Renato Ghini Bettiol
CUNY Lehman College and Graduate Center
https://www.lehman.edu/faculty/rbettiol
✉ r.bettiol@lehman.cuny.edu

Annotated bibliography

This is a list of my papers and preprints divided into the following broad topics:

I. Curvature and topology (in general dimension)
II. Curvature and topology (in dimensions 3 and 4)
III. Ricci flow
IV. Minimal surfaces and Constant Mean Curvature hypersurfaces
V. Conformal geometry
VI. Spectral geometry
VII. Flat manifolds
VIII. Equivariant methods in Geometric Variational Problems

A short description accompanies each entry, highlighting its key contributions and contextualizing the main results obtained. To access the full text, please click on the *MathReviews* or *arXiv* links, if available.

I. Curvature and topology (in general dimension)

(1) **Curvature operators and rational cobordism** (with M. Goodman)
submitted, arXiv:2212.07548

Generalizing the celebrated $\hat{A}$ genus obstruction to positive scalar curvature on spin manifolds of Lichnerowicz, we determine linear inequalities on the eigenvalues of curvature operators that imply vanishing of twisted $\hat{A}$ genera, where the twisting bundle is any prescribed parallel bundle of tensors. These curvature conditions are shown to be preserved by high-codimension surgeries.

(2) **Convex algebraic geometry of curvature operators** (with M. Kummer and R. Mendes)
SIAM J. Appl. Algebra Geom. 5 (2021), no. 2, 220-228, MR 4252070, arXiv:1908.03713

This paper establishes foundational links between curvature operators of Riemannian n-manifolds with $\sec \geq k$ or $\sec \leq k$ and the emerging field of Convex Algebraic Geometry, which originates from a coalescence of ideas in Convex Geometry, Optimization, and Algebraic Geometry. We determine in which dimensions n the convex semialgebraic set of such curvature operators is a *spectrahedron* or a *spectrahedral shadow*; in particular, for $n \geq 5$, these give new counter-examples to the Helton–Nie Conjecture. We also provide efficient algorithms to test membership in such sets, and discuss applications to information geometry and geometric data analysis.

(3) **Sectional curvature and Weitzenböck formulae** (with R. Mendes)
Indiana Univ. Math. J. 71 (2022), no. 3, 1209–1242, MR 4448583, arXiv:1708.09033

Motivated by an observation of Hitchin regarding positivity of curvature operators, we prove that sectional curvature bounds $\sec \geq k$ and $\sec \leq k$ are respectively equivalent to positive/negative-semidefiniteness of curvature terms in the Weitzenböck formulae for traceless symmetric p-tensors for all $p \geq 2$. We introduce a symmetric version of the Kulkarni-Nomizu product that enables to easily compute these curvature terms with explicit formulae. As an application, we prove that if a 4-manifold with indefinite intersection form (such as $S^2 \times S^2$) has $\sec > 0$, then the set of points where its curvature operator is not positive-definite has at least two connected components with nonempty interior. This provides some of the only symmetry-free evidence currently available in support of a (widely expected) negative answer to the famous question of Hopf from the 1920s on whether $S^2 \times S^2$ admits metrics with $\sec > 0$, see also (8) and (10).

(4) **Strongly nonnegative curvature** (with R. Mendes)
Math. Ann. 368 (2017), no. 3-4, 971-986. MR 3673642, arXiv:1511.07899

All currently known constructions of manifolds with sec ≥ 0 are shown to yield a stronger condition: their curvature operator can be modified with a 4-form to become a positive-semidefinite operator. This work is a natural continuation of the systematic study initiated in paper (6).

(5) **Flag manifolds with strongly positive curvature** (with R. Mendes)
Math. Z. 280 (2015), no. 3-4, 1031-1046. MR 3369365, arXiv:1412.0039

We give a complete description of the moduli space of homogeneous metrics with strongly positive curvature on $W^6 = \mathsf{SU}(3)/T^2$, $W^{12} = \mathsf{Sp}(3)/\mathsf{Sp}(1)^3$, and $W^{24} = \mathsf{F}_4/\mathsf{Spin}(8)$, which are respectively the manifolds of complete flags in $\mathbb{C}^3$, $\mathbb{H}^3$, and $\mathbb{C}\mathrm{a}^3$. Together with (6), this concludes the classification of simply connected homogeneous spaces with strongly positive curvature.

