}^3$ can be identified as the subspace of $\mathfrak{sa}_3(\mathbb{C}\mathrm{a})$ spanned by the 24 matrices

$$\mathbb{1}_r,\ \mathrm{I}_r,\ \mathrm{J}_r,\ \mathrm{K}_r,\ \mathrm{L}_r,\ \mathrm{M}_r,\ \mathrm{N}_r,\ \mathrm{O}_r, \qquad 1 \leq r \leq 3,$$

where $\mathbb{1}_r$, I_r, J_r, K_r are as in (9.15) and (9.16), and the remaining matrices L_r, M_r, N_r, O_r, $1 \leq r \leq 3$, are defined analogously, using the other imaginary octonion units l, m, n, and o. Since $\mathfrak{m}_r$, $1 \leq r \leq 3$, are irreducible and nonisomorphic, Schur's Lemma implies that G-invariant metrics on G/H are parametrized by 3 positive numbers. We denote these by $\vec{s} = (s_1, s_2, s_3)$, so that the corresponding metric is given by

$$\mathrm{g}_{\vec{s}} := s_1^2\, Q|_{\mathfrak{m}_1} \oplus s_2^2\, Q|_{\mathfrak{m}_2} \oplus s_3^2\, Q|_{\mathfrak{m}_3}, \quad \text{at } T_{(e\mathsf{H})}\mathsf{G}/\mathsf{H} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3.$$

Remark 9.7. The Lie algebra of the intermediate subgroup K is given by $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}_3$. As $\mathrm{g}_{\vec{s}}$ is $\mathrm{Ad}(\mathsf{K})$-invariant if and only $s_1 = s_2$, the projection (9.18) is a Riemannian submersion if and only if[6] $s_1 = s_2$. Note that the metrics defined in (9.3) are given by $\mathrm{g}_{\vec{s}}$ with $\vec{s} = (1, 1, t)$, hence clearly satisfy the latter.

A modified curvature operator of $(W^{24}, \mathrm{g}_{\vec{s}})$ is an H-equivariant operator on $\wedge^2\mathfrak{m}$. Thus, by Schur's Lemma, it decomposes as a block diagonal operator, whose blocks correspond to the isotypic components[7] of $\wedge^2\mathfrak{m}$. Let $\oplus^n V$ be one such isotypic component, where V is an irreducible H-representation of dimension d that appears n times in the decomposition of $\wedge^2\mathfrak{m}$ into irreducible factors. In a suitable basis, the block of a modified curvature operator corresponding to $\oplus^n V$ is of the form $A \otimes \mathrm{Id}$, where A

[6] More generally, for each $1 \leq r \leq 3$, there exists a copy of K in G whose Lie algebra is $\mathfrak{h} \oplus \mathfrak{m}_r$. Each such choice for K determines a bundle map (9.18) which is a Riemannian submersion onto the corresponding Cayley plane if and only if $s_{r+1} = s_{r+2}$, where indices are taken modulo 3.

[7] Recall that an H-representation can be decomposed as the sum of irreducible factors, and some of these factors may be isomorphic (i.e., appear with multiplicity larger than 1). An *isotypic component* of such a representation is the sum of all copies of an irreducible factor. For irreducible factors that have multiplicity 1, the isotypic component is just the irreducible factor itself.

is an $n \times n$ symmetric matrix and Id is the $d \times d$ identity matrix. In particular, such a block is positive-definite if and only if A is positive-definite, and analogously for positive-semidefiniteness. In order to compute each of the symmetric matrices A, we choose a *representative* vector $v_1 \neq 0$ in one copy of V, and then produce additional representatives $v_2, v_3, \ldots$ in the remaining copies of V by taking images of v_1 under isomorphisms of representations. Proceeding in this way through all isotypic components, we obtain a *complete list* of representatives $\{v_i\}$, such that the restriction of $R+\omega$ to the space of representatives $\operatorname{span}\{v_i\}$ is positive-definite if and only if $R+\omega$ is positive-definite, and analogously for positive-semidefiniteness.

A computation with weights shows that the representation of $\mathsf{Spin}(8)$ on $\wedge^2\mathfrak{m}$ decomposes as the sum of 3 copies of an irreducible representation of dimension 28, 3 distinct irreducible representations of dimension 56, and 3 distinct irreducible representations of dimension 8. The following 9 representative vectors form a *complete list*, as described above (in what follows, subindices r are always taken modulo 3):

$$\begin{aligned}
&\tfrac{1}{\sqrt{2}}\big(\mathrm{J}_r \wedge \mathrm{L}_r + \mathrm{K}_r \wedge \mathrm{M}_r\big), && 1 \leq r \leq 3,\\
&\tfrac{1}{2\sqrt{2}}\big(\mathbb{1}_{r+1} \wedge \mathbb{1}_{r+2} - \mathrm{I}_{r+1} \wedge \mathrm{I}_{r+2} - \mathrm{J}_{r+1} \wedge \mathrm{J}_{r+2} - \mathrm{K}_{r+1} \wedge \mathrm{K}_{r+2}\\
&\qquad - \mathrm{L}_{r+1} \wedge \mathrm{L}_{r+2} - \mathrm{M}_{r+1} \wedge \mathrm{M}_{r+2} - \mathrm{N}_{r+1} \wedge \mathrm{N}_{r+2} - \mathrm{O}_{r+1} \wedge \mathrm{O}_{r+2}\big), && 1 \leq r \leq 3,\\
&\tfrac{1}{\sqrt{2}}\big(\mathbb{1}_{r+1} \wedge \mathbb{1}_{r+2} + \mathrm{O}_{r+1} \wedge \mathrm{O}_{r+2}\big), && 1 \leq r \leq 3.
\end{aligned}$$

Furthermore, the following determine a basis of the H-invariant elements of $\wedge^4\mathfrak{m}$:

$$\begin{aligned}
\omega_r :=\, &(\mathbb{1}_{r+1} \wedge \mathrm{I}_{r+1} - \mathrm{J}_{r+1} \wedge \mathrm{K}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{I}_{r+2} + \mathrm{J}_{r+2} \wedge \mathrm{K}_{r+2})\\
&+(\mathbb{1}_{r+1} \wedge \mathrm{J}_{r+1} + \mathrm{I}_{r+1} \wedge \mathrm{K}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{J}_{r+2} - \mathrm{I}_{r+2} \wedge \mathrm{K}_{r+2})\\
&+(\mathbb{1}_{r+1} \wedge \mathrm{K}_{r+1} - \mathrm{I}_{r+1} \wedge \mathrm{J}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{K}_{r+2} + \mathrm{I}_{r+2} \wedge \mathrm{J}_{r+2})\\
&+(\mathbb{1}_{r+1} \wedge \mathrm{L}_{r+1} + \mathrm{I}_{r+1} \wedge \mathrm{M}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{L}_{r+2} - \mathrm{I}_{r+2} \wedge \mathrm{M}_{r+2})\\
&+(\mathbb{1}_{r+1} \wedge \mathrm{M}_{r+1} - \mathrm{I}_{r+1} \wedge \mathrm{L}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{M}_{r+2} + \mathrm{I}_{r+2} \wedge \mathrm{L}_{r+2})
\end{aligned}$$

$$+(\mathbb{1}_{r+1} \wedge \mathrm{N}_{r+1} - \mathrm{I}_{r+1} \wedge \mathrm{O}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{N}_{r+2} + \mathrm{I}_{r+2} \wedge \mathrm{O}_{r+2})$$

$$+(\mathbb{1}_{r+1} \wedge \mathrm{O}_{r+1} + \mathrm{I}_{r+1} \wedge \mathrm{N}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{O}_{r+2} - \mathrm{I}_{r+2} \wedge \mathrm{N}_{r+2})$$

$$+(\mathbb{1}_{r+1} \wedge \mathrm{I}_{r+1} + \mathrm{J}_{r+1} \wedge \mathrm{K}_{r+1}) \wedge (\mathrm{L}_{r+2} \wedge \mathrm{M}_{r+2} - \mathrm{N}_{r+2} \wedge \mathrm{O}_{r+2})$$

$$+(\mathbb{1}_{r+1} \wedge \mathrm{J}_{r+1} - \mathrm{I}_{r+1} \wedge \mathrm{K}_{r+1}) \wedge (\mathrm{L}_{r+2} \wedge \mathrm{N}_{r+2} + \mathrm{M}_{r+2} \wedge \mathrm{O}_{r+2})$$

$$+(\mathbb{1}_{r+1} \wedge \mathrm{K}_{r+1} + \mathrm{I}_{r+1} \wedge \mathrm{J}_{r+1}) \wedge (\mathrm{L}_{r+2} \wedge \mathrm{O}_{r+2} - \mathrm{M}_{r+2} \wedge \mathrm{N}_{r+2})$$

$$-(\mathbb{1}_{r+1} \wedge \mathrm{L}_{r+1} - \mathrm{I}_{r+1} \wedge \mathrm{M}_{r+1}) \wedge (\mathrm{J}_{r+2} \wedge \mathrm{N}_{r+2} + \mathrm{K}_{r+2} \wedge \mathrm{O}_{r+2})$$

$$-(\mathbb{1}_{r+1} \wedge \mathrm{M}_{r+1} + \mathrm{I}_{r+1} \wedge \mathrm{L}_{r+1}) \wedge (\mathrm{J}_{r+2} \wedge \mathrm{O}_{r+2} - \mathrm{K}_{r+2} \wedge \mathrm{N}_{r+2})$$

$$+(\mathbb{1}_{r+1} \wedge \mathrm{N}_{r+1} + \mathrm{I}_{r+1} \wedge \mathrm{O}_{r+1}) \wedge (\mathrm{J}_{r+2} \wedge \mathrm{L}_{r+2} - \mathrm{K}_{r+2} \wedge \mathrm{M}_{r+2})$$

$$+(\mathbb{1}_{r+1} \wedge \mathrm{O}_{r+1} - \mathrm{I}_{r+1} \wedge \mathrm{N}_{r+1}) \wedge (\mathrm{J}_{r+2} \wedge \mathrm{M}_{r+2} + \mathrm{K}_{r+2} \wedge \mathrm{L}_{r+2})$$

$$-(\mathrm{J}_{r+1} \wedge \mathrm{L}_{r+1} - \mathrm{K}_{r+1} \wedge \mathrm{M}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{N}_{r+2} - \mathrm{I}_{r+2} \wedge \mathrm{O}_{r+2})$$

$$-(\mathrm{J}_{r+1} \wedge \mathrm{M}_{r+1} + \mathrm{K}_{r+1} \wedge \mathrm{L}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{O}_{r+2} + \mathrm{I}_{r+2} \wedge \mathrm{N}_{r+2})$$

$$+(\mathrm{J}_{r+1} \wedge \mathrm{N}_{r+1} + \mathrm{K}_{r+1} \wedge \mathrm{O}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{L}_{r+2} + \mathrm{I}_{r+2} \wedge \mathrm{M}_{r+2})$$

$$+(\mathrm{J}_{r+1} \wedge \mathrm{O}_{r+1} - \mathrm{K}_{r+1} \wedge \mathrm{N}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{M}_{r+2} - \mathrm{I}_{r+2} \wedge \mathrm{L}_{r+2})$$

$$-(\mathrm{L}_{r+1} \wedge \mathrm{M}_{r+1} - \mathrm{N}_{r+1} \wedge \mathrm{O}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{I}_{r+2} - \mathrm{J}_{r+2} \wedge \mathrm{K}_{r+2})$$

$$-(\mathrm{L}_{r+1} \wedge \mathrm{N}_{r+1} + \mathrm{M}_{r+1} \wedge \mathrm{O}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{J}_{r+2} + \mathrm{I}_{r+2} \wedge \mathrm{K}_{r+2})$$

$$-(\mathrm{L}_{r+1} \wedge \mathrm{O}_{r+1} - \mathrm{M}_{r+1} \wedge \mathrm{N}_{r+1}) \wedge (\mathbb{1}_{r+2} \wedge \mathrm{K}_{r+2} - \mathrm{I}_{r+2} \wedge \mathrm{J}_{r+2})$$

$$-(\mathrm{J}_{r+1} \wedge \mathrm{L}_{r+1} + \mathrm{K}_{r+1} \wedge \mathrm{M}_{r+1}) \wedge (\mathrm{J}_{r+2} \wedge \mathrm{L}_{r+2} + \mathrm{K}_{r+2} \wedge \mathrm{M}_{r+2})$$

$$-(\mathrm{J}_{r+1} \wedge \mathrm{M}_{r+1} - \mathrm{K}_{r+1} \wedge \mathrm{L}_{r+1}) \wedge (\mathrm{J}_{r+2} \wedge \mathrm{M}_{r+2} - \mathrm{K}_{r+2} \wedge \mathrm{L}_{r+2})$$

$$-(\mathrm{J}_{r+1} \wedge \mathrm{N}_{r+1} - \mathrm{K}_{r+1} \wedge \mathrm{O}_{r+1}) \wedge (\mathrm{J}_{r+2} \wedge \mathrm{N}_{r+2} - \mathrm{K}_{r+2} \wedge \mathrm{O}_{r+2})$$

$$-(\mathrm{J}_{r+1} \wedge \mathrm{O}_{r+1} + \mathrm{K}_{r+1} \wedge \mathrm{N}_{r+1}) \wedge (\mathrm{J}_{r+2} \wedge \mathrm{O}_{r+2} + \mathrm{K}_{r+2} \wedge \mathrm{N}_{r+2})$$

$$-(\mathrm{L}_{r+1} \wedge \mathrm{M}_{r+1} + \mathrm{N}_{r+1} \wedge \mathrm{O}_{r+1}) \wedge (\mathrm{L}_{r+2} \wedge \mathrm{M}_{r+2} + \mathrm{N}_{r+2} \wedge \mathrm{O}_{r+2})$$

$$-(\mathrm{L}_{r+1} \wedge \mathrm{N}_{r+1} - \mathrm{M}_{r+1} \wedge \mathrm{O}_{r+1}) \wedge (\mathrm{L}_{r+2} \wedge \mathrm{N}_{r+2} - \mathrm{M}_{r+2} \wedge \mathrm{O}_{r+2})$$

$$-(\mathrm{L}_{r+1} \wedge \mathrm{O}_{r+1} + \mathrm{M}_{r+1} \wedge \mathrm{N}_{r+1}) \wedge (\mathrm{L}_{r+2} \wedge \mathrm{O}_{r+2} + \mathrm{M}_{r+2} \wedge \mathrm{N}_{r+2}),\ 1 \leq r \leq 3.$$

Thus, a general invariant 4-form $\omega \in \wedge^4\mathfrak{m}$ has coordinates $\vec{a} = (a_1, a_2, a_3)$, such that

$$\omega = a_1\,\omega_1 + a_2\,\omega_2 + a_3\,\omega_3. \tag{9.19}$$

The restriction $\widehat{R}(\vec{s}, \vec{a})$ of a modified curvature operator $(R + \omega)\colon \wedge^2\mathfrak{m} \to \wedge^2\mathfrak{m}$ to the subspace spanned by the above representative vectors can be computed using formula (8.20). The result is the block diagonal matrix $\widehat{R} = \operatorname{diag}\big(\widehat{R}_1,\ \widehat{R}_2,\ \widehat{R}_3\big)$, where the blocks, listed in the same order as the representatives, are given by:

$$\widehat{R}_1(\vec{s}, \vec{a}) := \begin{pmatrix} 4s_1 & \frac{s}{s_3} - 2a_3 & \frac{s}{s_2} - 2a_2 \\ \frac{s}{s_3} - 2a_3 & 4s_2 & \frac{s}{s_1} - 2a_1 \\ \frac{s}{s_2} - 2a_2 & \frac{s}{s_1} - 2a_1 & 4s_3 \end{pmatrix}, \tag{9.20}$$

$$\widehat{R}_2(\vec{s}, \vec{a}) := \operatorname{diag}\big(s_{r+1} + s_{r+2} - s_r + \tfrac{3s}{2s_r} + 7a_r,\ 1 \leq r \leq 3\big), \tag{9.21}$$

$$\widehat{R}_3(\vec{s}, \vec{a}) := \operatorname{diag}\Big(\tfrac{(s_{r+1} - s_{r+2})^2 - s_r^2}{2s_r} - a_r,\ 1 \leq r \leq 3\Big), \tag{9.22}$$

where

$$s := 2(s_1 s_2 + s_1 s_3 + s_2 s_3) - (s_1^2 + s_2^2 + s_3^2). \tag{9.23}$$

Therefore, $(W^{24}, \mathsf{g}_{\vec{s}})$ has strongly positive curvature if and only if there exists $\vec{a}$ such that the above block diagonal matrix $\widehat{R}(\vec{s}, \vec{a})$ is positive-definite. Our strategy to prove that for certain $\vec{s}$ there exists $\vec{a}$ such that $\widehat{R}(\vec{s}, \vec{a})$ is positive-definite has two steps, employing the usual first-order argument (Lemma 9.6). First, we find a very special $\vec{a}_0 = \vec{a}_0(\vec{s})$ such that $\widehat{R}\big(\vec{s}, \vec{a}_0\big)$ is positive-semidefinite;[8] second, we prove that (in some cases) there exists $\vec{a}_1 = \vec{a}_1(\vec{s})$ such that the corresponding ω' given by (9.19) has positive-definite restriction to the kernel of $\widehat{R}\big(\vec{s}, \vec{a}_0\big)$. In this way, $\widehat{R}(\vec{s}, \vec{a})$ is positive-definite by setting $\vec{a} = \vec{a}_0 + \varepsilon\,\vec{a}_1$ for $\varepsilon > 0$ sufficiently small. As a byproduct

[8] $\vec{a}_0$ is *very special* in the sense that if $\vec{s}$ is such that there exists $\vec{a}$ for which $\widehat{R}(\vec{s}, \vec{a})$ is positive-semidefinite, then $\widehat{R}(\vec{s}, \vec{a}_0)$ is positive-semidefinite (see Theorem 9.9).

of this method, we determine both the moduli spaces of F_4-invariant metrics $\mathrm{g}_{\vec{s}}$ with strongly nonnegative and positive curvature. The corresponding moduli spaces of metrics $\mathrm{g}_{\vec{s}}$ with $\sec \geq 0$ and $\sec > 0$ can be written in terms of the polynomials

$$p_r(\vec{s}) := (s_{r+1} - s_{r+2})^2 + 2s_r(s_{r+1} + s_{r+2}) - 3s_r^2, \tag{9.24}$$

which satisfy $p_1(\vec{s}) + p_2(\vec{s}) + p_3(\vec{s}) = s$, see (9.23). More precisely, the following result was proved by Valiev [98, Thm. 2], see also Püttmann [78, Thm. 3.1].

