CLASSICAL MECHANICS

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1 Introduction

Mechanics is part of physics studying motion of material bodies or conditions of their equilibrium. The latter is the subject of *statics* that is important in engineering. General properties of motion of bodies regardless of the source of motion (in particular, the role of constraints) belong to *kinematics*. Finally, motion caused by forces or interactions is the subject of *dynamics*, the biggest and most important part of mechanics.

Concerning systems studied, mechanics can be divided into mechanics of *material points*, mechanics of *rigid bodies*, mechanics of *elastic bodies*, and mechanics of fluids: *hydro*- and *aerodynamics*. At the core of each of these areas of mechanics is the equation of motion, Newton's second law. Mechanics of material points is described by ordinary differential equations (ODE). One can distinguish between mechanics of one or few bodies and mechanics of many-body systems. Mechanics of rigid bodies is also described by ordinary differential equations, including positions and velocities of their centers and the angles defining their orientation. Mechanics of elastic bodies and fluids (that is, mechanics of continuum) is more complicated and described by partial differential equation. In many cases mechanics of continuum is coupled to thermodynamics, especially in aerodynamics. The subject of this course are systems described by ODE, including particles and rigid bodies.

There are two limitations on classical mechanics. First, speeds of the objects should be much smaller than the speed of light, $v \ll c$, otherwise it becomes *relativistic mechanics*. Second, the bodies should have a sufficiently large mass and/or kinetic energy. For small particles such as electrons one has to use *quantum mechanics*.

Regarding theoretical approaches, mechanics splits into three parts: Newtonian, Lagrangian, and Hamiltonian mechanics. Newtonian mechanics is most straightforward in its formulation and is based on Newton's second law. It is efficient in most cases, especially for consideration of particles under the influence of forces. Lagrangian mechanics is more sophisticated and based of the least action principle. It is efficient for consideration of more general mechanical systems having constraints, in particular, mechanisms. Hamiltonian mechanics is even more sophisticated less practical in most cases. Its significance is in bridging classical mechanics to quantum mechanics.

In this course we will consider Newtonian, Lagrangian, and Hamiltonian mechanics, as well as some advanced additional topics.

Part I

Newtonian Mechanics

The basis of Newtonian mechanics are Newton's laws, especially second Newton's law being the equation of motion of a particle of mass m subject to the influence of a force $\mathbf{F}$

$$m\ddot{\mathbf{r}} = \mathbf{F}. \tag{1}$$

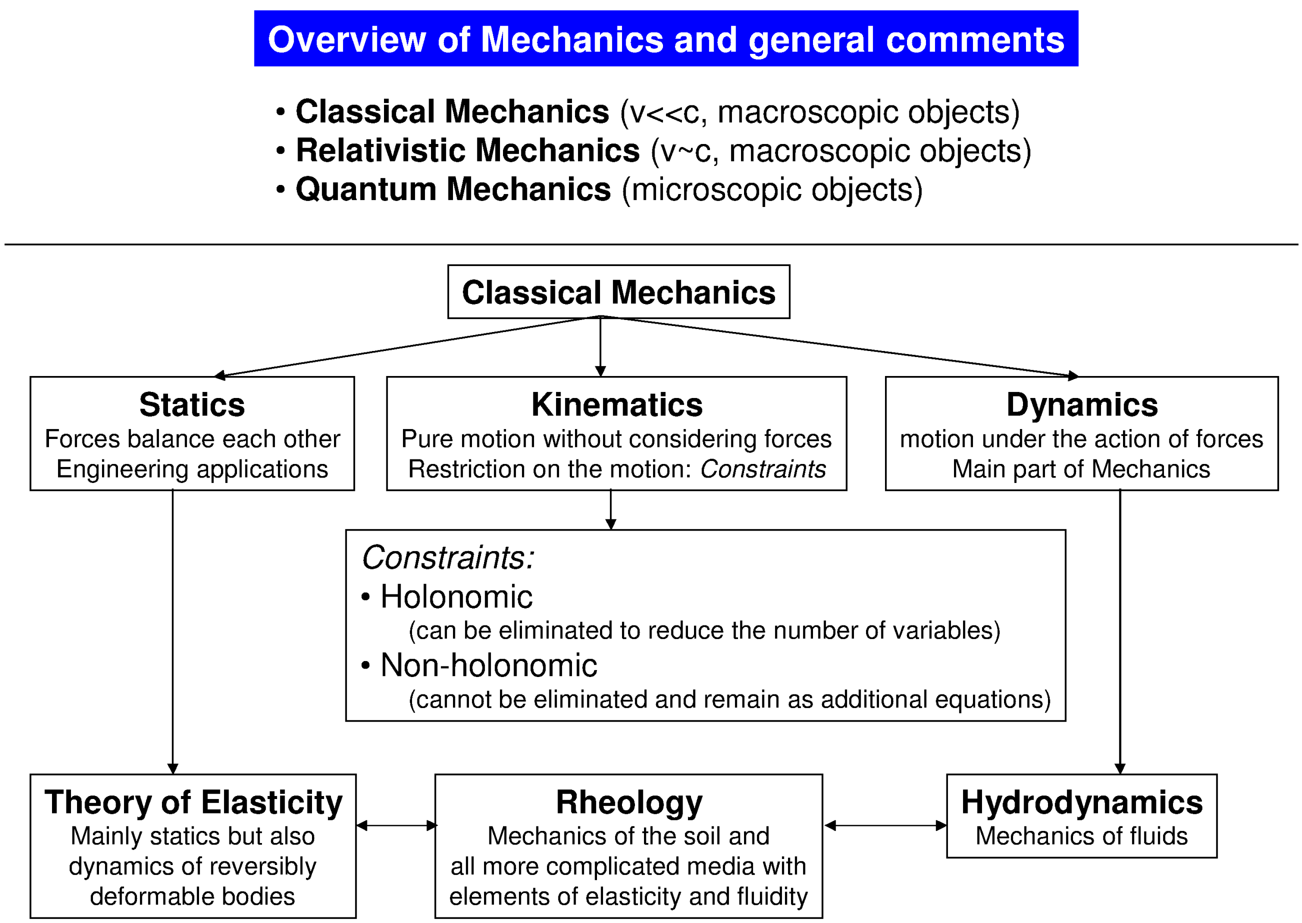

Figure 1: Overview of mechanics

Here $\ddot{\mathbf{r}} \equiv d^2\mathbf{r}/dt^2 \equiv \partial_t^2\mathbf{r}$ is the double time derivative of the position vector $\mathbf{r}$ of the particle, that is, its acceleration $\mathbf{a}$. This second-order differential equation can be written as the system of two first-order differential equations

