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Parametric resonance

December 1, 2015

Consider a harmonic oscillator with time-dependent frequency

$$\ddot{x} + \omega_0^2(t) x = 0. \tag{1}$$

Let the latter be

$$\omega_0(t) = \omega_0[1 + \alpha \cos(\omega t)], \qquad \alpha \ll 1. \tag{2}$$

As we shall see, if ω is close to $2\omega_0$, the amplitude of oscillations will exponentially increase with time. This phenomenon is called *parametric resonance* since a parameter of the problem, in this case oscillator's frequency, is changing. It is convenient to set

$$\omega = 2\omega_0 + \epsilon \tag{3}$$

with a small resonance detuning ϵ. Since $\alpha \ll 1$, we can simplify the equation of motion to

$$\ddot{x} + \omega_0^2 \left\{1 + 2\alpha \cos\left[(2\omega_0 + \epsilon) t\right]\right\} x = 0, \tag{4}$$

dropping the term $\sim \alpha^2$. The solution of this equation can be searched for in the form

$$x(t) = a(t) \cos\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] + b(t) \sin\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right]. \tag{5}$$

The functions $a(t)$ and $b(t)$ can be expected to change slowly with time, for small α and ϵ, thus their second derivatives can be dropped. The derivatives of x are given by

$$\begin{aligned} \dot{x} &= \dot{a} \cos\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] + \dot{b} \sin\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] \\ &\quad - a \left(\omega_0 + \frac{\epsilon}{2}\right) \sin\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] + b \left(\omega_0 + \frac{\epsilon}{2}\right) \cos\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] \end{aligned} \tag{6}$$

and

$$\begin{aligned} \ddot{x} &\cong -2\dot{a} \left(\omega_0 + \frac{\epsilon}{2}\right) \sin\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] + 2\dot{b} \left(\omega_0 + \frac{\epsilon}{2}\right) \cos\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] \\ &\quad - a \left(\omega_0 + \frac{\epsilon}{2}\right)^2 \cos\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] - b \left(\omega_0 + \frac{\epsilon}{2}\right)^2 \sin\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] \end{aligned} \tag{7}$$

or, dropping small terms again,

$$\begin{aligned} \ddot{x} &\cong -\omega_0{}^2 x - a\omega_0 \epsilon \cos\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] - b\omega_0 \epsilon \sin\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] \\ &\quad - 2\dot{a}\omega_0 \sin\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] + 2\dot{b}\omega_0 \cos\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right]. \end{aligned} \tag{8}$$

Products of trigonometric functions in Eq. (4) should be reduced to single trigonometric functions with combined arguments:

$$\begin{aligned} \cos\left[(2\omega_0 + \epsilon) t\right] \cos\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] &= \frac{1}{2} \left(\cos\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] + \cos\left[3\left(\omega_0 + \frac{\epsilon}{2}\right) t\right]\right) \\ \cos\left[(2\omega_0 + \epsilon) t\right] \sin\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] &= \frac{1}{2} \left(-\sin\left[\left(\omega_0 + \frac{\epsilon}{2}\right) t\right] + \sin\left[3\left(\omega_0 + \frac{\epsilon}{2}\right) t\right]\right). \end{aligned} \tag{9}$$

Here functions with triple arguments should be dropped as it can be shown that these terms cancel if the expression for $x(t)$ containing terms of this kind is used. After this Eq. (4) takes the form

$$\left[-a\epsilon + 2\dot{b} + \alpha\omega_0 a\right]\omega_0 \cos\left[\left(\omega_0 + \frac{\epsilon}{2}\right)t\right] + \left[-b\epsilon - 2\dot{a} - \alpha\omega_0 b\right]\omega_0 \sin\left[\left(\omega_0 + \frac{\epsilon}{2}\right)t\right] = 0. \tag{10}$$

This equation is fulfilled if and only if the coefficients in front of the sin and cos functions are zero:

$$\begin{aligned} 2\dot{a} + (\alpha\omega_0 + \epsilon)\, b &= 0 \\ (\alpha\omega_0 - \epsilon)\, a + 2\dot{b} &= 0. \end{aligned} \tag{11}$$

The solution of this system of equations can be searched for in the form

$$a(t) = a_0 e^{\mu t}, \qquad b(t) = b_0 e^{\mu t}. \tag{12}$$

The increment of the parametric resonance μ satisfies the characteristic equation

$$\begin{vmatrix} 2\mu & \alpha\omega_0 + \epsilon \\ \alpha\omega_0 - \epsilon & 2\mu \end{vmatrix} = (2\mu)^2 - (\alpha\omega_0)^2 + \epsilon^2 = 0. \tag{13}$$

Thus

$$\mu = \pm\frac{1}{2}\sqrt{(\alpha\omega_0)^2 - \epsilon^2}. \tag{14}$$

In the frequency range near the resonance

$$-\alpha\omega_0 < \epsilon < \alpha\omega_0, \tag{15}$$

see Eq. (3), μ takes positive and negative real values. The root $\mu > 0$ corresponds to the parametric instability that manifests itself in the exponential increase of the amplitude of oscillations. Note that inside the window of parametric instability the oscillator locks into the frequency $\omega/2$ that differs from its own frequency ω_0. Outside the instability window, μ is imaginary,

$$\mu = \mathrm{i}\Omega, \qquad \Omega = \frac{1}{2}\sqrt{\epsilon^2 - (\alpha\omega_0)^2}. \tag{16}$$

In this case the solution $x(t)$ contains the terms such as

$$\sin\left[\left(\omega_0 + \frac{\epsilon}{2} - \Omega\right)t\right] \qquad \text{and} \qquad \cos\left[\left(\omega_0 + \frac{\epsilon}{2} - \Omega\right)t\right], \tag{17}$$

that is, the oscillation frequency is neither ω_0 nor $\omega/2$. One can see that in the limit $\alpha \to 0$ the oscillator oscillates at its own frequency ω_0.

If the oscillator is damped with damping constant γ, then straightforward generalization of the consideration above leads to

$$a(t) = a_0 e^{(\mu-\gamma)t}, \qquad b(t) = b_0 e^{(\mu-\gamma)t}, \tag{18}$$

where μ is the same as above. Then the condition of the parametric instability has the form

$$\mu^* \equiv \mu_+ - \gamma = \frac{1}{2}\sqrt{(\alpha\omega_0)^2 - \epsilon^2} - \gamma > 0. \tag{19}$$

The parametric-resonance window becomes now

$$\epsilon^2 < (\alpha\omega_0)^2 - (2\gamma)^2 \tag{20}$$

that is narrower than the undamped result of Eq. (15). This window disappears at

$$\alpha\omega_0 = 2\gamma \tag{21}$$

that defines the threshold of the parametric resonance on the amplitude of the perturbation α. This condition provides a method of measuring the damping constant γ.

A physical example of parametric resonance is a pendulum with its length l oscillating in time. A person sitting on a swing and shifting his/her center of mass up and down with the double frequency is doing exactly this. Near the vertical position ($\varphi = 0$) l is being decreased by Δl, while near turning points ($\varphi = \pm\varphi_{\max}$) l is being increased by Δl. Working against gravity, the person is injecting the energy $\Delta E = 2mg\Delta l\,(1 - \cos\varphi_{\max})$ during the period of motion. The greater is the amplitude $\varphi_{\max}$, the higher is the energy input. This is why the amplitude is increasing exponentially.

In spite of the exponential increase of the amplitude, overall parametric resonance is a weaker phenomenon than a regular resonance caused by an applied sinusoidal force. With comparable forcings, it takes a longer time to develop a given amplitude in conditions of parametric resonance than regular resonance.