Theorem 9.8. *The homogeneous space* $(W^{24}, \mathrm{g}_{\vec{s}})$ *has* $\sec \geq 0$ *if and only if* $p_r(\vec{s}) \geq 0$, $r = 1, 2, 3$*; and has* $\sec > 0$ *if and only if* $p_r(\vec{s}) > 0$, $r = 1, 2, 3$, *and* s_r *are not all equal.*

We use this to prove the following result, which establishes the above first step.

Theorem 9.9. *The following are equivalent for the homogeneous space* $(W^{24}, \mathrm{g}_{\vec{s}})$*:*

(i) $\mathrm{g}_{\vec{s}}$ *has strongly nonnegative curvature,*

(ii) $\sec_{\mathrm{g}_{\vec{s}}} \geq 0$,

(iii) $p_r(\vec{s}) \geq 0$, *for* $r = 1, 2, 3$,

(iv) $R + \omega_0$ *is positive-semidefinite, where* $\omega_0 \in \wedge^4 \mathfrak{m}$ *is given by* (9.19) *setting*

$$a_r := \frac{(s_{r+1} - s_{r+2})^2 - s_r^2}{2s_r}, \qquad 1 \leq r \leq 3. \tag{9.25}$$

Proof. The implications (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) are trivial (see Section 8.1), and the equivalence (ii) $\Leftrightarrow$ (iii) follows from Theorem 9.8.

We now prove the crucial implication (iii) $\Rightarrow$ (iv). It is a direct verification that $\widehat{R}_3(\vec{s}, \vec{a}_0) = 0$, cf. (9.22) and (9.25). Furthermore, a simple calculation shows

$$\widehat{R}_2(\vec{s}, \vec{a}_0) = \operatorname{diag}\left(\tfrac{2}{s_r} p_r(\vec{s}),\ 1 \leq r \leq 3\right), \tag{9.26}$$

which is hence positive-semidefinite by condition (iii), see (9.21) and (9.24). If all s_r are equal, direct inspection shows that the first block $\widehat{R}_1(\vec{s}, \vec{a}_0)$ is positive-semidefinite. Otherwise, it follows from (9.20) and (9.25) that the 2×2 principal minors of $\widehat{R}_1(\vec{s}, \vec{a}_0)$ are equal to $4s(s_{r+1}-s_{r+2})^2$, $1 \leq r \leq 3$, hence nonnegative. Moreover, $\widehat{R}_1(\vec{s}, \vec{a}_0)v = 0$ where

$$v := \left(\frac{s_2 - s_3}{s_1}, \frac{s_3 - s_1}{s_2}, \frac{s_1 - s_2}{s_3}\right)^{\mathtt{t}}. \tag{9.27}$$

Since its diagonal entries are positive, Sylvester's criterion implies that $\widehat{R}_1(\vec{s}, \vec{a}_0)$ is positive-semidefinite. Thus, since all its blocks are positive-semidefinite, the block diagonal matrix $\widehat{R}(\vec{s}, \vec{a}_0)$ is positive-semidefinite. As noted above, this is equivalent to the entire modified curvature operator $R+\omega_0$ being positive-semidefinite, concluding the proof that (iv) holds. □

Remark 9.10. Note that the positive-semidefiniteness of the second block $\widehat{R}_2(\vec{s}, \vec{a}_0)$ is equivalent to the conditions $\mathfrak{p}_r(\vec{s}) \geq 0$ for the homogeneous metric $\mathrm{g}_{\vec{s}}$ to have $\sec \geq 0$. This can be easily verified through computations analogous to (9.26). In fact, (9.25) was originally discovered through the observation that positive-semidefiniteness of the second and third blocks $\widehat{R}_2(\vec{s}, \vec{a})$ and $\widehat{R}_3(\vec{s}, \vec{a})$ is equivalent to $a_r \in I_{r,\vec{s}}$, for certain intervals $I_{r,\vec{s}}$ that are nonempty if and only if $\mathfrak{p}_r(\vec{s}) \geq 0$. The choice (9.25) corresponds to setting a_r equal to the left endpoint of $I_{r,\vec{s}}$, which conveniently implies that $\widehat{R}_3(\vec{s}, \vec{a}_0) = 0$. Finally, note that this observation yields a proof that (i) $\Rightarrow$ (iii) which is independent of Theorem 9.8.

We are now ready to establish the second step in the strategy discussed above:

Theorem 9.11. *The homogeneous space $(W^{24}, \mathrm{g}_{\vec{s}})$ has strongly positive curvature if and only if s_r are pairwise distinct and $\mathfrak{p}_r(\vec{s}) > 0$, $r = 1, 2, 3$.*

Proof. Suppose $(W^{24}, \mathrm{g}_{\vec{s}})$ has strongly positive curvature, hence also $\sec_{\mathrm{g}_{\vec{s}}} > 0$. From Theorem 9.8, it follows[9] that s_r are not all equal and $\mathfrak{p}_r(\vec{s}) > 0$, $r = 1, 2, 3$. Further-

[9]Alternatively, positive-definiteness of the second and third blocks $\widehat{R}_2(\vec{s}, \vec{a})$ and $\widehat{R}_3(\vec{s}, \vec{a})$ directly

more, if any two s_r coincided, then (9.18) would be a Riemannian submersion, (see Remark 9.7), contradicting Theorems 8.16 and 8.22.

Conversely, assume that s_r are pairwise distinct and $p_r(\vec{s}) > 0$, $r = 1, 2, 3$. Recall that the third block $\widehat{R}_3(\vec{s}, \vec{a}_0)$ vanishes identically, and the second block $\widehat{R}_2(\vec{s}, \vec{a}_0)$ is positive-definite by (9.26). The first block $\widehat{R}_1(\vec{s}, \vec{a}_0)$ is positive-semidefinite, with kernel spanned by the vector v in (9.27). The restriction of $\omega' = a_1' \,\omega_1 + a_2' \,\omega_2 + a_3' \,\omega_3$, see (9.19), to the corresponding subspace of representatives reduces to multiplication by the scalar

$$v^{\mathsf{t}} \begin{pmatrix} 0 & -2a_3' & -2a_2' \\ -2a_3' & 0 & -2a_1' \\ -2a_2' & -2a_1' & 0 \end{pmatrix} v = \frac{4}{s_1 s_2 s_3} \sum_{r=1}^{3} \big(s_r(s_r - s_{r+1})(s_r - s_{r+2})\big) a_r'.$$

If s_r are pairwise different, the product of the coefficients of a_r' in the above sum is

$$\prod_{r=1}^{3} \big(s_r(s_r - s_{r+1})(s_r - s_{r+2})\big) = -s_1 s_2 s_3 (s_1 - s_2)^2 (s_1 - s_3)^2 (s_2 - s_3)^2 < 0,$$

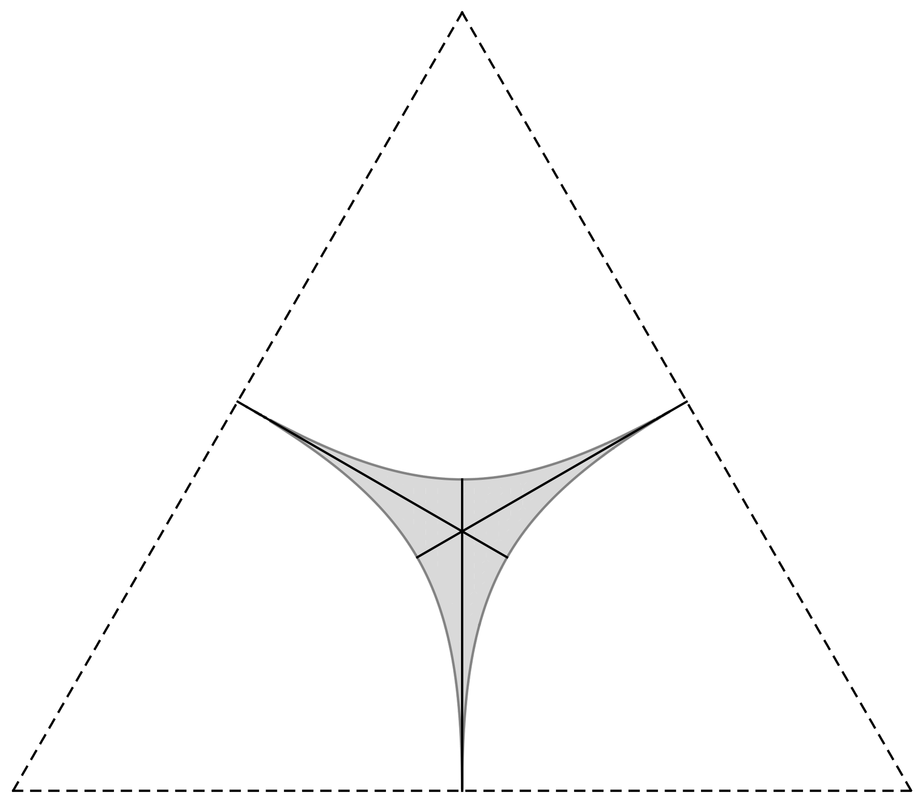
hence at least one coefficient is negative. Setting the corresponding a_r' to be sufficiently negative and $a_{r+1}', a_{r+2}' < 0$ small, the 4-form ω' becomes positive-definite on the subspace of representatives associated to $\ker \widehat{R}(\vec{s}, \vec{a}_0)$. Thus, the first-order perturbation $\widehat{R}(\vec{s}, \vec{a}_0 + \varepsilon\, \vec{a}_1)$, where $\vec{a}_1 = (a_1', a_2', a_3')$, is positive-definite for sufficiently small $\varepsilon > 0$. Therefore, $(W^{24}, \mathrm{g}_{\vec{s}})$ has strongly positive curvature. □

The above result not only proves that W^{24} admits F_4-invariant (homogeneous) metrics with strongly positive curvature, but also characterizes the moduli space of such metrics, as claimed. The intersection of the subset $\{\vec{s} \in \mathbb{R}^3 : p_r(\vec{s}) \geq 0\}$ with the affine plane $\{\vec{s} \in \mathbb{R}^3 : s_1 + s_2 + s_3 = 1\}$, that corresponds to a volume normalization,

---

implies that $p_r(\vec{s}) > 0$, $r = 1, 2, 3$, cf. Remark 9.10; and simultaneous positive-definiteness of the first block $\widehat{R}_1(\vec{s}, \vec{a})$ is impossible if the s_r are all equal, since its determinant is negative.

is the following shaded region:



where dotted segments indicate where $s_r = 0$ (i.e., the boundary of the positive octant), and continuous segments indicate where two of the s_r are equal.[10] Taking into account the permutations of the entries of $\vec{s}$, the above shaded region consists of 6 isometric copies of this moduli space.

Notice that an F_4-invariant metric has strongly positive curvature if and only if it has $\sec > 0$ and (9.18) is not a Riemannian submersion (see Remark 9.7). In other words, the *only* F_4-invariant metrics with $\sec > 0$ that fail to have strongly positive curvature are the metrics of the form (9.3) considered in Wallach's Theorem, which

[10]The continuous segments intersect at the point $\vec{s} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, which corresponds to the normal homogeneous metric $\frac{1}{3}Q|_{\mathfrak{m}}$ on W^{24}.

correspond to the 3 continuous segments in the above picture.

9.6 Aloff-Wallach spaces $W^7_{k,\ell}$

Let the Lie groups $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ be given by $\mathsf{S}^1_{k,\ell} \subset \mathsf{U}(2) \subset \mathsf{SU}(3)$, where:

$$\mathsf{U}(2) = \left\{ \operatorname{diag}\left(A, \det \overline{A}\right) \in \mathsf{SU}(3) : A \in \mathsf{U}(2) \right\}$$
$$\mathsf{S}^1_{k,\ell} = \left\{ \operatorname{diag}(z^k, z^\ell, \overline{z}^{k+\ell}) : z \in \mathsf{S}^1 \right\}$$

For convenience of notation, set $r \in (0,1]$ and $s \in (1,3]$ to be the numbers

$$r := k/\ell \quad \text{and} \quad s := 1 + r + r^2.$$

Up to the appropriate equivalences, the nontrivial cases are given by $0 < k \leq \ell$ and $\gcd(k,\ell) = 1$. The corresponding homogeneous bundle (9.1) is

$$S^3/\mathbb{Z}_{k+\ell} \longrightarrow W^7_{k,\ell} \longrightarrow \mathbb{C}P^2.$$

The base $\mathbb{C}P^2$ is a CROSS, the fiber $S^3/\mathbb{Z}_{k+\ell}$ has constant positive curvature and dimension ≤ 3, and $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair.[11]

The only other condition needed to apply Theorem 9.5 is strong fatness. Up to rescaling, the bi-invariant metric on $\mathfrak{g} = \mathfrak{su}(3)$ is given by (9.14). The Q-orthogonal complements defined in (9.2) are $\mathfrak{m} \cong \mathbb{C}^2$ and $\mathfrak{p} \cong \mathbb{R} \oplus \mathbb{C}$, and can be identified as the subspaces $\mathfrak{m} = \operatorname{span}\{\mathbb{1}_1, \mathrm{I}_1, \mathbb{1}_2, \mathrm{I}_2\}$ and $\mathfrak{p} = \operatorname{span}\{V_r, \mathbb{1}_3, \mathrm{I}_3\}$ of $\mathfrak{su}(3)$, where in addition to the matrices (9.15), we define:

$$V_r := \operatorname{diag}\left(\tfrac{(2+r)i}{\sqrt{3s}}, -\tfrac{(2r+1)i}{\sqrt{3s}}, \tfrac{(r-1)i}{\sqrt{3s}} \right)$$

[11] We remark that $(\mathfrak{k}, \mathfrak{h})$ is *not* a symmetric pair in this case.