$$\begin{aligned} \dot{\mathbf{r}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= \mathbf{F}/m \end{aligned} \tag{2}$$

where

$$\mathbf{v} = \dot{\mathbf{r}} \tag{3}$$

is particle's velocity. The force $\mathbf{F}$ can depend on particle's position and velocity, as well as explicitly on time: $\mathbf{F} = \mathbf{F}(\mathbf{r}, \mathbf{v}, t)$.

In the absence of the force (free particle), the solution of the above equation of motion is

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0 \\ \mathbf{r} &= \mathbf{r}_0 + \mathbf{v}_0 t, \end{aligned} \tag{4}$$

where $\mathbf{v}_0$ and $\mathbf{r}_0$ are integration constants of the ODE above, being physically the initial conditions: velocity and position at $t = 0$. This is first Newton's law, now having only a historical meaning.

Newton's second law is a differential equation for in general three-component vector variable $\mathbf{r} \equiv (x, y, z)$. Correspondingly Eq. (1) can be written in components as a system of three equations

$$\begin{aligned} m\ddot{x} &= F_x \\ m\ddot{y} &= F_y \\ m\ddot{z} &= F_z. \end{aligned} \tag{5}$$

In general, these three equations are not independent and may be coupled via the force depending on all three position and/or velocity components.

Systems of N particles are described by in generally coupled system of ODE's consisting of second Newton's laws for each particle

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i, \qquad i = 1, \ldots, N. \tag{6}$$

Each force $\mathbf{F}_i$ can be represented as the sum of the external force $\mathbf{F}_i^{\text{ext}}$ and inter-particle interaction forces $\mathbf{f}_{ij}$,

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_j \mathbf{f}_{ij}. \tag{7}$$

According to Newton's third law, interaction forces are anti-symmetric,

$$\mathbf{f}_{ij} = -\mathbf{f}_{ji}. \tag{8}$$

2 Mechanics of a single particle

Here we consider basic examples of solution of equations of motion for a single particle applying different basic mathematical methods.

2.1 Motion with a constant force

For $\mathbf{F} = \text{const}$ in Eq. (1) one readily finds the solution

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0 + \mathbf{a}t \\ \mathbf{r} &= \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2, \end{aligned} \tag{9}$$

where $\mathbf{a} = \mathbf{F}/m$ is constant acceleration. The validity of this solution can be checked, in particular, by differentiation over time. The trajectory $\mathbf{r}(t)$ (a parabola) is confined to the plane, orientation of which is specified by the two vectors $\mathbf{v}_0$ and $\mathbf{a}$. Thus this motion is effectively two-dimensional. It is convenient to choose the coordinate system such that the trajectory is in xy plane while $z = 0$.

2.2 Motion with a viscous damping

If a body is moving in a viscous fluid or in the air at a small enough speed, it is experiencing a drag force. For a symmetric particle's shape, the drag force is opposite to its velocity,

$$\mathbf{F}_v = -\alpha \mathbf{v}. \tag{10}$$

Newton's second law with a drag force, is usually written it explicitly as,

$$m\ddot{\mathbf{r}} + \alpha \dot{\mathbf{r}} = \mathbf{F}, \tag{11}$$

where $\mathbf{F}$ means all other forces. This equation can be rewritten as

$$\ddot{\mathbf{r}} + \Gamma \dot{\mathbf{r}} = \mathbf{F}/m, \tag{12}$$

where $\Gamma \equiv \alpha/m$ is a characteristic relaxation rate, measured in s^{-1}. One can rewrite this equation via the velocity,

$$\dot{\mathbf{v}} + \Gamma \mathbf{v} = \mathbf{F}/m. \tag{13}$$

After solving this first-order ODE, one can find $\mathbf{r}(t)$ by simple integration, $\mathbf{r}(t) = \int dt \mathbf{v}(t)$.

Let us start with the uniform ODE with $\mathbf{F} = 0$, i.e.,

$$\dot{\mathbf{v}} + \Gamma \mathbf{v} = 0. \tag{14}$$

In this case the motion is one-dimensional along a line in $3d$ space. It is convenient to choose the coordinates so that the motion is along x axis, while $y = z = 0$. Using, for brevity, the notation $v_x = v$, one obtains the equation

$$\dot{v} + \Gamma v = 0 \tag{15}$$

that can be solved as follows:

$$\begin{aligned} \frac{dv}{v} &= -\Gamma dt \\ \int \frac{dv}{v} &= -\Gamma \int dt \\ \ln v &= -\Gamma t + \mathrm{const} \end{aligned} \tag{16}$$

that finally yields

$$v = v_0 e^{-\Gamma t}. \tag{17}$$

Returning to the general vector form, one obtains

$$\mathbf{v} = \mathbf{v}_0 e^{-\Gamma t}. \tag{18}$$

Now, integrating this equation yields

$$\mathbf{r} = \mathbf{r}_0 + \frac{\mathbf{v}_0}{\Gamma} \left(1 - e^{-\Gamma t}\right). \tag{19}$$

This method of solution works for a class of nonlinear first-order ODE.