(6) **Strongly positive curvature** (with R. Mendes)
Ann. Global Anal. Geom. 53 (2018), no. 3, 287-309, MR 3785699, arXiv:1403.2117

This paper initiates a systematic study of manifolds with *strongly positive curvature*, i.e., manifolds whose curvature operator can be modified with a 4-form to become positive-definite. This is an intermediate curvature condition between sec > 0 and positive-definite curvature operator, and it is shown to be preserved under Riemannian submersions and Cheeger deformations.

II. Curvature and topology (in dimensions 3 and 4)

(7) **Extremality and rigidity for scalar curvature in dimension four** (with M. Goodman)
submitted, arXiv:2205.00543

Following Gromov, a Riemannian manifold is *area-extremal* if any modification that increases scalar curvature must decrease the area of some tangent 2-plane. We prove that large classes of compact 4-manifolds, with or without boundary, with sec ≥ 0 are area-extremal. We also show that all regions with sec > 0 on 4-manifolds are locally area-extremal. Our results apply to 4-manifolds with general holonomy, while similar previous results (which also rely on index theory for twisted Dirac operators) only apply to spheres or manifolds with special holonomy.

(8) **Geography of pinched four-manifolds** (with M. Kummer and R. Mendes)
Comm. Anal. Geom., to appear, arXiv:2106.02138

We prove several new restrictions on the Euler characteristic and signature of oriented 4-manifolds with (positively or negatively) pinched sectional curvature, building on algebro-geometric results from (2) with convex optimization tools. In particular, we quantify diameter and volume bounds of Gromov, and refine estimates of Berger. A striking consequence is that simply connected 4-manifolds with $0.161 \cong \frac{1}{1+3\sqrt{3}} \leq \sec \leq 1$ are homeomorphic to S^4 or $\mathbb{C}P^2$, cf. (3) and (10).

(9) **Four-dimensional manifolds with positive biorthogonal curvature**
Asian J. Math 21 (2017), no. 2, 391-396, MR 3672264, arXiv:1502.02270

Closed simply connected 4-manifolds that admit a metric with positive biorthogonal curvature ($\sec^\perp > 0$) are classified, up to homeomorphism. Recall that a manifold has $\sec^\perp > 0$ if given any two orthogonal tangent planes, the average of their sectional curvatures is positive. Building on my previous work (10), I show that the connected sums of any number of copies of $S^2 \times S^2$, $\mathbb{C}P^2$, and $\overline{\mathbb{C}P}^2$ has $\sec^\perp > 0$. Thus, up to changing their smooth structures, it is equivalent for closed simply connected 4-manifolds to admit metrics with $\sec^\perp > 0$, $\mathrm{Ric} > 0$, and $\mathrm{scal} > 0$.

(10) **Positive biorthogonal curvature on $S^2 \times S^2$**
Proc. Amer. Math. Soc. 142 (2014), no. 12, 4341-4353. MR 3267002, arXiv:1210.0043

Motivated by a question of Hopf (see (3)), we show that $S^2 \times S^2$ satisfies an intermediate curvature condition between Ric > 0 and sec > 0. Namely, it carries metrics for which the average of the sectional curvatures of any two planes tangent at the same point but separated by a minimum distance in the Grassmannian is positive. In particular, $S^2 \times S^2$ carries metrics with $\sec^\perp > 0$.

(11) **Three-manifolds with many flat planes** (with B. Schmidt)
Trans. Amer. Math. Soc. 370 (2018), no. 1, 669-693. MR 3717993, arXiv:1407.4165

We prove that a complete 3-manifold M has higher Euclidean rank (i.e., all of its geodesics γ carry a nonzero parallel Jacobi field orthogonal to $\dot\gamma$) if and only if its universal covering splits isometrically as $\widetilde{M} = N \times \mathbb{R}$. To our knowledge, this is the first rank rigidity result that does not require any hypotheses on the curvature or volume of M. The difference between higher Euclidean rank and other notions of an abundance of flat planes is also explored. I am currently mentoring a doctoral student who is working on extensions of this result to higher dimensions.

III. Ricci flow

(12) **Ricci flow does not preserve positive sectional curvature in dimension four**
(with A. Krishnan)
Calc. Var. Partial Differential Equations 62 (2023), no. 1, Paper No. 13, MR 4505156, arXiv:21..

We show that S^4 and $\mathbb{C}P^2$ admit metrics with sec > 0 that eventually lose this property when evolved via Ricci flow. This result builds on our earlier work (13); namely, we prove that Grove–Ziller metrics on these manifolds are C^∞-limits of metrics with sec > 0.