The Lie bracket operator $L\colon \mathfrak{m}\otimes\mathfrak{p}\to\mathfrak{m}$ defined in (9.5) is determined by:

$[\cdot,\cdot]$	V_r	$\mathbb{1}_3$	I_3
$\mathbb{1}_1$	$r\sqrt{\frac{3}{s}}\,\mathrm{I}_1$	$\mathbb{1}_2$	$-\mathrm{I}_2$
I_1	$-r\sqrt{\frac{3}{s}}\,\mathbb{1}_1$	$-\mathrm{I}_2$	$-\mathbb{1}_2$
$\mathbb{1}_2$	$\sqrt{\frac{3}{s}}\,\mathrm{I}_2$	$-\mathbb{1}_1$	I_1
I_2	$-\sqrt{\frac{3}{s}}\,\mathbb{1}_2$	I_1	$\mathbb{1}_1$

Thus, by (9.6), $\ker F=\ker L$ is spanned by the following 8 vectors of $\mathfrak{m}\otimes\mathfrak{p}$:

$$
\begin{gathered}
-\sqrt{\tfrac{s}{3}}\,\mathrm{I}_2\wedge V_r+\mathrm{I}_1\wedge\mathrm{I}_3,\qquad \sqrt{\tfrac{s}{3}}\,\mathbb{1}_2\wedge V_r+\mathrm{I}_1\wedge\mathbb{1}_3,\qquad -r\sqrt{\tfrac{3}{s}}\,\mathbb{1}_2\wedge\mathbb{1}_3+\mathrm{I}_1\wedge V_r,\\
\sqrt{\tfrac{s}{3}}\,\mathbb{1}_2\wedge V_r+\mathbb{1}_1\wedge\mathrm{I}_3\qquad \sqrt{\tfrac{s}{3}}\,\mathrm{I}_2\wedge V_r+\mathbb{1}_1\wedge\mathbb{1}_3,\qquad -r\sqrt{\tfrac{3}{s}}\,\mathbb{1}_2\wedge\mathrm{I}_3+\mathbb{1}_1\wedge V_r,\\
\mathbb{1}_2\wedge\mathbb{1}_3+\mathrm{I}_2\wedge\mathrm{I}_3,\qquad -\mathbb{1}_2\wedge\mathrm{I}_3+\mathrm{I}_2\wedge\mathbb{1}_3.
\end{gathered}
$$

Consider the symmetric operator induced by the H-invariant 4-form $\tau_{a,b}\in\wedge^2\mathfrak{m}\otimes\wedge^2\mathfrak{p}$,

$$
\begin{aligned}
\tau_{a,b}={}&(a\,\mathbb{1}_1\wedge\mathrm{I}_1+b\,\mathbb{1}_2\wedge\mathrm{I}_2)\otimes(\mathbb{1}_3\wedge\mathrm{I}_3)\\
&+\sqrt{3}\,(\mathrm{I}_1\wedge\mathbb{1}_2+\mathbb{1}_1\wedge\mathrm{I}_2)\otimes(V_r\wedge\mathbb{1}_3)\\
&+\sqrt{3}\,(\mathbb{1}_1\wedge\mathbb{1}_2-\mathrm{I}_1\wedge\mathrm{I}_2)\otimes(V_r\wedge\mathrm{I}_3),
\end{aligned}
$$

where $a,b\in\mathbb{R}$ are parameters. The restriction $\tau_{a,b}\colon\ \ker F\to\ker F$ is block diagonal with 2 identical 4×4 blocks given by

$$
\operatorname{diag}\left(\begin{pmatrix}2\sqrt{s} & -a-2\sqrt{s}\\ -a-2\sqrt{s} & 2\sqrt{s}\end{pmatrix},\begin{pmatrix}\frac{6r}{\sqrt{s}} & \frac{\sqrt{3}br}{\sqrt{s}}\\ \frac{\sqrt{3}br}{\sqrt{s}} & -2b\end{pmatrix}\right)
$$

In particular, it follows that $\tau_{a,b}\colon \ker F \to \ker F$ is positive-definite if and only if

$$-4\sqrt{s} < a < 0 \quad \text{and} \quad -4\frac{\sqrt{s}}{r} < b < 0.$$

Since $r > 0$ and $s > 0$, there exist $a, b \in \mathbb{R}$ that satisfy the above, so that $\tau_{a,b}\colon \ker F \to \ker F$ is positive-definite. From Lemma 9.6, there exists $\varepsilon > 0$ such that $(F + \varepsilon\tau_{a,b})\colon \mathfrak{m}\otimes\mathfrak{p} \to \mathfrak{m}\otimes\mathfrak{p}$ is positive-definite, proving strong fatness. Therefore, by Theorem 9.5, all the homogeneous spaces $(W^7_{k,\ell}, \mathrm{g}_t)$, $k\ell(k+\ell) \neq 0$, have strongly positive curvature for all $0 < t < 1$.

9.7 Berger space B^7

Differently from the previous examples, the Berger space $B^7 = \mathsf{SO}(5)/\mathsf{SO}(3)$ does not admit a homogeneous fibration. The inclusion $\mathsf{SO}(3) \subset \mathsf{SO}(5)$ comes from the conjugation action of $\mathsf{SO}(3)$ on the space of symmetric traceless 3×3 matrices, which is identified with $\mathbb{R}^5$. Alternatively, we can use the double covering maps $\mathsf{Sp}(2) \to \mathsf{SO}(5)$ and $\mathsf{SU}(2) \to \mathsf{SO}(3)$ to write $B^7 = \mathsf{Sp}(2)/\mathsf{SU}(2)$. The inclusion $\mathsf{SU}(2) \subset \mathsf{Sp}(2)$ is such that the corresponding Lie subalgebra $\mathfrak{h} = \mathfrak{su}(2)$ of $\mathfrak{g} = \mathfrak{sp}(2)$ is spanned by the following three vectors:

$$\mathrm{I}_4 = \begin{pmatrix} i & 0 \\ 0 & 3i \end{pmatrix} \qquad \mathrm{J}_4 = \begin{pmatrix} 2j & -\sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix} \qquad \mathrm{K}_4 = \begin{pmatrix} 2k & \sqrt{3}i \\ \sqrt{3}i & 0 \end{pmatrix}$$

Up to rescaling, the bi-invariant metric on $\mathfrak{g} = \mathfrak{sp}(3)$ is given by

$$Q(X,Y) = -\tfrac{1}{10}\operatorname{Re}\operatorname{tr}(XY)$$

Consider the Q-orthogonal splitting $\mathfrak{sp}(2) = \mathfrak{su}(2) \oplus \mathfrak{m}$. The complement $\mathfrak{m} \cong \mathbb{R} \oplus \mathbb{C}^3$ can be identified as the subspace $\mathfrak{m} = \operatorname{span}\{V, \mathbb{1}_1, \mathrm{I}_1, \mathbb{1}_2, \mathrm{I}_2, \mathbb{1}_3, \mathrm{I}_3\}$, where we set:

$$V = \operatorname{diag}\big(3i, -i\big)$$

$$\mathbb{1}_1 = \sqrt{2}\begin{pmatrix} \sqrt{3}j & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbb{1}_2 = \sqrt{5}\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \qquad \mathbb{1}_3 = \sqrt{10}\begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix} \tag{9.28}$$

$$\mathrm{I}_1 = \sqrt{2}\begin{pmatrix} \sqrt{3}k & -i \\ -i & 0 \end{pmatrix} \qquad \mathrm{I}_2 = \sqrt{5}\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \qquad \mathrm{I}_3 = \sqrt{10}\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$$

The isotropy representation of $\mathsf{SU}(2)$ on $\mathfrak{m}$ is irreducible, hence there is a unique $\mathsf{Sp}(2)$-invariant metric, up to homotheties, which is known to have $\sec > 0$. We denote by g this normal homogeneous metric corresponding to $Q|_{\mathfrak{m}}$.

A computation with weights shows that the representation of $\mathsf{SU}(2)$ on $\wedge^2\mathfrak{m}$ decomposes as the sum of 3 irreducible representations with dimensions 3, 7, and 11. The following is a *complete list*[12] of representative vectors from each of the 3 irreducible factors:

$$\frac{1}{\sqrt{14}}\big(\mathbb{1}_1 \wedge \mathrm{I}_1 + 2\,\mathbb{1}_2 \wedge \mathrm{I}_2 + 3\,\mathbb{1}_3 \wedge \mathrm{I}_3\big),$$
$$\frac{1}{\sqrt{3}}\big(\mathbb{1}_1 \wedge \mathrm{I}_1 + \mathbb{1}_2 \wedge \mathrm{I}_2 - \mathbb{1}_3 \wedge \mathrm{I}_3\big),$$
$$\frac{1}{\sqrt{42}}\big(-5\,\mathbb{1}_1 \wedge \mathrm{I}_1 + 4\,\mathbb{1}_2 \wedge \mathrm{I}_2 - \mathbb{1}_3 \wedge \mathrm{I}_3\big).$$

Furthermore, up to rescaling, the following is the unique invariant element of $\wedge^4\mathfrak{m}$:

$$\begin{aligned}\omega = &-\mathbb{1}_1 \wedge \mathrm{I}_1 \wedge \mathbb{1}_2 \wedge \mathrm{I}_2 + \mathbb{1}_1 \wedge \mathrm{I}_1 \wedge \mathbb{1}_3 \wedge \mathrm{I}_3 + \mathbb{1}_2 \wedge \mathrm{I}_2 \wedge \mathbb{1}_3 \wedge \mathrm{I}_3 \\ &+ V \wedge \mathbb{1}_1 \wedge \mathbb{1}_2 \wedge \mathrm{I}_3 - V \wedge \mathbb{1}_1 \wedge \mathrm{I}_2 \wedge \mathbb{1}_3 - V \wedge \mathrm{I}_1 \wedge \mathbb{1}_2 \wedge \mathbb{1}_3 - V \wedge \mathrm{I}_1 \wedge \mathrm{I}_2 \wedge \mathrm{I}_3.\end{aligned}$$

Let $R \in S^2(\wedge^2\mathfrak{m})$ be the curvature operator of the normal homogeneous space (B^7, g), and $\alpha \in S^2(\wedge^2\mathfrak{m})$ the operator defined in (8.17). From (8.18), it follows that $R +$

[12] See the discussion in page 125, Section 9.5.

$3\mathfrak{b}(\alpha)$ is a positive-semidefinite operator. The restriction $\widehat{R}$ of the modified curvature operator $R+3\mathfrak{b}(\alpha)+a\,\omega$ to the subspace of $\wedge^2\mathfrak{m}$ spanned by the above representative vectors can be computed using formulas (8.18), (2.13), and the table of Lie brackets:

$\frac{1}{2}[\cdot,\cdot]$	V	$\mathbb{1}_1$	I_1	$\mathbb{1}_2$	I_2	$\mathbb{1}_3$
$\mathbb{1}_1$	$-\mathrm{I}_1-\sqrt{6}\mathrm{K}_4$					
I_1	$\mathbb{1}_1+\sqrt{6}\mathrm{J}_4$	$-V-\mathrm{I}_4$				
$\mathbb{1}_2$	$-\mathrm{I}_2$	$\mathbb{1}_3-\frac{\sqrt{10}}{2}\mathrm{J}_4$	$\mathrm{I}_3+\frac{\sqrt{10}}{2}\mathrm{K}_4$			
I_2	$\mathbb{1}_2$	$\mathrm{I}_3-\frac{\sqrt{10}}{2}\mathrm{K}_4$	$-\mathbb{1}_3-\frac{\sqrt{10}}{2}\mathrm{J}_4$	$-V-2\mathrm{I}_4$		
$\mathbb{1}_3$	I_3	$-\mathbb{1}_2$	I_2	$\mathbb{1}_1-\frac{\sqrt{6}}{2}\mathrm{J}_4$	$-\mathrm{I}_1+\frac{\sqrt{6}}{2}\mathrm{K}_4$	
I_3	$-\mathbb{1}_3$	$-\mathrm{I}_2$	$-\mathbb{1}_2$	$\mathrm{I}_1-\frac{\sqrt{6}}{2}\mathrm{K}_4$	$\mathbb{1}_1-\frac{\sqrt{6}}{2}\mathrm{J}_4$	$V-3\mathrm{I}_4$

The result of this computation is the following, where the blocks are listed in the same order as the representatives:

$$\widehat{R} = \operatorname{diag}\big(56+a,\, 3-2a,\, a\big), \tag{9.29}$$

so the modified curvature operator $R+3\mathfrak{b}(\alpha)+a\,\omega$ is positive-definite if $0<a<\frac{3}{2}$. Therefore, the homogeneous space (B^7,g) has strongly positive curvature.

9.8 Berger space B^{13}

Let the Lie groups $\mathsf{H}\subset\mathsf{K}\subset\mathsf{G}$ be given by $\mathsf{Sp}(2)\cdot\mathsf{S}^1\subset\mathsf{U}(4)\subset\mathsf{SU}(5)$, where:[13]

$$\mathsf{U}(4)=\big\{\operatorname{diag}(A,\det\overline{A})\in\mathsf{SU}(5): A\in\mathsf{U}(4)\big\}$$

$$\mathsf{Sp}(2)\cdot\mathsf{S}^1=\big\{\operatorname{diag}\big(z\,A,\overline{z}^4\big): A\in\mathsf{Sp}(2)\subset\mathsf{SU}(4),\ z\in\mathsf{S}^1\big\}$$

[13]Note that the Lie group $\mathsf{H}=\mathsf{Sp}(2)\cdot\mathsf{S}^1$ is isomorphic to $(\mathsf{Sp}(2)\times\mathsf{S}^1)/\mathbb{Z}_2$, where $\mathbb{Z}_2\cong\{(\pm\mathrm{Id},1)\}$.

The corresponding homogeneous bundle (9.1) is

$$\mathbb{R}P^5 \longrightarrow B^{13} \longrightarrow \mathbb{C}P^4.$$

The base $\mathbb{C}P^4$ is a CROSS, the fiber $\mathbb{R}P^5$ has constant positive curvature, and both $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{k}, \mathfrak{h})$ are symmetric pairs.

The only other condition needed to apply Theorem 9.5 is strong fatness. Up to rescaling, the bi-invariant metric on $\mathfrak{g} = \mathfrak{su}(5)$ is given by the same formula as (9.14). The Q-orthogonal complements defined in (9.2) are $\mathfrak{m} \cong \mathbb{C}^4$ and $\mathfrak{p} \cong \mathbb{R} \oplus \mathbb{C}^2$, and can be identified as the subspaces $\mathfrak{m} = \operatorname{span}\{\mathbb{1}_1, \mathrm{I}_1, \mathbb{1}_2, \mathrm{I}_2, \mathbb{1}_3, \mathrm{I}_3, \mathbb{1}_4, \mathrm{I}_4\}$ and $\mathfrak{p} = \operatorname{span}\{V, \mathbb{1}_5, \mathrm{I}_5, \mathbb{1}_6, \mathrm{I}_6\}$ of $\mathfrak{su}(5)$, where we set:

$$\mathbb{1}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{I}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{1}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{I}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{1}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \qquad \mathrm{I}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \end{pmatrix}$$

$$\mathbb{1}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \qquad \mathrm{I}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}$$

$$\mathbb{1}_5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{I}_5 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbb{1}_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{I}_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Lie bracket operator $L\colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m}$ defined in (9.5) is determined by:

$\sqrt{2}\,[\cdot,\cdot]$	V	$\mathbb{1}_5$	I_5	$\mathbb{1}_6$	I_6
$\mathbb{1}_1$	$-\mathrm{I}_1$	$\mathbb{1}_3$	$-\mathrm{I}_3$	$\mathbb{1}_4$	$-\mathrm{I}_4$
I_1	$\mathbb{1}_1$	I_3	$\mathbb{1}_3$	I_4	$\mathbb{1}_4$
$\mathbb{1}_2$	$-\mathrm{I}_2$	$-\mathbb{1}_4$	$-\mathrm{I}_4$	$\mathbb{1}_3$	I_3
I_2	$\mathbb{1}_2$	$-\mathrm{I}_4$	$\mathbb{1}_4$	I_3	$-\mathbb{1}_3$
$\mathbb{1}_3$	I_3	$-\mathbb{1}_1$	$-\mathrm{I}_1$	$-\mathbb{1}_2$	I_2
I_3	$-\mathbb{1}_3$	$-\mathrm{I}_1$	$\mathbb{1}_1$	$-\mathrm{I}_2$	$-\mathbb{1}_2$
$\mathbb{1}_4$	I_4	$\mathbb{1}_2$	$-\mathrm{I}_2$	$-\mathbb{1}_1$	$-\mathrm{I}_1$
I_4	$-\mathbb{1}_4$	I_2	$\mathbb{1}_2$	$-\mathrm{I}_1$	$\mathbb{1}_1$