For linear uniform ODE one can use a more powerful method based on searching the solution in the exponential form such as

$$\mathbf{v} \propto e^{\lambda t}. \tag{20}$$

Substitution into Eq. (14) yields the algebraic equation for λ

$$\lambda + \Gamma = 0 \tag{21}$$

having the solution $\lambda = -\Gamma$. Thus the solution of the ODE has the form

$$\mathbf{v} = \mathbf{C}e^{-\Gamma t} = \mathbf{v}_0 e^{-\Gamma t} \tag{22}$$

that coincides with Eq. (18).

In the presence of a constant force, e.g., gravity, Eq. (13) can be solved by a small modification of the method above. The problem has a non-trivial asymptotic solution that can be found by setting $\dot{\mathbf{v}} = 0$. This yields

$$\mathbf{v}_\infty = \mathbf{F}/(m\Gamma), \tag{23}$$

the stationary velocity at large times. Then one can rewrite Eq. (13) in terms of the new variable $\mathbf{u} \equiv \mathbf{v} - \mathbf{v}_\infty$ as

$$\dot{\mathbf{u}} + \Gamma\mathbf{u} = 0. \tag{24}$$

This is mathematically the same equation as before and it can be solved in a similar way. Finally one obtains

$$\mathbf{v} = \mathbf{v}_\infty + (\mathbf{v}_0 - \mathbf{v}_\infty)\, e^{-\Gamma t}. \tag{25}$$

Integration of this formula yields

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_\infty t + \frac{\mathbf{v}_0 - \mathbf{v}_\infty}{\Gamma}\left(1 - e^{-\Gamma t}\right). \tag{26}$$

Finally, Eq. (13) can be solved in quadratures for any time-dependent force $\mathbf{F}(t)$ by the method of variation of constants. This method allows to find the solution of a non-uniform ODE (or system of ODE), including ODE with coefficients depending on time, if the solution of the corresponding uniform ODE is known. Using this method, one searches for the solution of Eq. (13) in the form

$$\mathbf{v}(t) = \mathbf{C}(t)e^{-\Gamma t}. \tag{27}$$

Substituting this into Eq. (13), one obtains the equation for the "variable constant" $\mathbf{C}(t)$

$$\dot{\mathbf{C}}(t)e^{-\Gamma t} = \mathbf{F}(t)/m. \tag{28}$$

Solving for $\dot{\mathbf{C}}(t)$ and integrating the result, one obtains

$$\mathbf{C}(t) = \int dt' e^{\Gamma t'} \frac{\mathbf{F}(t')}{m}. \tag{29}$$

Substituting this into Eq. (27), one obtains

$$\mathbf{v}(t) = e^{-\Gamma t} \int dt' e^{\Gamma t'} \frac{\mathbf{F}(t')}{m}. \tag{30}$$

In this expression the indefinite integral contains an integration constant. One can work it out rewriting the result in terms of a definite integral as

$$\mathbf{v}(t) = \mathbf{v}_0 e^{-\Gamma t} + e^{-\Gamma t} \int_0^t dt' e^{\Gamma t'} \frac{\mathbf{F}(t')}{m}. \tag{31}$$

To check this formula, one can differentiate it over t, obtaining Eq. (13).

2.3 Harmonic oscillator

Harmonic oscillator in its basic form is a body of mass m attached to a spring with spring constant k. Also including viscous damping and generalizing Eq. (11), one writes down the equation

$$m\ddot{\mathbf{r}} + \alpha\dot{\mathbf{r}} + k\mathbf{r} = \mathbf{F}, \tag{32}$$

where $\mathbf{F}$ is an external force, as before. In the following we will consider the case of the body performing a linear motion along the x axis. Dividing by the mass, one obtains the equation

$$\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x = f, \tag{33}$$

where

$$\Gamma \equiv \frac{\alpha}{2m}, \qquad \omega_0 \equiv \sqrt{\frac{k}{m}}, \qquad f \equiv \frac{F}{m}. \tag{34}$$

Here we have defined Γ in a way different from above for the sake of simplicity of the formulas. ω_0 is the frequency of oscillations in the absence of damping.

Solution of the uniform equation ($f = 0$), in accordance with the general method, can be searched in the form

$$x(t) \propto e^{i\Omega t}. \tag{35}$$

The imaginary i has been inserted in anticipation of an oscillating motion of the system. Substituting this into Eq. (33), one obtains the quadratic equation

$$-\Omega^2 + 2i\Gamma\Omega + \omega_0^2 = 0 \tag{36}$$

having the solution

$$\Omega_\pm = i\Gamma \pm \tilde{\omega}_0, \qquad \tilde{\omega}_0 \equiv \sqrt{\omega_0^2 - \Gamma^2}. \tag{37}$$

Thus the solution of the ODE has the form

$$x(t) = C_+ e^{i\Omega_+ t} + C_- e^{i\Omega_- t}, \tag{38}$$

where $C_\pm$ are integration constants. Using the relation

$$e^{i\varphi} \equiv \cos\varphi + i\sin\varphi, \tag{39}$$

one can rewrite the result in explicitly real form

$$x(t) = C_1 e^{-\Gamma t}\cos\tilde{\omega}_0 t + C_2 e^{-\Gamma t}\sin\tilde{\omega}_0 t, \tag{40}$$

where $C_{1,2}$ is another set of integration constants. The latter can be found from the initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = v_0 \tag{41}$$

that is,

$$x(0) = C_1 = x_0 \tag{42}$$

and

$$\dot{x}(0) = -\Gamma C_1 + \tilde{\omega}_0 C_2 = v_0. \tag{43}$$

One finds

$$C_1 = x_0, \qquad C_2 = \frac{v_0 + \Gamma x_0}{\tilde{\omega}_0}. \tag{44}$$

Thus

$$x(t) = x_0 e^{-\Gamma t}\cos\tilde{\omega}_0 t + \frac{v_0 + \Gamma x_0}{\tilde{\omega}_0} e^{-\Gamma t}\sin\tilde{\omega}_0 t. \tag{45}$$

Let us look at the solution. According to Eq. (37), in the absence of damping, $\Gamma = 0$, the body is oscillating with the frequency ω_0. Damping reduces oscillation frequency that turns to zero at $\Gamma = \omega_0$. In the strong-damping limit $\Gamma > \omega_0$ the motion of the body is aperiodic.

