Assignment 3, with solutions

1 Factorizability of a distribution function

A hypothetic velocity distribution of an ideal gas has the form

$$G(v) = Ae^{-kv}.$$

Does $G(v)$ satisfy the molecular chaos postulate? Find A from the normalization condition. Find the most probable speed, average speed, and the rms speed. Find the distribution function $g(v_x)$. Check if G factorizes as $G(v) = g(v_x)g(v_y)g(v_z)$.

Solution: $G(v)$ satisfies the molecular chaos postulate because it does not depend on the directions of the velocities. The normalization condition for the distribution function is

$$1 = \int\int\int_{-\infty}^{\infty} dv_x dv_y dv_z G(v) = 4\pi \int_0^\infty v^2 dv G(v) = \int_0^\infty dv f(v).$$

Substituting the explicit form of $G(v)$, one obtains

$$1 = 4\pi A \int_0^\infty v^2 dv e^{-kv} = 4\pi A J_2(k),$$

where

$$J_2(k) = \int_0^\infty dv v^2 e^{-kv}.$$

Using

$$J_0(k) = \int_0^\infty dv e^{-kv} = \frac{1}{k},$$

one obtains

$$J_2(k) = \frac{d^2}{dk^2} J_0(k) = \frac{2}{k^3}.$$

Thus

$$A = \frac{1}{4\pi J_2(k)} = \frac{k^3}{8\pi}.$$

The most probable speed v_m is defined by the maximum of $f(v)$, that is,

$$\max_v \left(v^2 e^{-kv}\right).$$

Taking the derivative over v, one obtains

$$0 = 2ve^{-kv} - kv^2 e^{-kv}$$

and thus

$$v_m = \frac{2}{k}.$$

The average speed is given by

$$\bar{v} = \int_0^\infty dv v f(v) = 4\pi A \int_0^\infty dv v^3 e^{-kv} = \frac{k^3}{2} J_3(k).$$

Using

$$J_3(k) = -\frac{d}{dk} J_2(k) = \frac{6}{k^4},$$

one finally obtains

$$\bar{v} = \frac{3}{k}.$$

The average square speed is given by

$$\overline{v^2} = \int_0^\infty dv v^2 f(v) = 4\pi A \int_0^\infty dv v^4 e^{-kv} = \frac{k^3}{2} J_4(k).$$

Using

$$J_4(k) = -\frac{d}{dk} J_3(k) = \frac{24}{k^5},$$

one finally obtains

$$\overline{v^2} = \frac{12}{k^2} \tag{1}$$

and thus

$$v_{\rm rms} = \sqrt{\overline{v^2}} = \frac{2\sqrt{3}}{k}.$$

The distribution function for a single velocity component can be obtained by integrating $G(v)$ over the remaining velocity components

$$g(v_x) = \int\int_{-\infty}^{\infty} dv_y dv_z G\left(\sqrt{v_x^2 + v_y^2 + v_z^2}\right) = 2\pi A \int_0^\infty v_\perp dv_\perp e^{-k\sqrt{v_x^2 + v_\perp^2}},$$

where $v_\perp = \sqrt{v_y^2 + v_z^2}$. Changing to the new variable $u = v_\perp^2$, one obtains

$$g(v_x) = \pi A \int_0^\infty du e^{-k\sqrt{v_x^2 + u}} = \frac{2\pi A}{k^2} e^{-k|v_x|} \left(1 + k\,|v_x|\right)$$

or, finally,

$$g(v_x) = \frac{k}{4} e^{-k|v_x|} \left(1 + k\,|v_x|\right).$$

Obviously

$$G(v) \neq g(v_x) g(v_y) g(v_z),$$

that is, our distribution function is not factorizable and thus it is not a good distribution function.

Still, let us investigate $g(v_x)$ obtained. Start with checking the normalization:

$$\int_{-\infty}^{\infty} dv_x g(v_x) = 2\int_0^\infty dv_x g(v_x) = \frac{k}{2}\int_0^\infty dv_x e^{-kv_x}\left(1 + kv_x\right) = \frac{k}{2}\left(J_0(k) + kJ_1(k)\right) = \frac{k}{2}\left(\frac{1}{k} + k\frac{1}{k^2}\right) = 1,$$

as it should be. The average square velocity is

$$\overline{v_x^2} = \int_{-\infty}^{\infty} dv_x v_x^2 g(v_x) = \frac{k}{2}\left(J_2(k) + kJ_3(k)\right) = \frac{k}{2}\left(\frac{2}{k^3} + k\frac{6}{k^4}\right) = \frac{4}{k^2}.$$

This is in accord with Eq. (1) since $\overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2} = \overline{v^2}$.