Thus, by (9.6), $\ker F = \ker L$ is spanned by the following 32 vectors of $\mathfrak{m} \otimes \mathfrak{p}$:

$$-\mathrm{I}_1 \wedge V + \mathrm{I}_4 \wedge \mathrm{I}_6, \quad -\mathbb{1}_1 \wedge V + \mathrm{I}_4 \wedge \mathbb{1}_6, \quad -\mathrm{I}_2 \wedge V + \mathrm{I}_4 \wedge \mathrm{I}_5, \quad \mathbb{1}_2 \wedge V + \mathrm{I}_4 \wedge \mathbb{1}_5,$$

$$
\begin{array}{rrrr}
\mathbb{1}_1 \wedge \mathbb{1}_6 + \mathrm{I}_4 \wedge V, & -\mathbb{1}_1 \wedge V + \mathbb{1}_4 \wedge \mathrm{I}_6, & \mathrm{I}_1 \wedge V + \mathbb{1}_4 \wedge \mathbb{1}_6, & -\mathbb{1}_2 \wedge V + \mathbb{1}_4 \wedge \mathrm{I}_5, \\
-\mathrm{I}_2 \wedge V + \mathbb{1}_4 \wedge \mathbb{1}_5, & \mathbb{1}_1 \wedge \mathrm{I}_6 + \mathbb{1}_4 \wedge V, & \mathrm{I}_2 \wedge V + \mathrm{I}_3 \wedge \mathrm{I}_6, & -\mathbb{1}_2 \wedge V + \mathrm{I}_3 \wedge \mathbb{1}_6, \\
-\mathrm{I}_1 \wedge V + \mathrm{I}_3 \wedge \mathrm{I}_5, & -\mathbb{1}_1 \wedge V + \mathrm{I}_3 \wedge \mathbb{1}_5, & \mathbb{1}_1 \wedge \mathbb{1}_5 + \mathrm{I}_3 \wedge V, & \mathbb{1}_2 \wedge V + \mathbb{1}_3 \wedge \mathrm{I}_6, \\
\mathrm{I}_2 \wedge V + \mathbb{1}_3 \wedge \mathbb{1}_6, & -\mathbb{1}_1 \wedge V + \mathbb{1}_3 \wedge \mathrm{I}_5, & \mathrm{I}_1 \wedge V + \mathbb{1}_3 \wedge \mathbb{1}_5, & \mathbb{1}_1 \wedge \mathrm{I}_5 + \mathbb{1}_3 \wedge V, \\
\mathbb{1}_1 \wedge \mathbb{1}_5 + \mathrm{I}_2 \wedge \mathrm{I}_6, & \mathbb{1}_1 \wedge \mathrm{I}_5 + \mathrm{I}_2 \wedge \mathbb{1}_6, & -\mathbb{1}_1 \wedge \mathbb{1}_6 + \mathrm{I}_2 \wedge \mathrm{I}_5, & -\mathbb{1}_1 \wedge \mathrm{I}_6 + \mathrm{I}_2 \wedge \mathbb{1}_5, \\
\mathbb{1}_1 \wedge \mathrm{I}_5 + \mathbb{1}_2 \wedge \mathrm{I}_6, & -\mathbb{1}_1 \wedge \mathbb{1}_5 + \mathbb{1}_2 \wedge \mathbb{1}_6, & -\mathbb{1}_1 \wedge \mathrm{I}_6 + \mathbb{1}_2 \wedge \mathrm{I}_5, & \mathbb{1}_1 \wedge \mathbb{1}_6 + \mathbb{1}_2 \wedge \mathbb{1}_5, \\
-\mathbb{1}_1 \wedge \mathbb{1}_6 + \mathrm{I}_1 \wedge \mathrm{I}_6, & \mathbb{1}_1 \wedge \mathrm{I}_6 + \mathrm{I}_1 \wedge \mathbb{1}_6, & -\mathbb{1}_1 \wedge \mathbb{1}_5 + \mathrm{I}_1 \wedge \mathrm{I}_5, & \mathbb{1}_1 \wedge \mathrm{I}_5 + \mathrm{I}_1 \wedge \mathbb{1}_5.
\end{array}
$$

Consider the symmetric operator induced by the H-invariant 4-form $\tau \in \wedge^2\mathfrak{m} \otimes \wedge^2\mathfrak{p}$,

$$
\begin{aligned}
\tau = &-(\mathbb{1}_2 \wedge \mathrm{I}_2 - \mathbb{1}_1 \wedge \mathrm{I}_1) \otimes (\mathbb{1}_5 \wedge \mathrm{I}_5 + \mathbb{1}_6 \wedge \mathrm{I}_6) \\
&+ (\mathbb{1}_1 \wedge \mathbb{1}_2 + \mathrm{I}_1 \wedge \mathrm{I}_2) \otimes (\mathbb{1}_5 \wedge \mathbb{1}_6 - \mathrm{I}_5 \wedge \mathrm{I}_6) \\
&- (\mathbb{1}_1 \wedge \mathrm{I}_2 - \mathrm{I}_1 \wedge \mathbb{1}_2) \otimes (\mathbb{1}_5 \wedge \mathrm{I}_6 + \mathrm{I}_5 \wedge \mathbb{1}_6) \\
&- (\mathbb{1}_3 \wedge \mathrm{I}_3 - \mathbb{1}_4 \wedge \mathrm{I}_4) \otimes (\mathbb{1}_5 \wedge \mathrm{I}_5 - \mathbb{1}_6 \wedge \mathrm{I}_6) \\
&+ (\mathbb{1}_3 \wedge \mathbb{1}_4 + \mathrm{I}_3 \wedge \mathrm{I}_4) \otimes (\mathbb{1}_5 \wedge \mathbb{1}_6 + \mathrm{I}_5 \wedge \mathrm{I}_6) \\
&- (\mathbb{1}_3 \wedge \mathrm{I}_4 - \mathrm{I}_3 \wedge \mathbb{1}_4) \otimes (\mathbb{1}_5 \wedge \mathrm{I}_6 - \mathrm{I}_5 \wedge \mathbb{1}_6) \\
&+ (\mathbb{1}_1 \wedge \mathrm{I}_3 - \mathrm{I}_1 \wedge \mathbb{1}_3 - \mathbb{1}_2 \wedge \mathrm{I}_4 + \mathrm{I}_2 \wedge \mathbb{1}_4) \otimes (V \wedge \mathbb{1}_5) \\
&+ (\mathbb{1}_1 \wedge \mathbb{1}_3 + \mathrm{I}_1 \wedge \mathrm{I}_3 + \mathbb{1}_2 \wedge \mathbb{1}_4 + \mathrm{I}_2 \wedge \mathrm{I}_4) \otimes (V \wedge \mathrm{I}_5) \\
&+ (\mathbb{1}_1 \wedge \mathrm{I}_4 - \mathrm{I}_1 \wedge \mathbb{1}_4 + \mathbb{1}_2 \wedge \mathrm{I}_3 - \mathrm{I}_2 \wedge \mathbb{1}_3) \otimes (V \wedge \mathbb{1}_6) \\
&+ (\mathbb{1}_1 \wedge \mathbb{1}_4 + \mathrm{I}_1 \wedge \mathrm{I}_4 - \mathbb{1}_2 \wedge \mathbb{1}_3 - \mathrm{I}_2 \wedge \mathrm{I}_3) \otimes (V \wedge \mathrm{I}_6).
\end{aligned}
$$

The restriction $\tau\colon \ker F \to \ker F$ is the identity operator, hence positive-definite. From Lemma 9.6, there exists $\varepsilon > 0$ such that $(F + \varepsilon\tau)\colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m} \otimes \mathfrak{p}$ is positive-definite, proving strong fatness. Therefore, by Theorem 9.5, the homogeneous space (B^{13}, g_t) has strongly positive curvature for all $0 < t < 1$.

Remark 9.12. The normal homogeneous metric (corresponding to $t = 1$) on the Berger space B^{13} is known to have $\sec > 0$, by the work of Berger [8]. A direct computation of its modified curvature operator shows that this metric does not have strongly positive curvature.

This concludes the proof of Theorem 9.1, since all simply-connected homogeneous spaces that admit an invariant metric with $\sec > 0$ were verified to also admit an invariant metric with strongly positive curvature, except for the Cayley plane $\mathbb{C}\mathrm{a}P^2$.

9.9 Comments and perspectives beyond homogeneous spaces

We conclude this chapter with a few comments regarding the above classification of homogeneous spaces with strongly positive curvature, and a brief account on future perspectives to understand more general manifolds with strongly positive curvature.

Remark 9.13. Many homogeneous spaces occurring in Theorem 9.1, apart from the CROSS, are related via Riemannian submersions or totally geodesic immersions, which allows for alternative proofs that these spaces have strongly positive curvature. For instance, there is an embedding $W^6 \to W^{12}$ whose image is the fixed point set of an isometry, hence totally geodesic, see (10.3) for details. In particular, it follows from Proposition 8.15 that W^6 has strongly positive curvature, since W^{12} does. Analogously, by the Taimanov embedding $W^7_{1,1} \to B^{13}$, the Aloff-Wallach space $W^7_{1,1}$ has strongly positive curvature since B^{13} does. Finally, there are Riemannian submersions $W^7_{k,\ell} \to W^6$, so Theorem 8.16 provides yet another proof that W^6 has strongly positive curvature.

Remark 9.14. In Wallach's Theorem 9.2, under the extra hypothesis that $(\mathfrak{k}, \mathfrak{h})$ is a symmetric pair, i.e., $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$, the conclusion is that $(\mathsf{G}/\mathsf{H}, \mathrm{g}_t)$ has $\sec > 0$ for all $0 < t < 1$ *or* $1 < t < \frac{4}{3}$, see Wallach [102, p. 291]. However, even under this extra hypothesis, the second and third terms in (9.8) are negative-semidefinite if $1 < t < \frac{4}{3}$, which prevents the proof of Theorem 9.5 from extending to $1 < t < \frac{4}{3}$. In fact, there are examples of homogeneous spaces $(\mathsf{G}/\mathsf{H}, \mathrm{g}_t)$ that satisfy all of the above hypotheses (and hence have $\sec > 0$ for all $1 < t < \frac{4}{3}$) but do not have strongly positive curvature (or even strongly nonnegative curvature) for all $1 < t < \frac{4}{3}$, see Remark 10.15.

After concluding the classification of simply-connected homogeneous spaces that admit an invariant metric with strongly positive curvature, a natural question is to determine the *moduli spaces* of all such invariant metrics. This was done in Section 9.5 for W^{24}, and is done for the other Wallach flag manifolds in Section 10.1, following Bettiol and Mendes [14]. Furthermore, in the remainder of Chapter 10, the moduli spaces of Berger spheres with strongly positive curvature are studied.

Apart from homogeneous spaces, other examples of closed manifolds with $\sec > 0$ are given by *biquotients*. A biquotient $\mathsf{G}/\!/\mathsf{H}$ is the orbit space of a free isometric action of a Lie group $\mathsf{H} \subset \mathsf{G} \times \mathsf{G}$ on a compact Lie group G, given by $(h_1, h_2) \cdot g = h_1 g h_2^{-1}$. The quotient map $\mathsf{G} \to \mathsf{G}/\!/\mathsf{H}$ is a Riemannian submersion, and hence formula (8.16) can be applied to compute the modified curvature operator of biquotients. In particular, it follows from (2.16) and (8.16) that biquotients have strongly nonnegative curvature, cf. Section 8.4. Furthermore, there are infinite families of biquotients with $\sec > 0$, namely the Eschenburg spaces E^6 and $E^7_{k,\ell}$, and the Bazaĭkin spaces B^{13}_q, which are respectively generalizations of W^6, $W^7_{k,\ell}$ and B^{13}, see Ziller [112] for details. An infinite subfamily of Eschenburg spaces $E^7_{k,\ell}$ can be shown to have strongly positive curvature, with a limiting argument analogous to the original argument of Eschenburg [36]. In particular, since there are Riemannian submersions $E^7_{k,\ell} \to E^6$, this implies that also E^6 has strongly positive curvature. However, a complete verification that all known examples of biquotients with $\sec > 0$ also have strongly positive curvature is still under way.

Homogeneous spaces and biquotients aside, the only other known examples of closed manifolds with $\sec > 0$ are cohomogeneity one manifolds, as well as a proposed positively curved exotic sphere [76]. Grove, Wilking and Ziller [46] performed a systematic study of simply-connected closed cohomogeneity one manifolds with $\sec > 0$, which lead to a classification result with an infinite family of *candidate* manifolds. More precisely, they found two infinite families P_k^7, Q_k^7, $k \geq 2$, and an exceptional case R^7, of 7-dimensional manifolds which are the only manifolds different from homogeneous spaces and biquotients that could carry a cohomogeneity one metric with $\sec > 0$. Shortly after, Grove, Verdiani and Ziller [44] constructed an invariant metric *with strongly positive curvature* on the candidate P_2^7, which was also identified as an exotic T_1S^4. Dearricott [28] has independently found a metric with $\sec > 0$ on this manifold, using modified curvature operators in an indirect way. Recently, Verdiani and Ziller [100] showed that the manifold R^7 does not admit an invariant metric with $\sec > 0$. A better understanding of cohomogeneity one manifolds with strongly positive curvature would certainly contribute to advancing this theory.

The search for topological obstructions for closed manifolds to admit strongly positive curvature is another major component in advancing this theory, and complements the search for non-homogeneous examples. An important tool to find topological obstructions to curvature conditions is Hamilton's Ricci flow. In particular, the Ricci flow was used by Böhm and Wilking [17] in the classification of manifolds with positive-definite curvature operator, proving that this condition is preserved under the flow. In earlier work, Böhm and Wilking [16] provided the first example of a closed manifold with $\sec > 0$ that develops mixed Ricci curvature under the Ricci flow. This example is the Wallach flag manifold $(W^{12}, \mathsf{g}_{\vec{s}})$ with a homogeneous metric $\mathsf{g}_{\vec{s}}$ that can be chosen *arbitrarily close* to a metric of the form g_t discussed in Section 9.4, see Section 10.1 for details. In particular, $\mathsf{g}_{\vec{s}}$ can be chosen to have strongly positive curvature, since this is an open condition. Therefore, similarly to $\sec > 0$, strongly positive curvature is *not preserved* under the Ricci flow.

Another natural attempt to find topological obstructions to strongly positive curvature is related to the Gauss-Bonnet integrand of a curvature operator, and the so-called *algebraic Hopf conjecture*, related to the Hopf Problem II. Given an operator $R \in S(\wedge^2 V)$, where $\dim V = 2n$, its *Gauss-Bonnet integrand* $\chi(R)$ is given by:

$$\chi(R) = \sum_{\sigma,\tau\in\mathfrak{S}_{2n}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\prod_{i=1}^{2n-1}\left\langle R(e_{\sigma(i)}\wedge e_{\sigma(i+1)}), e_{\tau(i)}\wedge e_{\tau(i+1)}\right\rangle,$$

where $\mathfrak{S}_{2n}$ is the group of permutations of $2n$ symbols, $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ and $\{e_i\}$ is an orthonormal basis of V. It has been long known that if R is positive-definite, then $\chi(R) > 0$, see Kulkarni [61, p. 191]. By the Chern-Gauss-Bonnet Theorem, the integral over a closed manifold (M^{2n}, g) of the Gauss-Bonnet integrand $\chi(R)$ of its curvature operator is equal to the Euler characteristic $\chi(M)$, multiplied by a (positive) dimensional constant.

While the Hopf Problem II remains open in general, its *algebraic* variant, that asks if an algebraic curvature operator R in even dimensions with $\sec_R > 0$ (recall (8.2)) has $\chi(R) > 0$, is completely settled. Milnor proved in unpublished work that the algebraic Hopf conjecture holds in dimensions ≤ 4, see Chern [24]. In particular, in light of Proposition 8.9, this shows that curvature operators in dimensions ≤ 4 that have strongly positive curvature also have $\chi(R) > 0$. Geroch [39], and later Klembeck [60], provided counter-examples to the algebraic Hopf conjecture in dimensions ≥ 6, which are algebraic curvature operators in even dimensions ≥ 6 that have $\sec_R > 0$ but $\chi(R) \leq 0$. It is not hard to verify that these examples have strongly positive curvature, hence are also counter-examples to the *strong version* of the algebraic Hopf conjecture. Nevertheless, it is not known whether closed manifolds with strongly positive curvature and even dimension ≥ 6 have $\chi(M) > 0$.