Let us now consider the motion of the harmonic oscillator under the influence of external force. Using the method of variation of constants in Eq. (38), one searches for the solution in the form

$$x(t) = C_+(t)e^{i\Omega_+ t} + C_-(t)e^{i\Omega_- t}. \tag{46}$$

The "variable constants" satisfy the system of equations

$$\begin{aligned} \dot{C}_+(t)x_+(t) + \dot{C}_-(t)x_-(t) &= 0 \\ \dot{C}_+(t)\dot{x}_+(t) + \dot{C}_-(t)\dot{x}_-(t) &= f(t). \end{aligned} \tag{47}$$

Its solution is

$$\dot{C}_+(t) = \frac{\begin{vmatrix} 0 & x_-(t) \\ f(t) & \dot{x}_-(t) \end{vmatrix}}{\begin{vmatrix} x_+(t) & x_-(t) \\ \dot{x}_+(t) & \dot{x}_-(t) \end{vmatrix}} = \frac{-f(t)x_-(t)}{x_+(t)\dot{x}_-(t) - \dot{x}_+(t)x_-(t)}. \tag{48}$$

Using

$$x_+(t)\dot{x}_-(t) - \dot{x}_+(t)x_-(t) = i\left(\Omega_- - \Omega_+\right)e^{i(\Omega_- + \Omega_+)t} = -2i\tilde{\omega}_0 e^{-2\Gamma t} \tag{49}$$

one obtains

$$\dot{C}_+(t) = -\frac{i}{2}\frac{f(t)}{\tilde{\omega}_0}e^{i\Omega_- t + 2\Gamma t} = -\frac{i}{2}\frac{f(t)}{\tilde{\omega}_0}e^{-i\Omega_+ t} \tag{50}$$

and, similarly,

$$\dot{C}_-(t) = \frac{i}{2}\frac{f(t)}{\tilde{\omega}_0}e^{i\Omega_+ t + 2\Gamma t} = \frac{i}{2}\frac{f(t)}{\tilde{\omega}_0}e^{-i\Omega_- t}. \tag{51}$$

Integrating these two formulas and substituting the result in Eq. (46), one obtains

$$x(t) = x_{\mathrm{free}}(t) + x_{\mathrm{forced}}(t), \tag{52}$$

where $x_{\mathrm{free}}(t)$ is the solution of the uniform ODE describing the free oscillator and given by Eq. (45) and

$$x_{\mathrm{forced}}(t) = -\frac{i}{2\tilde{\omega}_0}e^{i\Omega_+ t}\int_0^t dt' f(t')e^{-i\Omega_+ t'} + \frac{i}{2\tilde{\omega}_0}e^{i\Omega_- t}\int_0^t dt' f(t')e^{-i\Omega_- t'} \tag{53}$$

is the response to the external force. The latter can be simplified as

$$\begin{aligned} x_{\mathrm{forced}}(t) &= \frac{i}{2\tilde{\omega}_0}\int_0^t dt' f(t')\left[-e^{i\Omega_+(t-t')} + e^{i\Omega_-(t-t')}\right] \\ &= \frac{i}{2\tilde{\omega}_0}\int_0^t dt' f(t')e^{-\Gamma(t-t')}\left[-e^{i\tilde{\omega}_0(t-t')} + e^{-i\tilde{\omega}_0(t-t')}\right] \\ &= \frac{1}{\tilde{\omega}_0}\int_0^t dt' f(t')e^{-\Gamma(t-t')}\sin\left[\tilde{\omega}_0(t-t')\right]. \end{aligned} \tag{54}$$

One can check that $x_{\mathrm{forced}}(0) = \dot{x}_{\mathrm{forced}}(0) = 0$, that is, the forced solution is independent of the initial conditions and does not change the form of $x_{\mathrm{free}}(t)$.

Let us consider the important case of a sinusoidal force

$$f(t) = f_0 \sin \omega t, \tag{55}$$

applied starting from $t = 0$. To compute $x_{\text{forced}}(t)$, it is convenient to convert everything into the exponential form, after which integration simplifies:

$$\begin{aligned} x_{\text{forced}}(t) &= \frac{f_0}{4\tilde{\omega}_0} \int_0^t dt' \left(e^{i\omega t'} - e^{-i\omega t'} \right) e^{-\Gamma(t-t')} \left[-e^{i\tilde{\omega}_0 (t-t')} + e^{-i\tilde{\omega}_0 (t-t')} \right] \\ &= \frac{f_0}{4\tilde{\omega}_0} \int_0^t dt' e^{-\Gamma(t-t')} \left[-e^{i\omega t'} e^{i\tilde{\omega}_0 (t-t')} + e^{i\omega t'} e^{-i\tilde{\omega}_0 (t-t')} + c.c. \right], \end{aligned} \tag{56}$$

where *c.c.* is complex conjugate. Further one proceeds as

$$\begin{aligned} x_{\text{forced}}(t) &= \frac{f_0}{4\tilde{\omega}_0} \int_0^t dt' \left[-e^{(-\Gamma + i\tilde{\omega}_0)t} e^{(\Gamma - i\tilde{\omega}_0 + i\omega)t'} + e^{(-\Gamma - i\tilde{\omega}_0)t} e^{(\Gamma + i\tilde{\omega}_0 + i\omega)t'} \right] + c.c. \\ &= \frac{f_0}{4\tilde{\omega}_0} \left[-e^{(-\Gamma + i\tilde{\omega}_0)t} \frac{e^{(\Gamma - i\tilde{\omega}_0 + i\omega)t} - 1}{\Gamma - i\tilde{\omega}_0 + i\omega} + e^{(-\Gamma - i\tilde{\omega}_0)t} \frac{e^{(\Gamma + i\tilde{\omega}_0 + i\omega)t} - 1}{\Gamma + i\tilde{\omega}_0 + i\omega} \right] + c.c. \\ &= \frac{f_0}{4\tilde{\omega}_0} \left[-\frac{e^{i\omega t} - e^{(-\Gamma + i\tilde{\omega}_0)t}}{\Gamma - i\tilde{\omega}_0 + i\omega} + \frac{e^{i\omega t} - e^{(-\Gamma - i\tilde{\omega}_0)t}}{\Gamma + i\tilde{\omega}_0 + i\omega} \right] + c.c. \end{aligned} \tag{57}$$