(13) **Four-dimensional cohomogeneity one Ricci flow and nonnegative sectional curvature**
(with A. Krishnan)
Comm. Anal. Geom. 27 (2019), no. 3, 511-527, MR 4003002, arXiv:1606.00778

We show that the so-called Grove–Ziller metrics with sec ≥ 0 on S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ immediately lose sec ≥ 0 if evolved via Ricci flow. This behavior, which indicates limitations of Ricci flow beyond dimension 3 (where sec ≥ 0 is preserved), was previously known to happen only in dimensions ≥ 6 or in noncompact manifolds. The proof relies on studying Ricci flow on cohomogeneity one manifolds, where it reduces to a system of PDEs in 2 variables. This joint work was one of the pillars of A. Krishnan's UPenn doctoral thesis under W. Ziller from 2019.

IV. Minimal surfaces and Constant Mean Curvature hypersurfaces

(14) **Bifurcating minimal tori in ellipsoids** (with P. Piccione)
in preparation

In a follow-up to our recent work (15), we show that sufficiently elongated 3-dimensional ellipsoids with torus symmetry admit at least two geometrically distinct embedded smooth minimal tori.

(15) **Nonplanar minimal spheres in ellipsoids of revolution** (with P. Piccione)
submitted, arXiv:2111.14995

We use global bifurcation techniques to prove that sufficiently elongated 3-dimensional ellipsoids of revolution admit arbitrarily many geometrically distinct nonplanar embedded smooth minimal 2-spheres. This refines the negative answer recently obtained by Haslhofer–Ketover to a question of Yau from 1987. More precisely, we quantify the growth rate of the number of such minimal spheres, and describe their asymptotic behavior as the ellipsoids converge to a cylinder.

(16) **Deformations of free boundary CMC hypersurfaces** (with P. Piccione and B. Santoro)
J. Geom. Anal. 27 (2017), no. 4, 3254-3284. MR 3708014, arXiv:1411.0354

We show that a free boundary hypersurface in M with constant mean curvature H that is equivariantly nondegenerate can be deformed into unique (up to rigid motions) free boundary hypersurfaces with constant mean curvature close to H inside M with ambient metrics close to the original one, provided their Killing fields vary smoothly, cf. (31). As applications, we deform free boundary CMC disks and annuli in the unit ball of space forms with varying curvature.

(17) **Delaunay-type hypersurfaces in cohomogeneity one manifolds** (with P. Piccione)
Int. Math. Res. Not. IMRN, 2016, no. 10, 3124-3162. MR 3551832, arXiv:1306.6043

Delaunay surfaces are highly symmetric CMC surfaces of space forms. We use bifurcation theory and cohomogeneity one actions to construct infinitely many Delaunay-type hypersurfaces in a large class of manifolds, ranging from S^n, $\mathbb{C}P^n$, and $\mathbb{H}P^n$, cf. (25), to Kervaire exotic spheres.

V. Conformal geometry

(18) **Multiplicity of singular solutions to the fractional Yamabe problem on spheres** (with M. d. M. González and A. Maalaoui)
in preparation

We prove nonuniqueness results including the existence of uncountably many solutions to the Singular γ-Fractional Yamabe Problem on $S^n \setminus S^1$ for all $n \geq 4$ and $0 < \gamma \lesssim \frac{n}{2} - 1$, extending results in (22) for the Yamabe Problem ($\gamma = 1$) and in (20) for the constant Q-curvature problem ($\gamma = 2$). The *fractional* Yamabe problem corresponds to a nonlocal PDE involving a scattering operator with the same principal symbol as the fractional Laplacian $(-\Delta)^\gamma$, with $0 < \gamma < \frac{n}{2}$.

(19) **Bifurcating metrics of constant Q-curvature in four dimensions** (with S. Sbiti)
in preparation

Nonuniqueness results are proven for the constant Q-curvature problem on certain 4-manifolds, including products of surfaces, showing that some phenomena observed in (20) also occur in dimension 4, despite the exponential nonlinearity of the Q-curvature PDE in that dimension. This project was part of S. Sbiti's doctoral work at UPenn, where I co-advised him with W. Ziller.

(20) **Nonuniqueness of conformal metrics with constant Q-curvature** (with P. Piccione and Y. Sire)
Int. Math. Res. Not. IMRN 2021, no. 9, 6967-6992, MR 4251294, arXiv:1806.01373

This paper establishes the first nonuniqueness results for metrics with constant Q-curvature in a given conformal class, in dimensions ≥ 5. In addition to some analogies with the Yamabe problem, adapting techniques from (21) and (23), we observe new phenomena which is only possible in the absence of the maximum principle; for instance, we find bifurcating branches of metrics with constant Q-curvature issuing from homogeneous metrics with $Q < 0$ and $\mathrm{scal} < 0$.