CHAPTER 10

MODULI SPACES

In this chapter, we analyze the moduli spaces of homogeneous metrics with strongly positive curvature and strongly nonnegative curvature on Wallach flag manifolds (following Bettiol and Mendes [14]) and on Berger spheres. These results provide concrete applications of several techniques developed in Chapters 8 and 9.

The Wallach flag manifolds W^6, W^{12}, and W^{24} are total spaces of sphere bundles

$$S^2\big(\tfrac{1}{2}\big) \to W^6 \to \mathbb{C}P^2, \quad S^4\big(\tfrac{1}{2}\big) \to W^{12} \to \mathbb{H}P^2, \quad S^8\big(\tfrac{1}{2}\big) \to W^{24} \to \mathbb{C}\mathrm{a}P^2, \tag{10.1}$$

and were studied in Sections 9.3, 9.4, and 9.5, where they were proved to have strongly positive curvature. The latter contains a complete description of the above mentioned moduli spaces for W^{24}, see Theorems 9.9 and 9.11. In Section 10.1, we apply similar techniques to determine these moduli spaces also for W^6 and W^{12}, following [14].

The Berger spheres are odd-dimensional spheres obtained by rescaling the round metric by λ in the vertical direction of one of the Hopf bundles

$$S^1 \to S^{2n+1} \to \mathbb{C}P^n, \quad S^3 \to S^{4n+3} \to \mathbb{H}P^n, \quad S^7 \to S^{15} \to S^8\big(\tfrac{1}{2}\big). \tag{10.2}$$

More precisely, a Berger metric on the total space of one of the above bundles is of the form $\lambda\, \mathrm{g}_{\mathcal{V}} \oplus \mathrm{g}_{\mathcal{H}}$, where $\mathrm{g}_{\mathcal{V}} \oplus \mathrm{g}_{\mathcal{H}}$ is the round metric, and $\mathrm{g}_{\mathcal{V}}$ and $\mathrm{g}_{\mathcal{H}}$ respectively denote its vertical and horizontal parts. These metrics are homogeneous and have $\sec > 0$ if and only if $0 < \lambda < \frac{4}{3}$, see Verdiani and Ziller [99]. In Sections 10.2, 10.3 and 10.4, we prove that the above Hopf bundles are strongly fat (recall Definition 9.4), and use

Theorem 9.5 to conclude that certain Berger metrics have strongly positive curvature, analogously to what was previously done for other homogeneous spaces in Chapter 9. However, the range of Berger metrics for which this method works is somewhat unsatisfactory, since it does not include the round metric, $\lambda = 1$. This is due to the fact that the normal homogeneous metric on these spheres is a Berger metric with shrunk Hopf fibers, see (10.10), (10.14) and (10.21). For the first two bundles in (10.2), an argument with Cheeger deformations using Proposition 8.18 gives a more satisfactory range (see Propositions 10.7 and 10.11). For the third bundle in (10.2), a direct computation provides a complete description of the moduli spaces (see Proposition 10.14). In particular, this yields examples of homogeneous spaces with $\sec > 0$ that do not have strongly nonnegative curvature, see Remark 10.15.

10.1 Wallach flag manifolds

The Wallach flag manifolds W^6, W^{12}, and W^{24} have appeared in Sections 9.3, 9.4, and 9.5 respectively. These are the manifolds of complete flags in $\mathbb{K}^3$, where $\mathbb{K}$ is one of the real normed division algebras: $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{C}\mathrm{a}$, respectively. An analogous construction can be carried out with $\mathbb{K} = \mathbb{R}$, giving rise to the real flag manifold W^3, which can also be considered here (but on which the notion of strongly positive curvature coincides with $\sec > 0$ by dimensional reasons, see Corollary 8.10).

Recall that a *complete flag* $\mathcal{F}$ in $\mathbb{K}^3$ is a sequence of linear subspaces

$$\mathcal{F} = \big\{\{0\} \subset F^1 \subset F^2 \subset \mathbb{K}^3\big\},$$

where $\dim_{\mathbb{K}} F^i = i$. The natural projection map

$$\pi\colon W^{3\dim\mathbb{K}} \to \mathbb{K}P^2, \quad \pi(\mathcal{F}) = F^1,$$

is a submersion, whose fiber $\pi^{-1}(F^1)$ over a $\mathbb{K}$-line F^1 consists of all $\mathbb{K}$-planes F^2 that contain F^1. Such $\mathbb{K}$-planes are in one-to-one correspondence with the $\mathbb{K}$-lines in the orthogonal complement $(F^1)^\perp$, hence $\pi^{-1}(F^1)$ can be identified with $\mathbb{K}P^1$. This makes $W^{3\dim\mathbb{K}}$ the total space of a sphere bundle over $\mathbb{K}P^2$, see (10.1). The isometry group G of each projective plane $\mathbb{K}P^2$ above acts transitively on $W^{3\dim\mathbb{K}}$. Furthermore, the induced subaction of the isotropy subgroup K of a $\mathbb{K}$-line F^1 is transitive on the corresponding fiber $\pi^{-1}(F^1)$. Altogether, each sphere bundle (10.1) is a homogeneous bundle of the form (9.1), with Lie groups $\mathsf{H}\subset\mathsf{K}\subset\mathsf{G}$ given by:

G/H	$\mathbb{K}$	H	K	G
W^3	$\mathbb{R}$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathsf{O}(2)$	$\mathsf{SO}(3)$
W^6	$\mathbb{C}$	T^2	$\mathsf{U}(2)$	$\mathsf{SU}(3)$
W^{12}	$\mathbb{H}$	$\mathsf{Sp}(1)\mathsf{Sp}(1)\mathsf{Sp}(1)$	$\mathsf{Sp}(2)\mathsf{Sp}(1)$	$\mathsf{Sp}(3)$
W^{24}	$\mathbb{C}\mathrm{a}$	$\mathsf{Spin}(8)$	$\mathsf{Spin}(9)$	F_4

Let Q be the bi-invariant metric on G described in Sections 9.3, 9.4, and 9.5 in each of the above cases, and let $\mathfrak{m}$ be the subspace such that $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ is a Q-orthogonal direct sum, as in Section 8.4. In all cases above, the $\mathrm{Ad}(\mathsf{H})$-representation $\mathfrak{m}$ has 3 irreducible factors $\mathfrak{m}=\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3$, each isomorphic to $\mathbb{K}$ as a real vector space. These irreducible factors can be parametrized by the skew-Hermitian matrices with zero diagonal entries

$$\mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}\ni(x_1,x_2,x_3)\longmapsto\begin{pmatrix}0 & x_3 & -\overline{x_2}\\ -\overline{x_3} & 0 & x_1\\ x_2 & -\overline{x_1} & 0\end{pmatrix}\in\mathfrak{m}_1\oplus\mathfrak{m}_2\oplus\mathfrak{m}_3.$$

The standard bases of $\mathbb{K}^3$ correspond to Q-orthonormal bases of $\mathfrak{m}$ via the above.

We denote the elements of such bases by

$$\mathbb{1}_r, \mathrm{I}_r, \mathrm{J}_r, \mathrm{K}_r, \mathrm{L}_r, \mathrm{M}_r, \mathrm{N}_r, \mathrm{O}_r, \qquad 1 \leq r \leq 3.$$

Explicit formulas for the first half of the above matrices are given in (9.15) and (9.16). Since $\mathfrak{m}_r$, $1 \leq r \leq 3$, are irreducible and nonisomorphic, Schur's Lemma implies that G-invariant metrics on G/H are parametrized by 3 positive numbers. We denote these by $\vec{s} = (s_1, s_2, s_3)$, so that the corresponding G-invariant metric is given by

$$\mathrm{g}_{\vec{s}} := s_1^2\, Q|_{\mathfrak{m}_1} \oplus s_2^2\, Q|_{\mathfrak{m}_2} \oplus s_3^2\, Q|_{\mathfrak{m}_3}, \quad \text{at } T_{(e\mathsf{H})}\mathsf{G}/\mathsf{H} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3,$$

which is the same notation used in Section 9.5 for the case $\mathbb{K} = \mathbb{C}\mathrm{a}$.

For any fixed $\vec{s}$, there are natural inclusions

$$(W^3, \mathrm{g}_{\vec{s}}) \hookrightarrow (W^6, \mathrm{g}_{\vec{s}}) \hookrightarrow (W^{12}, \mathrm{g}_{\vec{s}}) \hookrightarrow (W^{24}, \mathrm{g}_{\vec{s}}) \tag{10.3}$$

whose images are the fixed point set of an isometry, and hence totally geodesic. Thus, by Proposition 8.15 and Theorem 9.9, all the above homogeneous spaces have strongly nonnegative curvature if[1] $p_r(\vec{s}) \geq 0$, for $r = 1, 2, 3$, since $(W^{24}, \mathrm{g}_{\vec{s}})$ does. Furthermore, by Proposition 8.15 and Theorem 9.11, these homogeneous spaces have strongly positive curvature if s_r are pairwise distinct and $p_r(\vec{s}) > 0$, for $r = 1, 2, 3$. However, this does not account for *all* homogeneous metrics with strongly positive curvature on W^3, W^6 and W^{12}; e.g., the metrics $\mathrm{g}_{\vec{s}}$ on W^6 and W^{12} have strongly positive curvature if $\vec{s} = (1, 1, t)$ and $0 < t < 1$, see Sections 9.3 and 9.4.

Using methods similar to those in the proof of Theorems 9.9 and 9.11 we now provide a complete description of these other moduli spaces. The case of the real flag manifold $(W^3, \mathrm{g}_{\vec{s}})$ is omitted, since it is locally isometric to S^3 with the Berger

[1] Recall that $p_r(\vec{s})$ are the polynomials defined in (9.24), see also the figure in page 131.

metric $\lambda_1\, \mathrm{g}_{\mathcal{V}_1} \oplus \lambda_2\, \mathrm{g}_{\mathcal{V}_2} \oplus \lambda_3\, \mathrm{g}_{\mathcal{V}_3}$, for which the moduli space of homogeneous metrics with $\sec > 0$ (and hence strongly positive curvature) is well-known,[2] see e.g. [99].

Theorem 10.1. *The following are equivalent for* $(W^6, \mathrm{g}_{\vec{s}})$, $(W^{12}, \mathrm{g}_{\vec{s}})$ *and* $(W^{24}, \mathrm{g}_{\vec{s}})$*:*

(i) $\mathrm{g}_{\vec{s}}$ *has strongly nonnegative curvature,*

(ii) $\sec_{\mathrm{g}_{\vec{s}}} \geq 0$,

(iii) $p_r(\vec{s}) \geq 0$, *for* $r = 1, 2, 3$.

Proof. The fact that (ii) and (iii) are equivalent was proved by Valiev [98, Thm. 2], see also Püttmann [78, Thm. 3.1] and Theorem 9.8. The equivalence of (i) and (iii) was proved in Theorem 9.9 for the case of W^{24}. The cases of W^6 and W^{12} also follow from Theorem 9.9, since their modified curvature operator is a restriction to the corresponding subspace of the modified curvature operator of W^{24}, due to the totally geodesic embeddings (10.3), see Proposition 8.15. □

Remark 10.2. The fact that the above moduli spaces for W^6, W^{12} and W^{24} coincide can be explained using a generalization of the embeddings (10.3), by showing that planes with extremal sectional curvature can be moved via isometries to become tangent to certain totally geodesic submanifolds, see Wilking [106].

Restrictions of a positive-semidefinite modified curvature operator of W^{24} to subspaces corresponding to W^6 or W^{12} are clearly positive-semidefinite, but may also be positive-definite in these subspaces. This is what happens if two of the s_r coincide:

Theorem 10.3. *The following are equivalent for* $(W^6, \mathrm{g}_{\vec{s}})$ *and* $(W^{12}, \mathrm{g}_{\vec{s}})$*:*

(i) $\mathrm{g}_{\vec{s}}$ *has strongly positive curvature,*

(ii) $\sec_{\mathrm{g}_{\vec{s}}} > 0$,

(iii) s_r *are* not all equal, *and* $p_r(\vec{s}) > 0$, *for* $r = 1, 2, 3$.

---

[2] These are left-invariant metrics on $\mathsf{SU}(2) \cong S^3$ and have $\sec > 0$ if and only if $p_r(\vec{\lambda}) > 0$, where $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial, and the fact that (ii) and (iii) are equivalent was proved by Valiev [98, Thm. 2], see also Püttmann [78, Thm. 3.1] and Theorem 9.8. In what follows, we prove the crucial implication (iii) $\Rightarrow$ (i). The case of W^6 follows from that of W^{12}, since modified curvature operators on the former are restrictions of modified curvature operators on the latter.

The H-representation $\wedge^2\mathfrak{m}$ on W^{12} decomposes into a total of 9 isotypic components. More precisely, there are 2 copies each of 3 irreducible nonisomorphic representations of dimension 3, 3 distinct irreducible representations of dimension 12, and 3 distinct irreducible representations of dimension 4. The following 12 representative vectors form a *complete list* (defined in page 125), where indices are taken modulo 3:

$$\begin{aligned}
&\tfrac{1}{\sqrt{2}}\big(\mathbb{1}_{r+1}\wedge \mathrm{I}_{r+1} - \mathrm{J}_{r+1}\wedge \mathrm{J}_{r+1}\big),\ \tfrac{1}{\sqrt{2}}\big(\mathbb{1}_{r+2}\wedge \mathrm{I}_{r+2} + \mathrm{J}_{r+2}\wedge \mathrm{J}_{r+2}\big), && 1\leq r\leq 3,\\
&\tfrac{1}{2}\big(\mathbb{1}_{r+1}\wedge \mathbb{1}_{r+2} - \mathrm{I}_{r+1}\wedge \mathrm{I}_{r+2} - \mathrm{J}_{r+1}\wedge \mathrm{J}_{r+2} - \mathrm{J}_{r+1}\wedge \mathrm{J}_{r+2}\big), && 1\leq r\leq 3,\\
&\tfrac{1}{\sqrt{2}}\big(\mathbb{1}_{r+1}\wedge \mathbb{1}_{r+2} + \mathrm{J}_{r+1}\wedge \mathrm{J}_{r+2}\big), && 1\leq r\leq 3.
\end{aligned}$$

Furthermore, the following determine a basis of the H-invariant elements of $\wedge^4\mathfrak{m}$:

$$\begin{aligned}
\phi_r :=\ & (\mathbb{1}_{r+1}\wedge \mathrm{I}_{r+1} - \mathrm{J}_{r+1}\wedge \mathrm{J}_{r+1})\wedge(\mathbb{1}_{r+2}\wedge \mathrm{I}_{r+2} + \mathrm{J}_{r+2}\wedge \mathrm{J}_{r+2})\\
&+(\mathbb{1}_{r+1}\wedge \mathrm{J}_{r+1} + \mathrm{I}_{r+1}\wedge \mathrm{J}_{r+1})\wedge(\mathbb{1}_{r+2}\wedge \mathrm{J}_{r+2} - \mathrm{I}_{r+2}\wedge \mathrm{J}_{r+2})\\
&+(\mathbb{1}_{r+1}\wedge \mathrm{J}_{r+1} - \mathrm{I}_{r+1}\wedge \mathrm{J}_{r+1})\wedge(\mathbb{1}_{r+2}\wedge \mathrm{J}_{r+2} + \mathrm{I}_{r+2}\wedge \mathrm{J}_{r+2}), && 1\leq r\leq 3,\\
\psi_r :=\ & \mathbb{1}_r\wedge \mathrm{I}_r\wedge \mathrm{J}_r\wedge \mathrm{J}_r, && 1\leq r\leq 3.
\end{aligned}$$

Thus, an invariant 4-form $\omega\in\wedge^4\mathfrak{m}$ has coordinates $\big(\vec{a},\vec{b}\big) = (a_1,a_2,a_3,b_1,b_2,b_3)$, such that:

$$\omega = a_1\,\phi_1 + a_2\,\phi_2 + a_3\,\phi_3 + b_1\,\psi_1 + b_2\,\psi_2 + b_3\,\psi_3. \tag{10.4}$$

The restriction $\widehat{R}\big(\vec{s},\vec{a},\vec{b}\big)$ of a modified curvature operator $(R+\omega)\colon \wedge^2\mathfrak{m}\to\wedge^2\mathfrak{m}$

to the subspace spanned by the above representative vectors can be computed using formula (8.20). The result is the block diagonal matrix $\widehat{R} = \operatorname{diag}\big(\widehat{R}_1,\ \widehat{R}_2,\ \widehat{R}_3\big)$, where the blocks, listed in the same order as the representatives, are given by:

$$\widehat{R}_1\big(\vec{s},\vec{a},\vec{b}\big) := \operatorname{diag}\left(\begin{pmatrix} 4s_{r+1}-b_{r+1} & -\frac{s}{s_r}+2a_r \\ -\frac{s}{s_r}+2a_r & 4s_{r+2}+b_{r+2} \end{pmatrix},\ 1\leq r\leq 3\right), \tag{10.5}$$

$$\widehat{R}_2\big(\vec{s},\vec{a},\vec{b}\big) := \operatorname{diag}\left(s_{r+1}+s_{r+2}-s_r+\tfrac{s}{2s_r}+3a_r,\ 1\leq r\leq 3\right), \tag{10.6}$$

$$\widehat{R}_3\big(\vec{s},\vec{a},\vec{b}\big) := \operatorname{diag}\left(\tfrac{(s_{r+1}-s_{r+2})^2-s_r^2}{2s_r}-a_r,\ 1\leq r\leq 3\right), \tag{10.7}$$

where $s = 2(s_1s_2+s_1s_3+s_2s_3)-(s_1^2+s_2^2+s_3^2)$, cf. (9.20), (9.21), (9.22) and (9.23).