The first term in this expression is the so-called resonant term in which the denominator becomes small for ω close to ω_0. The other term is non-resonant term that differs from the first one by replacement $\tilde{\omega}_0 \Rightarrow -\tilde{\omega}_0$. It is sufficient to compute one of these terms, then the other one can be easily obtained from the first one. Let us calculate the resonance term. Adding *c.c.* amounts to doubling the real part of the expression and annihilating its imaginary part. Shortcutting the non-resonant term as $\ldots$, one proceeds as

$$\begin{aligned} x_{\text{forced}}(t) &= -\frac{f_0}{4\tilde{\omega}_0} \frac{e^{i\omega t} - e^{(-\Gamma + i\tilde{\omega}_0)t}}{\Gamma - i\tilde{\omega}_0 + i\omega} + c.c. + \ldots \\ &= -\frac{f_0}{4\tilde{\omega}_0} \frac{\left[e^{i\omega t} - e^{(-\Gamma + i\tilde{\omega}_0)t} \right] (\Gamma + i\tilde{\omega}_0 - i\omega)}{(\omega - \tilde{\omega}_0)^2 + \Gamma^2} + c.c. + \ldots \\ &= -\frac{f_0}{4\tilde{\omega}_0} \frac{\Gamma \left[e^{i\omega t} - e^{(-\Gamma + i\tilde{\omega}_0)t} + c.c. \right] + i (\tilde{\omega}_0 - \omega) \left[e^{i\omega t} - e^{(-\Gamma + i\tilde{\omega}_0)t} - c.c. \right]}{(\omega - \tilde{\omega}_0)^2 + \Gamma^2} + \ldots \\ &= -\frac{f_0}{2\tilde{\omega}_0} \frac{\Gamma \left[\cos(\omega t) - \cos(\tilde{\omega}_0 t) e^{-\Gamma t} \right] - (\tilde{\omega}_0 - \omega) \left[\sin(\omega t) - \sin(\tilde{\omega}_0 t) e^{-\Gamma t} \right]}{(\omega - \tilde{\omega}_0)^2 + \Gamma^2} + \ldots \end{aligned} \tag{58}$$

At the times longer than the relaxation time of the oscillator

$$\tau \equiv \frac{1}{\Gamma} \tag{59}$$

the terms in the above formula that are oscillating at oscillator's own frequency $\tilde{\omega}_0$ die out and only the forced terms oscillating at the frequency ω remain,

$$x_{\text{forced}}(t) = \frac{f_0}{2\tilde{\omega}_0} \left[-\frac{\Gamma \cos(\omega t) + (\omega - \tilde{\omega}_0) \sin(\omega t)}{(\omega - \tilde{\omega}_0)^2 + \Gamma^2} + \frac{\Gamma \cos(\omega t) + (\omega + \tilde{\omega}_0) \sin(\omega t)}{(\omega + \tilde{\omega}_0)^2 + \Gamma^2} \right]. \tag{60}$$

Here the terms with $\cos(\omega t)$ is shifted by quarter of the period with respect to the harmonic force. Exactly at resonance, $\omega = \tilde{\omega}_0$, only this term in the resonant part of the expression becomes dominant and reaches its maximum. Near the resonance, $|\omega - \tilde{\omega}_0| \sim \Gamma \ll \omega_0$, the resonant term is large, and the much smaller non-resonant term can be neglected.

The power absorbed by the oscillator can be calculated via the work of the external force done on the oscillator,

$$P_{\rm abs} = \frac{1}{T}\int_0^T dt \frac{dA}{dt} = \frac{1}{T}\int_0^T dt F(t)\dot{x}(t), \tag{61}$$

where $T \equiv 2\pi/\omega$. In the stationary state near the resonance, $x(t)$ is given by the first term of Eq. (60) and $F(t) = mf_0 \sin\omega t$. One obtains

$$P_{\rm abs} = \frac{1}{T}\int_0^T dt \frac{mf_0^2}{2\omega_0}\frac{\Gamma}{(\omega-\omega_0)^2+\Gamma^2}\omega\sin^2\omega t \cong \frac{mf_0^2}{4}\frac{\Gamma}{(\omega-\omega_0)^2+\Gamma^2}. \tag{62}$$

Here approximation $\omega \cong \tilde{\omega}_0 \cong \omega_0$ was used that is mandatory since the whole approach ignoring the non-resonant term is valid near the resonance only under that condition that the resonance is narrow, $\Gamma \ll \omega_0$.

The stationary (settled) state of the oscillator under the influence of a harmonic force can be obtained in a much easier way by just searching the solution in the form

$$x(t) = A\sin\omega t + B\cos\omega t. \tag{63}$$

Pre-calculating

$$\dot{x}(t) = A\omega\cos\omega t - B\omega\sin\omega t \tag{64}$$

$$\ddot{x}(t) = -\omega^2 x(t) \tag{65}$$

and inserting it into Eq. (33) using Eq. (55), one obtains

$$-A\omega^2\sin\omega t - B\omega^2\cos\omega t + 2\Gamma A\omega\cos\omega t - 2\Gamma B\omega\sin\omega t + A\omega_0^2\sin\omega t + B\omega_0^2\cos\omega t = f_0\sin\omega t. \tag{66}$$