(21) **Infinitely many solutions to the Yamabe problem on noncompact manifolds** (with P. Piccione)
Ann. Inst. Fourier (Grenoble) 68 (2018), no. 2, 589-609, MR 3803113, arXiv:1603.07788

This is the last in a series of joint papers with P. Piccione, establishing multiplicity results for the Yamabe problem in various settings. The main novelty here is a topological argument combining

Aubin's inequality and fundamental groups with infinite profinite completion to show that a very large class of noncompact product manifolds (including $S^n \times \mathbb{H}^d$, $2 \leq d < n$, and $S^n \times \mathbb{R}^d$, $n \geq 2$, $d \geq 1$) carries infinitely many complete conformal metrics with constant scalar curvature. As a corollary, there are infinitely many solutions to the Singular Yamabe Problem on $S^n \setminus S^k$ for all $0 \leq k < (n-2)/2$. This generalizes the result in our previous paper (22), that dealt with $k = 1$, to the maximal range of dimensions k where nonuniqueness is possible for the Yamabe problem.

(22) **Bifurcation of periodic solutions to the singular Yamabe problem on spheres** (with P. Piccione and B. Santoro)
J. Differential Geom. 103 (2016), no. 2, 191-205. MR 3504948, arXiv:1401.7071

In this paper, we show that there are infinitely many solutions to the Singular Yamabe Problem on $S^n \setminus S^1$, $n \geq 5$, that is, infinitely many complete metrics with constant scalar curvature conformal to the round (incomplete) metric. This is proved with bifurcation techniques, using that there exist hyperbolic metrics with arbitrarily many eigenvalues of the Laplacian in $[1/4, 1/4 + \varepsilon]$ on any surface of genus ≥ 2. Due to Mostow's Rigidity, this approach does not apply to $S^n \setminus S^k$ if $k \neq 1$. Those cases are handled with different techniques in (21), see also (18).

(23) **Multiplicity of solutions to the Yamabe problem on collapsing Riemannian submersions** (with P. Piccione)
Pacific J. Math. 266 (2013), no. 1, 1-21. MR 3105774, arXiv:1304.5510

We prove there is a sequence of bifurcation instants for collapsing families of homogeneous metrics on G/H, provided this is the total space of a homogeneous fiber bundle $\mathsf{K}/\mathsf{H} \to \mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{K}$ with $\mathrm{scal}_{\mathsf{K}/\mathsf{H}} > 0$ and either $\mathsf{H} \triangleleft \mathsf{K}$ or $\mathsf{K} \triangleleft \mathsf{G}$; extending the main result in (24) to other topologies.

(24) **Bifurcation and local rigidity of homogeneous solutions to the Yamabe problem on spheres** (with P. Piccione)
Calc. Var. Partial Differential Equations 47 (2013), no. 3-4, 789-807. MR 3070564, arXiv:1107..

We analyze the local stability (as solutions to the Yamabe problem) of Berger metrics on spheres, i.e., metrics obtained by shrinking the round metric in the vertical directions of the Hopf bundles $S^1 \to S^{2n+1} \to \mathbb{C}P^n$, $S^3 \to S^{4n+3} \to \mathbb{H}P^n$, or $S^7 \to S^{15} \to S^8(1/2)$. These account for almost all the homogeneous metrics on spheres, see (26). We find that all Berger metrics on S^{2n+1} are locally rigid, except for the round metric. In contrast, we find sequences of bifurcation instants on S^{4n+3} and S^{15}; in particular, these manifolds carry infinitely many non-homogeneous metrics with constant scalar curvature that are conformal and arbitrarily close to homogeneous metrics.

VI. Spectral geometry

(25) **Full Laplace spectrum of distance spheres in symmetric spaces of rank one** (with E. Lauret and P. Piccione)
Bull. Lond. Math. Soc., to appear, arXiv:2012.02349

We use modern Lie-theoretic methods to explicitly compute the full spectrum of the Laplacian on homogeneous spheres which occur as geodesic distance spheres in (compact or noncompact) symmetric spaces of rank one. We provide a single unified formula for all eigenvalues, and also for their multiplicities. As an application, we find all resonant radii for distance spheres in such spaces, i.e., radii at which there is bifurcation of embedded constant mean curvature spheres, refining earlier results from (17) on the existence of Delaunay-type spheres in CROSSes.