Therefore, $(W^{12},\mathrm{g}_{\vec{s}})$ has strongly positive curvature if and only if there exist $\vec{a}$ and $\vec{b}$ such that the above block diagonal matrix $\widehat{R}\big(\vec{s},\vec{a},\vec{b}\big)$ is positive-definite. Since we are assuming that $p_r(\vec{s})>0$, for $r=1,2,3$, we know from Theorem 10.1 that there exist $\vec{a}_0$ and $\vec{b}_0$ such that $\widehat{R}\big(\vec{s},\vec{a}_0,\vec{b}_0\big)$ is positive-semidefinite. In fact, it follows from the totally geodesic immersions (10.3) and Theorem 9.9 that this holds if we set $\vec{b}_0=(0,0,0)$ and $\vec{a}_0$ to be as in (9.25), i.e., such that $\widehat{R}_3\big(\vec{s},\vec{a}_0,\vec{b}_0\big)$ vanishes identically. Direct inspection shows that the second block $\widehat{R}_2\big(\vec{s},\vec{a}_0,\vec{b}_0\big)$ is positive-definite. As to the first block $\widehat{R}_1\big(\vec{s},\vec{a}_0,\vec{b}_0\big)$, note that

$$\det\begin{pmatrix} 4s_{r+1} & -\frac{s}{s_r}+2a_r \\ -\frac{s}{s_r}+2a_r & 4s_{r+2} \end{pmatrix} = \frac{4s}{s_r^2}(s_{r+1}-s_{r+2})^2. \tag{10.8}$$

If, on the one hand, s_r are pairwise different, then $\widehat{R}_1\big(\vec{s},\vec{a}_0,\vec{b}_0\big)$ is positive-definite. In this case, for any $a_r'<0$, the 4-form $\omega'=a_1'\phi_1+a_2'\phi_2+a_3'\phi_3$ becomes positive-definite on the subspace of representatives corresponding to $\ker\widehat{R}\big(\vec{s},\vec{a}_0,\vec{b}_0\big)$. Thus, the first-order perturbation $\widehat{R}\big(\vec{s},\vec{a}_0+\varepsilon\vec{a}',\vec{b}_0\big)$, where $\vec{a}'=(a_1',a_2',a_3')$, is positive-definite for sufficiently small $\varepsilon > 0$, by Lemma 9.6. If, on the other hand, $s_r \neq s_{r+1} = s_{r+2}$, then the 2×2 matrix in (10.8) has kernel spanned by $w := (1,1)^{\mathtt{t}}$. The restriction of

$$\omega' = a_1' \, \phi_1 + a_2' \, \phi_2 + a_3' \, \phi_3 + b_1' \, \psi_1 + b_2' \, \psi_2 + b_3' \, \psi_3$$

to the corresponding subspace of representatives reduces to multiplication by

$$w^{\mathtt{t}} \begin{pmatrix} -b_{r+1}' & 2a_r' \\ 2a_r' & b_{r+2}' \end{pmatrix} w = 4a_r' - b_{r+1}' + b_{r+2}'.$$

Setting $a_r' < 0$, $b_r' = b_{r+1}' = 0$ and $b_{r+2}' > -4a_r'$, the above ω' becomes positive-definite on the subspace of representatives corresponding to $\ker \widehat{R}(\vec{s}, \vec{a}_0, \vec{b}_0)$. Similarly to the previous case, the first-order perturbation $\widehat{R}(\vec{s}, \vec{a}_0 + \varepsilon \vec{a}', \vec{b}_0 + \varepsilon \vec{b}')$ is positive-definite for sufficiently small $\varepsilon > 0$, by Lemma 9.6.

Thus, we conclude that $(W^{12}, \mathrm{g}_{\vec{s}})$ has strongly positive curvature, so (i) holds. □

Remark 10.4. Even though (10.3) are totally geodesic isometric embeddings between the Wallach flag manifolds, note that the block diagonal matrix $\widehat{R}$ for W^{12} with blocks (10.5), (10.6), and (10.7) is *not* a submatrix of the block diagonal matrix $\widehat{R}$ for W^{24} with blocks (9.20), (9.21), and (9.22). This is due to the lists of representative vectors for W^{12} *not*[3] being a sublist of the one for W^{24}. We also remark that the invariant 4-forms (10.4) with $\vec{b} = (0,0,0)$ coincide with the projection of the invariant 4-forms (9.19) to $\wedge^4$ of the subspace tangent to W^{12}.

Remark 10.5. The upshot of Theorems 9.11 and 10.3 is that a homogenous metric on W^6 or W^{12} has strongly positive curvature if and only if it has $\sec > 0$, while a homogenous metric on W^{24} has strongly positive curvature if and only if it has $\sec > 0$ and does not submerge onto $\mathbb{C}\mathrm{a}P^2$.

[3] In fact, such a choice is not possible due to the differences in the corresponding representations.

Note that some of the metrics with strongly positive curvature on W^6 and W^{12} that cannot be obtained by combining Proposition 8.15 and Theorem 9.11, namely those corresponding to $\vec{s} = (1, 1, t)$, $0 < t < 1$, had been found using Theorem 9.5. The only extra information given by Theorem 10.3 is that those corresponding to $\vec{s} = (1, 1, t)$, $1 < t < \frac{4}{3}$ also have strongly positive curvature, cf. Remark 9.14. We remark that $t \in (0, 1) \cup \left(1, \frac{4}{3}\right)$ parametrizes the entire line segments in the figure of page 131, excluding the central point where they intersect (which corresponds to the normal homogeneous metric).

10.2 Berger spheres S^{2n+1}

The Hopf bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ is a homogeneous bundle as in (9.1), where the Lie groups $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ are given by

$$\mathsf{SU}(n) \subset \mathrm{S}(\mathsf{U}(n)\mathsf{U}(1)) \subset \mathsf{SU}(n+1), \tag{10.9}$$

and the inclusions are the usual ones as block matrices. The base $\mathbb{C}P^n$ is a CROSS, and $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair. The fiber S^1 has dimension 1, and can be considered to have constant positive curvature for the purposes of applying Theorem 9.5, since it trivially verifies the appropriate conditions due to $\wedge^2\mathbb{R} = \{0\}$.

Up to rescaling, the bi-invariant metric Q on $\mathfrak{g} = \mathfrak{su}(n+1)$ is given by formula (9.14). The homogeneous metric g_t defined in (9.3) is the Berger metric

$$\mathsf{g}_t = \tfrac{t(n+1)}{2n}\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}, \tag{10.10}$$

see, e.g., Grove and Ziller [47, Table 2.4]. In particular, notice that the normal homogeneous metric g_1 is *never* the round metric, except on $\mathsf{SU}(2) \cong S^3$. Furthermore, up to rescaling, the Berger metrics $\lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$, $\lambda > 0$, parametrize the space of

$\mathsf{SU}(n+1)$-invariant metrics on $S^{2n+1} = \mathsf{SU}(n+1)/\mathsf{SU}(n)$.

The Q-orthogonal complements defined in (9.2) are $\mathfrak{m} \cong \mathbb{C}^n$ and $\mathfrak{p} \cong \mathbb{R}$, and can be identified as the following subspaces of $\mathfrak{su}(n+1)$.

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & \dots & 0 & z_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & z_n \\ -\overline{z_1} & \dots & -\overline{z_n} & 0 \end{pmatrix} : z_j \in \mathbb{C} \right\}, \tag{10.11}$$

$$\mathfrak{p} = \operatorname{span}\big\{\operatorname{diag}\big(-i, \dots, -i, n\,i\big)\big\}. \tag{10.12}$$

Note that $\mathfrak{m} \otimes \mathfrak{p} \cong \mathbb{C}^n \otimes \mathbb{R} \cong \oplus^n(\mathbb{C} \otimes \mathbb{R})$, and there is a corresponding decomposition of the Lie bracket operator $L_n\colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m}$ defined in (9.5) as $L_n = \oplus^n L$, where $L = L_1$. Thus, the operator $F_n\colon \mathfrak{m}\otimes\mathfrak{p} \to \mathfrak{m}\otimes\mathfrak{p}$ defined in (9.6) decomposes as $F_n = \oplus^n F$, where $F = F_1$. Set $n = 1$, and let $\mathbb{1}$ and I be the matrices obtained by setting z_1 equal to 1 and i on (10.11), respectively. Furthermore, let $\mathrm{I}_\mathfrak{p} = \frac{1}{\sqrt{2}}\operatorname{diag}(-i, i)$, see (10.12). The Lie bracket operator $L\colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m}$ is determined by:

$\frac{1}{\sqrt{2}}[\cdot,\cdot]$	$\mathrm{I}_\mathfrak{p}$
$\mathbb{1}$	I
I	$-\mathbb{1}$

Thus, by (9.6), $\ker F = \ker L = \{0\}$, hence F is positive-definite. Therefore, also $F_n = \oplus^n F$ is positive-definite, hence the homogenous bundles $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ are strongly fat.[4]

Instead of applying Theorem 9.5 directly, notice that since $\wedge^2\mathfrak{p} = \{0\}$, the modified curvature operator $\widehat{R}$ in (9.9) is block diagonal, $\widehat{R} = \operatorname{diag}\big(\widehat{R}_{11}, \widehat{R}_{22}\big)$. From (9.10)

[4]Notice that $\wedge^2\mathfrak{p} = \{0\}$, so the only way in which the operator $(F+\tau)$, with $\tau \in \wedge^2\mathfrak{m} \otimes \wedge^2\mathfrak{p}$, can be positive-definite is if F is positive-definite, since $\tau = 0$.

and (9.11), the first block $\widehat{R}_{11}$ can be made positive-definite for all $0 < t < \frac{4}{3}$, using that the base $\mathbb{C}P^n$ has strongly positive curvature; and the second block $\widehat{R}_{22}$ is a multiple of F_n, hence positive-definite. Thus, by (10.10), we conclude that the Berger spheres $\left(S^{2n+1}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\right)$ have strongly positive curvature for all $0 < \lambda < \frac{2(n+1)}{3n}$.

Remark 10.6. The above is not the maximal range of λ for which $\left(S^{2n+1}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\right)$ has strongly positive curvature, unless $n = 1$. Notice that, e.g., since the round metric $\lambda = 1$ has positive-definite curvature operator (which is an open condition), the metrics $\lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$ have strongly positive curvature for λ in an open interval around $\lambda = 1$, for all $n \geq 1$. This is not detected by the methods of Theorem 9.5 because the normal homogeneous metric g_1 on $S^{2n+1} = \mathsf{SU}(n+1)/\mathsf{SU}(n)$ is a Berger metric that was already shrunk in the vertical directions of the Hopf bundle, see (10.10).

Although the above application of Theorem 9.5 is conceptually interesting, since it uses strong fatness of the Hopf bundle, a stronger result follows from Proposition 8.18.

Proposition 10.7. *The Berger spheres* $\left(S^{2n+1}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\right)$ *have strongly positive curvature for all* $0 < \lambda \leq 1$.

Proof. It suffices to prove that the Berger metrics $\lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$, $0 < \lambda \leq 1$, are Cheeger deformations of the round metric, and apply Proposition 8.18 and Theorem 8.20. Since $\mathsf{H} = \mathsf{SU}(n)$ is a normal subgroup of $\mathsf{K} = \mathsf{S}(\mathsf{U}(n)\mathsf{U}(1))$ in (10.9), there is an isometric K-action on $\mathsf{G}/\mathsf{H} = S^{2n+1}$ given by $k \cdot g\mathsf{H} := gk^{-1}\mathsf{H}$. The orbits[5] of this action are the fibers of $\pi\colon \mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{K}$, i.e., the Hopf fibers. The corresponding Cheeger deformation has the effect of shrinking the round metric in the vertical directions of the Hopf bundle. More precisely, the Cheeger deformation of the round metric after time $s > 0$ is the Berger metric $\frac{1}{1+s}\, \mathsf{g}_{\mathcal{V}} + \mathsf{g}_{\mathcal{H}}$, see (4.8). $\square$

Remark 10.8. We stress that this approach cannot be used in *any* of the examples

[5]The ineffective kernel of this action is H, so it descends to an isometric K/H-action on S^{2n+1}, which is simply the usual circle action with orbit space $\mathbb{C}P^n$.

studied in Sections 9.3, 9.4, 9.6 and 9.8 using Theorem 9.5, since H is not a normal subgroup of K in those cases.

Remark 10.9. Although the above is not a complete description of the moduli space of Berger spheres S^{2n+1} with strongly positive curvature, direct computations of modified curvature operators are feasible in low dimensions. The case $n = 1$ is trivial, since strongly positive curvature is equivalent to $\sec > 0$ in dimensions ≤ 4, see Corollary 8.10. The next case is $n = 2$, in which we have the Hopf bundle

$$S^1 \longrightarrow S^5 \longrightarrow \mathbb{C}P^2$$

A direct computation shows that the Berger metric $\lambda\, \mathrm{g}_{\mathcal{V}} \oplus \mathrm{g}_{\mathcal{H}}$ on S^5 has positive-definite curvature operator if and only if $0 < \lambda < \frac{6}{5} = 1.2$, and strongly positive curvature[6] if and only if $0 < \lambda < \frac{4}{3}$. Recall that $\lambda\, \mathrm{g}_{\mathcal{V}} \oplus \mathrm{g}_{\mathcal{H}}$ are known to have $\sec > 0$ if and only if $0 < \lambda < \frac{4}{3}$.

10.3 Berger spheres S^{4n+3}

The Hopf bundle $S^3 \to S^{4n+3} \to \mathbb{H}P^n$ is a homogeneous bundle as in (9.1), where the Lie groups $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ are given by

$$\mathsf{Sp}(n) \subset \mathsf{Sp}(n)\mathsf{Sp}(1) \subset \mathsf{Sp}(n+1), \tag{10.13}$$

and the inclusions are the usual ones as block matrices. The base $\mathbb{H}P^n$ is a CROSS, and $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair. The fiber S^3 has constant positive curvature and dimension ≤ 3.

Up to rescaling, the bi-invariant metric Q on $\mathfrak{g} = \mathfrak{sp}(n+1)$ is given by formula

[6]The only $\mathsf{SU}(3)$-invariant 4-forms on $S^5 = \mathsf{SU}(3)/\mathsf{SU}(2)$ are multiples of the (pull-back of) the volume form of the base $\mathbb{C}P^2$; recall Proposition 8.14.