Equating the coefficients in front of $\sin\omega t$ and $\cos\omega t$, one obtains the system of linear equations

$$\begin{aligned} -A\omega^2 - 2\Gamma B\omega + A\omega_0^2 &= f_0 \\ -B\omega^2 + 2\Gamma A\omega + B\omega_0^2 &= 0. \end{aligned} \tag{67}$$

From the second equation one obtains $A = B\left(\omega^2-\omega_0^2\right)/\left(2\Gamma\omega\right)$. Substituting it into the first equation yields

$$-B\frac{\left(\omega^2-\omega_0^2\right)^2}{2\Gamma\omega} - 2\Gamma\omega B = f_0 \tag{68}$$

and

$$B = -f_0\frac{2\Gamma\omega}{\left(\omega^2-\omega_0^2\right)^2+4\Gamma^2\omega^2}, \qquad A = -f_0\frac{\omega^2-\omega_0^2}{\left(\omega^2-\omega_0^2\right)^2+4\Gamma^2\omega^2}, \tag{69}$$

so that

$$x(t) = -f_0\frac{2\Gamma\omega\cos\omega t + \left(\omega^2-\omega_0^2\right)\sin\omega t}{\left(\omega^2-\omega_0^2\right)^2+4\Gamma^2\omega^2}. \tag{70}$$

This solution contains both resonant and nonresonant terms. Near resonance one can write

$$\omega^2-\omega_0^2 = (\omega-\omega_0)(\omega+\omega_0) \cong 2\omega_0(\omega-\omega_0) \tag{71}$$

and thus

$$x(t) \cong -\frac{f_0}{2\omega_0}\frac{\Gamma\cos\omega t + (\omega-\omega_0)\sin\omega t}{(\omega-\omega_0)^2+\Gamma^2}. \tag{72}$$

This coincides with the resonant term in Eq. (60). In general, one should be able to demonstrate that Eqs. (60) and (70) that are both exact solutions of the problem, are identical.

2.4 Charged particle in a magnetic field

Moving charged particle experiences the so-called Lorentz force from the magnetic field B,

$$F_L = q\left[\mathbf{v} \times \mathbf{B}\right]. \tag{73}$$

Since this force depends on the velocity, it is convenient to write the equation of motion in terms of the velocity, as was done in the case of the viscous drag force,

$$m\dot{\mathbf{v}} = q\left[\mathbf{v} \times \mathbf{B}\right]. \tag{74}$$

In the basic case of uniform magnetic field that will be considered below, z axis will be chosen along $\mathbf{B}$, so that $B_z = B$ and $B_x = B_y = 0$. The equation of motion in components has the form

$$\begin{aligned} \dot{v}_x &= \omega_c v_y && (75)\\ \dot{v}_y &= -\omega_c v_x && (76)\\ \dot{v}_z &= 0, && (77) \end{aligned}$$

where

$$\omega_c \equiv \frac{qB}{m} \tag{78}$$

is the cyclotron frequency. Differentiating the first equation over time and substituting the second equation, one obtains the second-order ODE for v_x

$$\ddot{v}_x + \omega_c^2 v_x = 0. \tag{79}$$

This is the harmonic-oscillator equation considered in the pervious section. Thus v_x will be oscillating in time. The same equation can be obtained for v_y, thus v_y will be oscillating, too. On the other hand, motion in the direction of the field is a free motion,

$$\dot{v}_z = 0, \qquad v_z = \text{const} = v_{z0}, \qquad z = z_0 + v_{z0}t. \tag{80}$$

The solution of Eq. (79) can be obtained from Eq. (40):

$$v_x(t) = C_{x1}\cos\omega_c t + C_{x2}\sin\omega_c t. \tag{81}$$

Integration constants obtained from the initial conditions have the form

$$\begin{aligned} C_{x1} &= v_x(0) = v_{x0} && (82)\\ C_{x2} &= \dot{v}_x(0)/\omega_c = v_y(0) = v_{y0}, && (83) \end{aligned}$$

thus the final solution has the form

$$v_x(t) = v_{x0}\cos\omega_c t + v_{y0}\sin\omega_c t. \tag{84}$$

Similarly one obtains

$$v_y(t) = v_{y0}\cos\omega_c t - v_{x0}\sin\omega_c t. \tag{85}$$

Let us calculate

$$\begin{aligned} v_x^2 + v_y^2 &= (v_{x0}\cos\omega_c t + v_{y0}\sin\omega_c t)^2 + (v_{y0}\cos\omega_c t - v_{x0}\sin\omega_c t)^2 \\ &= v_{x0}^2\cos^2\omega_c t_2 + 2v_{x0}v_{y0}\cos\omega_c t\,\sin\omega_c t + v_{y0}^2\sin^2\omega_c t \\ &+ v_{x0}^2\sin^2\omega_c t_2 - 2v_{x0}v_{y0}\sin\omega_c t\,\cos\omega_c t + v_{y0}^2\cos^2\omega_c t \\ &= v_{x0}^2 + v_{y0}^2 = \text{const}. && (86) \end{aligned}$$

Thus the vector (v_x, v_y) is rotating at the constant rate ω_c while the kinetic energy $E_k = m\mathbf{v}^2/2$ is conserved. Time dependence of x and y can be obtained by integration of the results for v_x and v_y. One obtains

$$\begin{aligned} x &= x_0 + \frac{v_{x0}}{\omega_c} \sin \omega_c t - \frac{v_{y0}}{\omega_c} \cos \omega_c t \\ y &= y_0 + \frac{v_{y0}}{\omega_c} \sin \omega_c t + \frac{v_{x0}}{\omega_c} \cos \omega_c t. \end{aligned} \tag{87}$$

This trajectory is circular with the center at an arbitrary (x_0, y_0). The radius of the circle R can be found by the calculation similar to the above:

$$R^2 = (x - x_0)^2 + (y - y_0)^2 = \frac{v_{x0}^2 + v_{y0}^2}{\omega_c^2}. \tag{88}$$

Thus the cyclotron radius is given by

$$R = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{qB}, \tag{89}$$

where $v_\perp = \sqrt{v_x^2 + v_y^2}$.