(26) **The first eigenvalue of a homogeneous CROSS** (with E. Lauret and P. Piccione)
J. Geom. Anal. 32 (2022), no. 3, Paper No. 76, MR 4363749, arXiv:2001.08471

We provide explicit formulae for the first eigenvalue of the Laplacian on a compact rank one symmetric space (CROSS) endowed with *any* homogeneous metric. As consequences, we prove that homogeneous metrics on CROSSes are isospectral if and only if they are isometric, and also discuss their stability (or lack thereof) as solutions to the Yamabe problem, see also (24).

VII. Flat manifolds

(27) **Subspace foliations and collapse of closed flat manifolds**
(with A. Derdzinski, R. Mossa, and P. Piccione)
Math. Nachr. 295 (2022), no. 12, 2338–2356, arXiv:2002.05757

This paper investigates the interplay between certain totally geodesic foliations of a closed flat manifold and its collapsed Gromov–Hausdorff limits, building on our earlier work (28). The main results explicitly identify such collapsed limits as flat orbifolds, and provide algebraic and geometric criteria to determine whether they are singular orbifolds or smooth manifolds.

(28) **Teichmüller theory and collapse of flat manifolds**
(with P. Piccione and A. Derdzinski)
Ann. Mat. Pura Appl. (4) 197 (2018), no. 4, 1247-1268, MR 3829569, arXiv:1705.08431

An algebraic description of the Teichmüller space and moduli space of flat Riemannian metrics on a closed manifold or orbifold is provided; in particular, we show that all flat manifolds admit non-homothetic flat deformations, while flat orbifolds may be rigid. We also analyze the boundaries of these spaces, consisting of the collapsed Gromov–Hausdorff limits of flat manifolds and orbifolds.

VIII. Equivariant methods in Geometric Variational Problems

(29) **Global bifurcation for a class of nonlinear ODEs** (with P. Piccione)
São Paulo J. Math. Sci. 16 (2022), no. 1, 486–507, MR 4426405, arXiv:2107.08181

We give a brief survey of global bifurcation techniques, and illustrate their use by finding multiple positive periodic solutions to second order quasilinear ODEs related to the Yamabe problem. As an application, we provide a bifurcation-theoretic proof of a classical nonuniqueness result for the Yamabe problem independently discovered by O. Kobayashi and R. Schoen in the 1980s.

(30) **Instability and bifurcation** (with P. Piccione)
Notices Amer. Math. Soc. 67 (2020), no. 11, 1679-1691, MR 4201907

This is a survey article, aimed at undergraduate students, that gives an overview of classical results in bifurcation theory and some applications to Geometric Analysis, including to multiplicity and symmetry-breaking results for geodesics, CMC hypersurfaces, and the Yamabe problem.

(31) **Deforming solutions of geometric variational problems with varying symmetry groups**
(with P. Piccione and G. Siciliano)
Transform. Groups 19 (2014), no. 4, 941-968. MR 3278856, arXiv:1403.4275

An abstract equivariant implicit function theorem is proven for variational problems invariant under a varying symmetry group, which is encoded as a bundle of Lie groups. Several low regularity extensions are discussed, with applications to CMC hypersurfaces and harmonic maps.

(32) **Equivariant bifurcation in geometric variational problems**
(with P. Piccione and G. Siciliano)
Progress in Nonlinear Differential Equations and Their Applications, Vol. 85 (2014), 103-133, Springer, MR 3330725, arXiv:1308.3268

We prove an extension tailored to geometric variational problems of a celebrated equivariant bifurcation result due to Smoller and Wasserman, discussing applications to CMC hypersurfaces.

(33) **On the equivariant implicit function theorem with low regularity and applications to geometric variational problems** (with P. Piccione and G. Siciliano)
Proc. Edinb. Math. Soc. (2) 58 (2015), no. 1, 53-80. MR 3333978, arXiv:1009.5721

An implicit function theorem is proven for functionals with low regularity, invariant under a local Lie group action, with applications to CMC hypersurfaces, harmonic maps, and closed geodesics.

(34) **Equivariant deformations of Hamiltonian stationary Lagrangian submanifolds**
(with P. Piccione, B. Santoro)
Mat. Contemp. 43 (2014), 61-88, MR 3426257, arXiv:1302.6970

We deform nondegenerate Hamiltonian stationary Lagrangian submanifolds of Kähler manifolds, varying the ambient almost complex structure and metric, and possibly leaving the Kähler realm.

(35) **Genericity of nondegenerate geodesics with general boundary conditions**
(with R. Giambò)
Topol. Methods in Nonlinear Anal. 35 (2010), no. 2, 339-365. MR 2676821, arXiv:0910.4175
We prove that semi-Riemannian geodesics are generically nondegenerate in several contexts.