(9.14). The homogeneous metric g_t defined in (9.3) is the Berger metric

$$\mathrm{g}_t = \tfrac{t}{2}\,\mathrm{g}_{\mathcal{V}} \oplus \mathrm{g}_{\mathcal{H}}, \tag{10.14}$$

see, e.g., Grove and Ziller [47, Table 2.4]. Furthermore, up to rescaling, there is a 3-parameter family of $\mathsf{Sp}(n+1)$-invariant metrics on $S^{4n+3} = \mathsf{Sp}(n+1)/\mathsf{Sp}(n)$, which includes the Berger metrics $\lambda\,\mathrm{g}_{\mathcal{V}} \oplus \mathrm{g}_{\mathcal{H}}$, $\lambda > 0$. The remaining $\mathsf{Sp}(n+1)$-invariant metrics are of the form $\lambda_1\,\mathrm{g}_{\mathcal{V}_1} \oplus \lambda_2\,\mathrm{g}_{\mathcal{V}_2} \oplus \lambda_3\,\mathrm{g}_{\mathcal{V}_3} \oplus \mathrm{g}_{\mathcal{H}}$, $\lambda_i > 0$, where $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$ is a Q-orthonormal decomposition of the vertical direction $\mathcal{V} \cong \mathfrak{sp}(1) \cong \mathbb{R}^3$. In what follows, we only consider the case $\lambda = \lambda_1 = \lambda_2 = \lambda_3$ corresponding to the Berger metric $\lambda\,\mathrm{g}_{\mathcal{V}} \oplus \mathrm{g}_{\mathcal{H}}$, since it has the largest isometry group after the round metric.[7]

The Q-orthogonal complements defined in (9.2) are $\mathfrak{m} \cong \mathbb{H}^n$ and $\mathfrak{p} \cong \mathfrak{sp}(1) \cong \mathbb{R}^3$, and can be identified as the following subspaces of $\mathfrak{sp}(n+1)$.

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & \dots & 0 & z_1 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & z_n \\ -\overline{z_1} & \dots & -\overline{z_n} & 0 \end{pmatrix} : z_j \in \mathbb{H} \right\}, \tag{10.15}$$

$$\mathfrak{p} = \big\{ \operatorname{diag}\big(0, \dots, 0, w\big) : w \in \operatorname{Im}\mathbb{H} \big\}. \tag{10.16}$$

The problem of verifying strong fatness can be reduced to the case $n = 1$. Indeed, $\mathfrak{m} \otimes \mathfrak{p} \cong \mathbb{H}^n \otimes \operatorname{Im}\mathbb{H} \cong \oplus^n(\mathbb{H} \otimes \operatorname{Im}\mathbb{H})$ and the Lie bracket operator $L_n \colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m}$ defined in (9.5) clearly decomposes as $L_n = \oplus^n L$, where $L = L_1$. Thus, the operator $F_n \colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m} \otimes \mathfrak{p}$ defined in (9.6) decomposes as $F_n = \oplus^n F$, where $F = F_1$.

[7] This ansatz considerably simplifies certain computations, through the use of Proposition 8.14. Recall that the identity component of the full isometry group of the above $\mathsf{Sp}(n+1)$-homogeneous metrics is either $\mathsf{Sp}(n+1)$, $\mathsf{Sp}(n+1)\mathsf{U}(1)$, $\mathsf{Sp}(n+1)\mathsf{Sp}(1)$ or $\mathsf{SO}(4n+4)$, according respectively to the cases in which λ_i are pairwise distinct, two of the λ_i coincide, $\lambda_1 = \lambda_2 = \lambda_3$, and $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

According to Definition 9.4, the homogeneous bundle (10.9) is strongly fat if there exists $\tau_n \in \wedge^2\mathfrak{m} \otimes \wedge^2\mathfrak{p}$ such that $(F_n + \tau_n)\colon \mathfrak{m}\otimes\mathfrak{p} \to \mathfrak{m}\otimes\mathfrak{p}$ is positive-definite. If there exists τ such that $F+\tau$ is positive definite, then $\tau_n = \oplus^n \tau$ is such that $F_n + \tau_n$ is positive-definite, verifying the above claim.

Assume that $n=1$, and let $\mathbb{1}$, I, J, and K be the matrices obtained by setting z_1 equal to 1, i, j, and k on (10.15), respectively. Furthermore, let $\mathrm{I}_\mathfrak{p}$, $\mathrm{J}_\mathfrak{p}$, and $\mathrm{K}_\mathfrak{p}$ be the matrices obtained by setting w equal to $\sqrt{2}i$, $\sqrt{2}j$, and $\sqrt{2}k$ on (10.16), respectively. The Lie bracket operator $L\colon \mathfrak{m}\otimes\mathfrak{p} \to \mathfrak{m}$ defined in (9.5) is determined by:

$\frac{1}{\sqrt{2}}[\cdot,\cdot]$	$\mathrm{I}_\mathfrak{p}$	$\mathrm{J}_\mathfrak{p}$	$\mathrm{K}_\mathfrak{p}$
$\mathbb{1}$	I	J	K
I	$-\mathbb{1}$	K	$-$J
J	$-$K	$-\mathbb{1}$	I
K	J	$-$I	$-\mathbb{1}$

Thus, by (9.6), $\ker F = \ker L$ is spanned by the following 8 vectors of $\mathfrak{m}\otimes\mathfrak{p}$:

$$\begin{array}{llll} \mathrm{I}\wedge\mathrm{I}_\mathfrak{p} - \mathrm{J}\wedge\mathrm{J}_\mathfrak{p}, & \mathbb{1}\wedge\mathrm{I}_\mathfrak{p} + \mathrm{K}\wedge\mathrm{J}_\mathfrak{p}, & \mathbb{1}\wedge\mathrm{J}_\mathfrak{p} + \mathrm{I}\wedge\mathrm{K}_\mathfrak{p}, & \mathbb{1}\wedge\mathrm{K}_\mathfrak{p} - \mathrm{I}\wedge\mathrm{J}_\mathfrak{p}, \\ \mathrm{I}\wedge\mathrm{I}_\mathfrak{p} - \mathrm{K}\wedge\mathrm{K}_\mathfrak{p}, & \mathbb{1}\wedge\mathrm{I}_\mathfrak{p} - \mathrm{J}\wedge\mathrm{K}_\mathfrak{p}, & \mathbb{1}\wedge\mathrm{J}_\mathfrak{p} - \mathrm{K}\wedge\mathrm{I}_\mathfrak{p}, & \mathbb{1}\wedge\mathrm{K}_\mathfrak{p} + \mathrm{J}\wedge\mathrm{I}_\mathfrak{p}. \end{array}$$

Consider the operator induced by the $\mathsf{Sp}(1)$-invariant[8] 4-form $\tau \in \wedge^2\mathfrak{m}\otimes\wedge^2\mathfrak{p}$,

$$\begin{aligned} \tau = {} & (\mathrm{J}\wedge\mathrm{K} - \mathbb{1}\wedge\mathrm{I}) \otimes (\mathrm{J}_\mathfrak{p}\wedge\mathrm{K}_\mathfrak{p}) \\ & + (\mathbb{1}\wedge\mathrm{J} + \mathrm{I}\wedge\mathrm{K}) \otimes (\mathrm{I}_\mathfrak{p}\wedge\mathrm{K}_\mathfrak{p}) \\ & + (\mathrm{I}\wedge\mathrm{J} - \mathbb{1}\wedge\mathrm{K}) \otimes (\mathrm{I}_\mathfrak{p}\wedge\mathrm{J}_\mathfrak{p}). \end{aligned} \tag{10.17}$$

[8] The corresponding 4-form $\tau \in \Omega^4(S^7)$ is invariant under the full isometry group $\mathsf{Sp}(2)\mathsf{Sp}(1)$ of the Berger metrics $\lambda\,\mathsf{g}_\mathcal{V} \oplus \mathsf{g}_\mathcal{H}$, $\lambda > 0$.

The restriction $\tau\colon \ker F \to \ker F$ is the identity operator, hence positive-definite. From Lemma 9.6, there exists $\varepsilon > 0$ such that $(F + \varepsilon\tau)\colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m} \otimes \mathfrak{p}$ is positive-definite. Thus, the homogeneous bundle $S^3 \to S^7 \to \mathbb{H}P^1$ is strongly fat, and hence so are all homogenous bundles $S^3 \to S^{4n+3} \to \mathbb{H}P^n$.

Therefore, by (10.14) and Theorem 9.5, we conclude that the Berger spheres $\big(S^{4n+3}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\big)$ have strongly positive curvature for all $0 < \lambda < \frac{1}{2}$.

Remark 10.10. Analogously to Remark 10.6, we observe that the above is not the maximal range of λ for which $\big(S^{4n+3}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\big)$ has strongly positive curvature, since, e.g., the round metric $\lambda = 1$ has positive-definite curvature operator and hence strongly positive curvature.

Although the above application of Theorem 9.5 is conceptually interesting, since it uses strong fatness of the Hopf bundle, a stronger result follows from Proposition 8.18.

Proposition 10.11. *The Berger spheres* $\big(S^{4n+3}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\big)$ *have strongly positive curvature for all* $0 < \lambda \leq 1$.

Proof. The proof is completely analogous to that of Proposition 10.7, using that $\mathsf{H} = \mathsf{Sp}(n)$ is a normal subgroup of $\mathsf{K} = \mathsf{Sp}(n)\mathsf{Sp}(1)$ in (10.13). □

Remark 10.12. Analogously to Remark 10.9, direct computations of modified curvature operators are feasible in low dimensions to yield complete descriptions of the appropriate moduli spaces of Berger metrics. Let us consider the case $n = 1$,

$$S^3 \longrightarrow S^7 \longrightarrow \mathbb{H}P^1.$$

A direct computation shows that the Berger metric $\lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$ has positive-definite curvature operator if and only if $\frac{1}{2} < \lambda < \lambda_1 \cong 1.202$, and strongly positive curvature[9]

[9]The only $\mathsf{Sp}(2)\mathsf{Sp}(1)$-invariant 4-forms on $S^7 = \mathsf{Sp}(2)/\mathsf{Sp}(1)$ are $a\,\tau + b\,\mathrm{vol}$, $a, b \in \mathbb{R}$, where vol is the (pull-back of the) volume form of the base $\mathbb{H}P^1$, and τ is given by (10.17); recall Proposition 8.14.

if and only if $0 < \lambda < \lambda_2 \cong 1.304$, where λ_1 and λ_2 are the real roots of the polynomials $p_1(\lambda) = 8\lambda^3 - 16\lambda^2 + 11\lambda - 4$ and $p_2(\lambda) = 25\lambda^3 - 60\lambda^2 + 48\lambda - 16$, respectively. Recall that these metrics have $\sec > 0$ if and only if $0 < \lambda < \frac{4}{3} \cong 1.333$.

Notice that the proper inclusions $(\frac{1}{2}, \lambda_1) \subsetneq (0, \lambda_2) \subsetneq (0, \frac{4}{3})$ correspond to proper inclusions of the classes of Berger metrics with, respectively, positive-definite curvature operator, strongly positive curvature, and $\sec > 0$. It is also straightforward to verify that the closures of these intervals, namely $\left[\frac{1}{2}, \lambda_1\right] \subsetneq \left[0, \lambda_2\right] \subsetneq \left[0, \frac{4}{3}\right]$ correspond to proper inclusions of the classes of Berger metrics with, respectively, positive-semidefinite curvature operator, strongly nonnegative curvature, and $\sec \geq 0$.

Finally, recall that S^7 admits a totally geodesic embedding into S^{4n+3} for all $n \geq 1$, where both are equipped with the Berger metric $\lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$. Thus, by Proposition 8.15, the Berger spheres $\left(S^{4n+3}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\right)$ with $n \geq 1$ and $\lambda_2 < \lambda < \frac{4}{3}$, are examples of homogeneous spaces with $\sec > 0$ that do not have strongly positive curvature, similarly to $\mathbb{C}\mathrm{a}P^2$, B^{13}, and W^{24}, see Theorem 8.22, Remark 9.12 and Theorems 9.8 and 9.11. However, we stress that, differently from the previous examples, the Berger spheres $\left(S^{4n+3}, \lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}\right)$, $\lambda_2 < \lambda < \frac{4}{3}$, do not have strongly nonnegative curvature.

We conclude this section mentioning a few important consequences of the above discussion for the bundle

$$\mathbb{C}P^1 \to \mathbb{C}P^{2n+1} \to \mathbb{H}P^n. \tag{10.18}$$

This is also a homogeneous bundle as in (9.1), where the Lie groups $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ are respectively given by

$$\mathsf{Sp}(n)\mathsf{U}(1) \subset \mathsf{Sp}(n)\mathsf{Sp}(1) \subset \mathsf{Sp}(n+1), \tag{10.19}$$

There are 10 linearly independent 4-forms on S^7 invariant under the smaller group $\mathsf{Sp}(2)$, including τ and vol. This makes the problem of determining the moduli space of strongly positive curvature homogeneous metrics with 3 different parameters $\{\lambda_1, \lambda_2, \lambda_3\}$ much more complicated.

and the inclusions are the usual ones as block matrices. Note that the smaller group $\mathsf{H}' = \mathsf{Sp}(n)$ for the Hopf bundle is a subgroup of the above $\mathsf{H} = \mathsf{Sp}(n)\mathsf{U}(1)$, see (10.13). Since $\mathsf{H}/\mathsf{H}' \cong S^1$, fiber and total space of the Hopf bundle are circle bundles over the corresponding fiber and total space of (10.18), that is, the diagram

$$\begin{array}{ccccc} S^3 & \longrightarrow & S^{4n+3} & \longrightarrow & \mathbb{H}P^n \\ \downarrow & & \downarrow & & \| \\ S^2 & \longrightarrow & \mathbb{C}P^{2n+1} & \longrightarrow & \mathbb{H}P^n \end{array}$$

is commutative, where the horizontal lines are the homogeneous bundles corresponding to (10.13) and (10.19), and the vertical arrows are projections of circle bundles. In particular, the complements $\mathfrak{p}'$ and $\mathfrak{p}$ such that $\mathfrak{k} = \mathfrak{h}' \oplus \mathfrak{p}' = \mathfrak{h} \oplus \mathfrak{p}$ satisfy $\mathfrak{p} \subset \mathfrak{p}'$. Thus, it follows that the fatness map $F\colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m} \otimes \mathfrak{p}$ for the bundle (10.18) is the restriction to $\mathfrak{m} \otimes \mathfrak{p}$ of the fatness map $F'\colon \mathfrak{m} \otimes \mathfrak{p}' \to \mathfrak{m} \otimes \mathfrak{p}'$ for the Hopf bundle. Since there exists τ' such that $F' + \tau'$ is positive-definite, it follows that $F + \tau$ is also positive-definite, where τ is the restriction of τ' to $\mathfrak{m} \otimes \mathfrak{p}$. Therefore, (10.18) is a strongly fat bundle for all $n \geq 1$.

The other hypotheses of Theorem 9.5 are also easily verified for (10.18). Thus, we conclude that the Berger-type metrics $\lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$ on $\mathbb{C}P^{2n+1}$ satisfy the following result, where $\mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$ is the standard metric, and $\mathsf{g}_{\mathcal{V}}$ and $\mathsf{g}_{\mathcal{H}}$ respectively denote its vertical and horizontal parts with respect to (10.18).

Proposition 10.13. *The Berger-type metrics* $\lambda\, \mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$ *on* $\mathbb{C}P^{2n+1}$ *have strongly positive curvature for all* $0 < \lambda \leq 1$*.*

The case $\lambda = 1$ does not follow from Theorem 9.5, but has been addressed in Theorem 8.20. Notice that Proposition 10.13 does not follow from Proposition 10.11. Indeed, equipping S^{4n+3} with any Berger metric, the projection $S^{4n+3} \to \mathbb{C}P^{2n+1}$ is a Riemannian submersion, where $\mathbb{C}P^{2n+1}$ is equipped with its standard metric. Moreover, notice that the method used in the proof of Propositons 10.7 and 10.11

does not apply to the above bundle (10.18), since $\mathsf{H} = \mathsf{Sp}(n)\mathsf{U}(1)$ is not normal in $\mathsf{K} = \mathsf{Sp}(n)\mathsf{Sp}(1)$, see (10.19).