3 Momentum and angular momentum

Momentum of a particle is defined as

$$\mathbf{p} = m\mathbf{v}. \tag{90}$$

It can be used to write Newton's second law in the form

$$\dot{\mathbf{p}} = \mathbf{F}. \tag{91}$$

In the absence of forces momentum is conserved, as well as the velocity. The above is not essentially new.

Significance of momentum becomes apparent if one considers a system of interacting particles and defines the total momentum as

$$\mathbf{P} = \sum_i \mathbf{p}_i. \tag{92}$$

Differentiating it over time one obtains

$$\dot{\mathbf{P}} = \sum_i \dot{\mathbf{p}}_i = \sum_i \mathbf{F}_i. \tag{93}$$

Separating the forces into external and internal, Eq. (7), and using Newton's third law, Eq. (8), one can see that only external forces change the momentum of the system,

$$\dot{\mathbf{P}} = \sum_i \mathbf{F}_i^{\text{ext}}. \tag{94}$$

In an isolated system there are only interaction forces between the particles that are changing momenta of individual particles, whereas the total momentum is conserved. Conservation of the total momentum is important, in particular, in collisions. During collisions usually systems can be considered as effectively isolated because collisions occur during a very short time, so that external forces cannot change momenta significantly during collisions. To the contrary, internal forces during collisions are large and important.

Angular momentum of a particle is defined by

$$\mathbf{l} = [\mathbf{r} \times \mathbf{p}], \tag{95}$$

where $\mathbf{r}$ is the position vector of the particle defined with respect to a particular frame or coordinate system. Thus angular momentum depends on the position of the origin of the frame. The time derivative of the angular momentum is given by

$$\dot{\mathbf{l}} = [\dot{\mathbf{r}} \times \mathbf{p}] + [\mathbf{r} \times \dot{\mathbf{p}}] = [\mathbf{r} \times \mathbf{F}] \equiv \boldsymbol{\tau}. \tag{96}$$

Here $\boldsymbol{\tau}$ is the torque that also depends on the choice of the frame origin. If the force is central, i.e., directed everywhere away from or towards a particular central point, one can choose a frame having the origin at this point, then the torque is zero and angular momentum is conserved.

The total angular momentum of the system is defined by

$$\mathbf{L} = \sum_i [\mathbf{r}_i \times \mathbf{p}_i] \,. \tag{97}$$

Its time derivative reads

$$\dot{\mathbf{L}} = \sum_i [\mathbf{r}_i \times \mathbf{F}_i] = \sum_i \left[\mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}\right] + \sum_i \left[\mathbf{r}_i \times \sum_j \mathbf{f}_{ij}\right] . \tag{98}$$

In the interaction term, renaming i and j and using Newton's third law, one can write

$$\dot{\mathbf{L}}_{\text{int}} = \sum_{ij} [\mathbf{r}_i \times \mathbf{f}_{ij}] = \sum_{ij} [\mathbf{r}_j \times \mathbf{f}_{ji}] = -\sum_{ij} [\mathbf{r}_j \times \mathbf{f}_{ij}] \,. \tag{99}$$

One can symmetrize $\dot{\mathbf{L}}_{\text{int}}$ as

$$\dot{\mathbf{L}}_{\text{int}} = \frac{1}{2} \sum_{ij} [(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{f}_{ij}] \,. \tag{100}$$

In nature interaction forces between particles are directed along the line connecting the particles. Thus the cross product in the formula above disappears. The change of the angular momentum of a system is entirely due to external torques,

$$\dot{\mathbf{L}} = \sum_i \left[\mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}\right] = \sum_i \boldsymbol{\tau}_i^{\text{ext}} = \boldsymbol{\tau}. \tag{101}$$

Consider another frame with the origin shifted by the vector $\mathbf{a}$, so that the positions of the particles in the old frame are given by

$$\mathbf{r}_i = \mathbf{r}'_i + \mathbf{a}. \tag{102}$$

For the total angular momentum one obtains

$$\mathbf{L} = \sum_i [(\mathbf{r}'_i + \mathbf{a}) \times \mathbf{p}_i] = \mathbf{L}' + [\mathbf{a} \times \mathbf{P}] \,. \tag{103}$$

Thus, if the total momentum $\mathbf{P}$ is zero, total angular momentum does not depend on the position of frame's origin. If the system is close to the origin of the primed frame but far from the origin of the original frame, one can say that the term $[\mathbf{a} \times \mathbf{P}]$ is the angular momentum corresponding to the motion of the system as the whole in the original frame.

Similarly for the torque one can write

$$\boldsymbol{\tau} = \sum_i [(\mathbf{r}'_i + \mathbf{a}) \times \mathbf{F}_i] = \boldsymbol{\tau}' + \mathbf{a} \times \sum_i \mathbf{F}_i. \tag{104}$$

If the net force acting on the system is zero, torque does not depend on the position of the frame's origin, $\boldsymbol{\tau} = \boldsymbol{\tau}'$.

4 Work, energy and potential forces

Infinitesimal work done by the force $\mathbf{F}$ on a particle undergoing displacement $d\mathbf{r}$ is defined by

$$\delta A = \mathbf{F} \cdot d\mathbf{r}. \tag{105}$$

In this formula δA is used instead of dA since work is not a function and the above is not its differential. Power W is work done per unit of time,

$$W = \frac{\delta A}{dt} = \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{F} \cdot \mathbf{v}. \tag{106}$$

If $\mathbf{F}$ is acting on a free particle, the latter will accelerate or decelerate, and the work on the particle will change its kinetic energy defined by

$$E_k = \frac{m\mathbf{v}^2}{2}. \tag{107}$$

Indeed, the infinitesimal change of the energy can be related to the infinitesimal work as

$$dE_k = m\mathbf{v} \cdot d\mathbf{v} = m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}dt = \mathbf{F} \cdot d\mathbf{r} = \delta A. \tag{108}$$