10.4 Berger sphere S^{15}

The Hopf bundle $S^7 \to S^{15} \to S^8\big(\frac{1}{2}\big)$ is a homogeneous bundle as in (9.1), where the Lie groups $\mathsf{H} \subset \mathsf{K} \subset \mathsf{G}$ are given by

$$\mathsf{Spin}(7) \subset \mathsf{Spin}(8) \subset \mathsf{Spin}(9). \tag{10.20}$$

The above inclusions are described in Section 9.5, and further details can be found in Grove and Ziller [47, p. 633-634] and Verdiani and Ziller [99, p. 476]. The base $S^8\big(\frac{1}{2}\big)$ is a CROSS, the fiber S^7 has constant positive curvature, and both $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{k}, \mathfrak{h})$ are symmetric pairs.

Up to rescaling, the bi-invariant metric Q on $\mathfrak{g} = \mathfrak{so}(9)$ is given by formula (9.14), where Re can be omitted. The homogeneous metric g_t defined in (9.3) is the Berger metric

$$\mathsf{g}_t = \tfrac{t}{4}\,\mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}, \tag{10.21}$$

see, e.g., Grove and Ziller [47, Table 2.4]. Furthermore, up to rescaling, the Berger metrics $\lambda\,\mathsf{g}_{\mathcal{V}} \oplus \mathsf{g}_{\mathcal{H}}$, $\lambda > 0$, parametrize the space of $\mathsf{Spin}(9)$-invariant metrics on $S^{15} = \mathsf{Spin}(9)/\mathsf{Spin}(7)$.

The Q-orthogonal complements defined in (9.2) are $\mathfrak{m} \cong \mathbb{R}^8$ and $\mathfrak{p} \cong \mathbb{R}^7$, and can be identified as the following subspaces of $\mathfrak{so}(9)$,

$$\begin{aligned}
\mathfrak{m} &= \operatorname{span}\big\{E_{19}, E_{29}, E_{39}, E_{49}, E_{59}, E_{69}, E_{79}, E_{89}\big\},\\
\mathfrak{p} &= \operatorname{span}\big\{V_1 := \tfrac{1}{2}(E_{15} + E_{26} + E_{37} + E_{48}), V_2 := \tfrac{1}{2}(E_{17} + E_{28} - E_{35} - E_{46}),\\
&\qquad V_3 := \tfrac{1}{2}(E_{13} - E_{24} - E_{57} + E_{68}), V_4 := \tfrac{1}{2}(E_{16} - E_{25} - E_{38} + E_{47}),
\end{aligned}$$

$$V_5 := \tfrac{1}{2}(E_{18} - E_{27} + E_{36} - E_{45}), V_6 := \tfrac{1}{2}(E_{12} + E_{34} - E_{56} - E_{78}),$$
$$V_7 := \tfrac{1}{2}(E_{14} + E_{23} - E_{58} - E_{67})\big\},$$

where $E_{ij} \in \mathfrak{so}(9)$ denotes the matrix with 1 in the (i,j)th entry and -1 in the (j,i)th entry. The Lie bracket operator $L\colon \mathfrak{m} \otimes \mathfrak{p} \to \mathfrak{m}$ defined in (9.5) is determined by:

$2[\cdot,\cdot]$	V_1	V_2	V_3	V_4	V_5	V_6	V_7
E_{19}	E_{59}	E_{79}	E_{39}	E_{69}	E_{89}	E_{29}	E_{49}
E_{29}	E_{69}	E_{89}	$-E_{49}$	$-E_{59}$	$-E_{79}$	$-E_{19}$	E_{39}
E_{39}	E_{79}	$-E_{59}$	$-E_{19}$	$-E_{89}$	E_{69}	E_{49}	$-E_{29}$
E_{49}	E_{89}	$-E_{69}$	E_{29}	E_{79}	$-E_{59}$	$-E_{39}$	$-E_{19}$
E_{59}	$-E_{19}$	E_{39}	$-E_{79}$	E_{29}	E_{49}	$-E_{69}$	$-E_{89}$
E_{69}	$-E_{29}$	E_{49}	E_{89}	$-E_{19}$	$-E_{39}$	E_{59}	$-E_{79}$
E_{79}	$-E_{39}$	$-E_{19}$	E_{59}	$-E_{49}$	E_{29}	$-E_{89}$	E_{69}
E_{89}	$-E_{49}$	$-E_{29}$	$-E_{69}$	E_{39}	$-E_{19}$	E_{79}	E_{59}

Thus, by (9.6), $\ker F = \ker L$ is spanned by the following 48 vectors of $\mathfrak{m} \otimes \mathfrak{p}$:

$$-E_{19}\wedge V_1 + E_{89}\wedge V_7, \quad -E_{19}\wedge V_2 + E_{89}\wedge V_6, \quad -E_{29}\wedge V_6 + E_{89}\wedge V_5,$$
$$-E_{19}\wedge V_3 + E_{89}\wedge V_4, \quad E_{19}\wedge V_4 + E_{89}\wedge V_3, \quad E_{19}\wedge V_6 + E_{89}\wedge V_2,$$
$$E_{19}\wedge V_7 + E_{89}\wedge V_1, \quad -E_{19}\wedge V_4 + E_{79}\wedge V_7, \quad E_{19}\wedge V_5 + E_{79}\wedge V_6,$$
$$-E_{19}\wedge V_6 + E_{79}\wedge V_5, \quad E_{19}\wedge V_7 + E_{79}\wedge V_4, \quad -E_{19}\wedge V_1 + E_{79}\wedge V_3,$$
$$-E_{29}\wedge V_6 + E_{79}\wedge V_2, \quad E_{19}\wedge V_3 + E_{79}\wedge V_1, \quad E_{19}\wedge V_2 + E_{69}\wedge V_7,$$
$$-E_{19}\wedge V_1 + E_{69}\wedge V_6, \quad E_{19}\wedge V_3 + E_{69}\wedge V_5, \quad -E_{29}\wedge V_6 + E_{69}\wedge V_4,$$
$$-E_{19}\wedge V_5 + E_{69}\wedge V_3, \quad -E_{19}\wedge V_7 + E_{69}\wedge V_2, \quad E_{19}\wedge V_6 + E_{69}\wedge V_1,$$
$$E_{19}\wedge V_5 + E_{59}\wedge V_7, \quad E_{19}\wedge V_4 + E_{59}\wedge V_6, \quad -E_{19}\wedge V_7 + E_{59}\wedge V_5,$$
$$-E_{19}\wedge V_6 + E_{59}\wedge V_4, \quad E_{19}\wedge V_2 + E_{59}\wedge V_3, \quad -E_{19}\wedge V_3 + E_{59}\wedge V_2,$$

$$
\begin{array}{rrr}
-E_{29}\wedge V_6+E_{59}\wedge V_1, & -E_{29}\wedge V_6+E_{49}\wedge V_7, & E_{19}\wedge V_3+E_{49}\wedge V_6,\\
E_{19}\wedge V_1+E_{49}\wedge V_5, & -E_{19}\wedge V_2+E_{49}\wedge V_4, & -E_{19}\wedge V_6+E_{49}\wedge V_3,\\
E_{19}\wedge V_4+E_{49}\wedge V_2, & -E_{19}\wedge V_5+E_{49}\wedge V_1, & E_{19}\wedge V_6+E_{39}\wedge V_7,\\
-E_{19}\wedge V_7+E_{39}\wedge V_6, & -E_{19}\wedge V_4+E_{39}\wedge V_5, & E_{19}\wedge V_5+E_{39}\wedge V_4,\\
-E_{29}\wedge V_6+E_{39}\wedge V_3, & E_{19}\wedge V_1+E_{39}\wedge V_2, & -E_{19}\wedge V_2+E_{39}\wedge V_1,\\
-E_{19}\wedge V_3+E_{29}\wedge V_7, & E_{19}\wedge V_2+E_{29}\wedge V_5, & E_{19}\wedge V_1+E_{29}\wedge V_4,\\
E_{19}\wedge V_7+E_{29}\wedge V_3, & -E_{19}\wedge V_5+E_{29}\wedge V_2, & -E_{19}\wedge V_4+E_{29}\wedge V_1.
\end{array}
$$

Consider the operator induced by the $\mathsf{Spin}(7)$-invariant 4-form $\tau \in \wedge^2\mathfrak{m}\otimes\wedge^2\mathfrak{p}$,

$$
\begin{aligned}
\tau =&-(E_{19}\wedge E_{59}-E_{29}\wedge E_{69}+E_{39}\wedge E_{79}-E_{49}\wedge E_{89})\otimes(V_2\wedge V_3)\\
&-(E_{19}\wedge E_{59}+E_{29}\wedge E_{69}-E_{39}\wedge E_{79}-E_{49}\wedge E_{89})\otimes(V_4\wedge V_6)\\
&-(E_{19}\wedge E_{59}-E_{29}\wedge E_{69}-E_{39}\wedge E_{79}+E_{49}\wedge E_{89})\otimes(V_5\wedge V_7)\\
&+(E_{19}\wedge E_{79}-E_{29}\wedge E_{89}-E_{39}\wedge E_{59}+E_{49}\wedge E_{69})\otimes(V_1\wedge V_3)\\
&+(E_{19}\wedge E_{79}-E_{29}\wedge E_{89}+E_{39}\wedge E_{59}-E_{49}\wedge E_{69})\otimes(V_4\wedge V_7)\\
&-(E_{19}\wedge E_{79}+E_{29}\wedge E_{89}+E_{39}\wedge E_{59}+E_{49}\wedge E_{69})\otimes(V_5\wedge V_6)\\
&-(E_{19}\wedge E_{39}+E_{29}\wedge E_{49}-E_{59}\wedge E_{79}-E_{69}\wedge E_{89})\otimes(V_1\wedge V_2)\\
&+(E_{19}\wedge E_{39}+E_{29}\wedge E_{49}+E_{59}\wedge E_{79}+E_{69}\wedge E_{89})\otimes(V_4\wedge V_5)\\
&-(E_{19}\wedge E_{39}-E_{29}\wedge E_{49}+E_{59}\wedge E_{79}-E_{69}\wedge E_{89})\otimes(V_6\wedge V_7)\\
&+(E_{19}\wedge E_{69}-E_{29}\wedge E_{59}+E_{39}\wedge E_{89}-E_{49}\wedge E_{79})\otimes(V_1\wedge V_6)\\
&-(E_{19}\wedge E_{69}+E_{29}\wedge E_{59}+E_{39}\wedge E_{89}+E_{49}\wedge E_{79})\otimes(V_2\wedge V_7) \qquad (10.22)\\
&-(E_{19}\wedge E_{69}+E_{29}\wedge E_{59}-E_{39}\wedge E_{89}-E_{49}\wedge E_{79})\otimes(V_3\wedge V_5)\\
&+(E_{19}\wedge E_{89}+E_{29}\wedge E_{79}-E_{39}\wedge E_{69}-E_{49}\wedge E_{59})\otimes(V_1\wedge V_7)
\end{aligned}
$$

$$+\left(E_{19}\wedge E_{89}-E_{29}\wedge E_{79}-E_{39}\wedge E_{69}+E_{49}\wedge E_{59}\right)\otimes(V_2\wedge V_6)$$
$$+\left(E_{19}\wedge E_{89}+E_{29}\wedge E_{79}+E_{39}\wedge E_{69}+E_{49}\wedge E_{59}\right)\otimes(V_3\wedge V_4)$$
$$-\left(E_{19}\wedge E_{29}-E_{39}\wedge E_{49}-E_{59}\wedge E_{69}+E_{79}\wedge E_{89}\right)\otimes(V_1\wedge V_4)$$
$$-\left(E_{19}\wedge E_{29}-E_{39}\wedge E_{49}+E_{59}\wedge E_{69}-E_{79}\wedge E_{89}\right)\otimes(V_2\wedge V_5)$$
$$+\left(E_{19}\wedge E_{29}+E_{39}\wedge E_{49}+E_{59}\wedge E_{69}+E_{79}\wedge E_{89}\right)\otimes(V_3\wedge V_7)$$
$$-\left(E_{19}\wedge E_{49}-E_{29}\wedge E_{39}-E_{59}\wedge E_{89}+E_{69}\wedge E_{79}\right)\otimes(V_1\wedge V_5)$$
$$+\left(E_{19}\wedge E_{49}-E_{29}\wedge E_{39}+E_{59}\wedge E_{89}-E_{69}\wedge E_{79}\right)\otimes(V_2\wedge V_4)$$
$$-\left(E_{19}\wedge E_{49}+E_{29}\wedge E_{39}+E_{59}\wedge E_{89}+E_{69}\wedge E_{79}\right)\otimes(V_3\wedge V_6).$$

The restriction $\tau\colon \ker F\to\ker F$ is the identity operator, hence positive-definite. From Lemma 9.6, there exists $\varepsilon>0$ such that $(F+\varepsilon\tau)\colon \mathfrak{m}\otimes\mathfrak{p}\to\mathfrak{m}\otimes\mathfrak{p}$ is positive-definite. Thus, the homogeneous bundle $S^7\to S^{15}\to S^8\left(\frac{1}{2}\right)$ is strongly fat.

Therefore, by (10.21) and Theorem 9.5, we conclude that the Berger sphere $\left(S^{15},\lambda\,\mathsf{g}_{\mathcal{V}}\oplus\mathsf{g}_{\mathcal{H}}\right)$ has strongly positive curvature for all $0<\lambda<\frac{1}{4}$. Once more, this is not the maximal range of λ on which there is strongly positive curvature.

Proposition 10.14. *The Berger sphere* $\left(S^{15},\lambda\,\mathsf{g}_{\mathcal{V}}\oplus\mathsf{g}_{\mathcal{H}}\right)$ *has positive-definite curvature operator if and only if* $\frac{1}{2}<\lambda<\lambda_3\cong 1.165$, *and strongly positive curvature if and only if* $0<\lambda<\lambda_4\cong 1.184$, *where* λ_3 *is the only real root of* $p_3(\lambda)=16\lambda^3-32\lambda^2+19\lambda-4$ *and* λ_4 *is the largest real root of* $p_4(\lambda)=289\lambda^3-612\lambda^2+360\lambda-48$.

Proof. Follows from a direct computation analogous to those in Remarks 10.9 and 10.12, that uses Proposition 8.14 and the fact that the only $\mathsf{Spin}(9)$-invariant 4-forms on $S^{15}=\mathsf{Spin}(9)/\mathsf{Spin}(7)$ are $a\,\tau+b\,\omega$, $a,b\in\mathbb{R}$, where τ is given by (10.22), and $\omega\in\wedge^4\mathfrak{m}$ is given by

$$\omega=E_{19}\wedge E_{29}\wedge E_{39}\wedge E_{49}-E_{59}\wedge E_{69}\wedge E_{79}\wedge E_{89}$$

$$- (E_{19} \wedge E_{29} + E_{39} \wedge E_{49}) \wedge (E_{59} \wedge E_{69} + E_{79} \wedge E_{89})$$
$$- (E_{19} \wedge E_{39} - E_{29} \wedge E_{49}) \wedge (E_{59} \wedge E_{79} - E_{69} \wedge E_{89})$$
$$- (E_{19} \wedge E_{49} + E_{29} \wedge E_{39}) \wedge (E_{59} \wedge E_{89} + E_{69} \wedge E_{79}). \qquad \square$$

Remark 10.15. Analogously to Remark 10.12, we observe that the proper inclusions $(\frac{1}{2}, \lambda_3) \subsetneq (0, \lambda_4) \subsetneq (0, \frac{4}{3})$ correspond to proper inclusions of the classes of Berger metrics with, respectively, positive-definite curvature operator, strongly positive curvature, and $\sec > 0$. It is also straightforward to verify that the closures of these intervals, $\left[\frac{1}{2}, \lambda_3\right] \subsetneq \left[0, \lambda_4\right] \subsetneq \left[0, \frac{4}{3}\right]$ correspond to proper inclusions of the classes of Berger metrics with, respectively, positive-semidefinite curvature operator, strongly nonnegative curvature, and $\sec \geq 0$. Thus, $\left(S^{15}, \lambda\, \mathrm{g}_{\mathcal{V}} \oplus \mathrm{g}_{\mathcal{H}}\right)$, $\lambda_4 < \lambda < \frac{4}{3}$, is a further example of homogeneous space with $\sec > 0$ without strongly nonnegative curvature.

Remark 10.16. Since the Hopf bundle $S^7 \to S^{15} \to S^8\left(\frac{1}{2}\right)$ is such that both $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{k}, \mathfrak{h})$ are symmetric pairs, see (10.20), Remark 10.15 implies that the strong version of Wallach's theorem (Theorem 9.5) does not extend to $1 < t < \frac{4}{3}$, see Remark 9.14.

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