Potential forces are forces that can be expressed via gradients of position-dependent potential energy U,

$$\mathbf{F} = -\nabla U(\mathbf{r}) = -\frac{\partial U}{\partial \mathbf{r}}. \tag{109}$$

Work done by potential forces is independent of the trajectory and depends only on the initial and final positions,

$$A_{12} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \frac{\partial U}{\partial \mathbf{r}} \cdot d\mathbf{r} = U(\mathbf{r}_1) - U(\mathbf{r}_2). \tag{110}$$

The work on a closed path is zero for potential forces,

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0, \tag{111}$$

because the change of U is zero. Eq. (110) can be rewritten as the definition of potential energy via the force,

$$U(\mathbf{r}) = U(\mathbf{r}_0) - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}. \tag{112}$$

Here $U(\mathbf{r}_0)$ is the (arbitrary) value of potential energy at the reference point $\mathbf{r}_0$. Potential energy a secondary or auxiliary physical quantity, not directly measurable and introduced via the force as a primary quantity, and it is defined up to an arbitrary additive constant. Using Eq. (112), one has first to be sure that the integral does not depend on the path. To check whether $\mathbf{F}$ is a potential force, one can use the Stokes' theorem to express the closed-path line integral as the flux through the surface spanned by the closed path,

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_S \operatorname{rot} \mathbf{F} \cdot d\mathbf{S}. \tag{113}$$

If

$$\operatorname{rot} \mathbf{F} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} = 0, \tag{114}$$

$\mathbf{F}$ is a potential force.

Not all forces are potential. For instance, friction forces are not. An example of potential forces is gravity near the Earth's surface,

$$U = mg\mathbf{r} \cdot \mathbf{e}_z + \text{const}, \tag{115}$$

where $\mathbf{e}_z$ is the unit vector directed up. The general gravitational force between two masses has the potential energy

$$U = G\frac{mM}{|\mathbf{r}|}. \tag{116}$$

Here the origin of the coordinate system is put at the big mass M (e.g., the sun) and $\mathbf{r}$ points to the position of the small mass m (e.g., the Earth). The gravitational force acting on m has the form

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{r}} = -G\frac{mM}{r^2}\frac{\mathbf{r}}{r}. \tag{117}$$

A special kind force if Lorentz force, Eq. (73). As it is not doing any work, there is no potential energy associated with it. For the same reason this force does not lead to dissipation of energy as friction forces.

The total energy of the particle

$$E = \frac{m\mathbf{v}^2}{2} + U(\mathbf{r}) \tag{118}$$

can be shown to be dynamically conserved (i.e., to be an *integral of motion*), if there are no other forces such as friction. Indeed,

$$\dot{E} = m\dot{\mathbf{v}} \cdot \mathbf{v} + \frac{\partial U}{\partial \mathbf{r}} \cdot \dot{\mathbf{r}} = (m\dot{\mathbf{v}} - \mathbf{F}) \cdot \mathbf{v} = 0 \tag{119}$$

via Newton's second law. Because of conservation of the total energy, systems with potential forces are called conservative. In the presence of the viscous friction force, Eq. (10), one obtains

$$\dot{E} = m\dot{\mathbf{v}} \cdot \mathbf{v} + \frac{\partial U}{\partial \mathbf{r}} \cdot \dot{\mathbf{r}} = \left(-\alpha\mathbf{v} - \frac{\partial U}{\partial \mathbf{r}} + \frac{\partial U}{\partial \mathbf{r}}\right) \cdot \mathbf{v} = -\alpha\mathbf{v}^2, \tag{120}$$

dissipation of energy.

Let us obtain the total energy E as the first integral of Newton's second law. Dot-multiplying it by $\mathbf{v}$ and manipulating the expressions, one obtains

$$0 = \left(m\dot{\mathbf{v}} + \frac{\partial U}{\partial \mathbf{r}}\right) \cdot \mathbf{v} = \frac{m}{2}\frac{d\mathbf{v}^2}{dt} + \frac{\partial U}{\partial \mathbf{r}} \cdot \dot{\mathbf{r}} = \frac{d}{dt}\left(\frac{m\mathbf{v}^2}{2} + U(\mathbf{r})\right). \tag{121}$$

Integrating this, one obtains Eq. (118) with $E = \text{const}$ as an integral of motion.

The total energy of a system of interacting particles has the form

$$E = \sum_i \frac{m_i\mathbf{v}_i}{2} + U(\{\mathbf{r}_i\}), \tag{122}$$

where the potential energy depends, in general, on the positions of all particles. In the absence of non-conservative forces, conservation of this many-body energy follows from Newton's second law, as above. In most cases, potential energy includes one-particle and two-particle terms,

$$U(\{\mathbf{r}_i\}) = \sum_i U_0(\mathbf{r}_i) + \frac{1}{2}\sum_{ij} V(|\mathbf{r}_i - \mathbf{r}_j|). \tag{123}$$

The factor $1/2$ in the interaction term is inserted to compensate for double counting of interacting pairs ij and ji, so that the interaction energy between each two particles is just V. Using

$$\frac{\partial|\mathbf{r}|}{\partial \mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}, \tag{124}$$

one obtains the corresponding force,

$$\mathbf{F}_i = -\frac{\partial U}{\partial \mathbf{r}_i} = -\frac{\partial U_0}{\partial \mathbf{r}_i} + \sum_j \mathbf{f}_{ij}, \tag{125}$$

where interaction forces are given by

$$\mathbf{f}_{ij} = -V'\left(|\mathbf{r}_i - \mathbf{r}_j|\right) \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{126}$$

Here V' is the derivative of the function over its argument. As it was said in the comment to Eq. (100), interaction forces are directed along the line connecting the two particles.

5 Center of mass, reduced mass

For any system of point masses one can define the center of mass (CM) or center of inertia as

$$\mathbf{R} = \frac{1}{M} \sum_i m_i \mathbf{r}_i, \tag{127}$$

where $M = \sum_i m_i$ is the total mass. For a solid body CM is defined by a corresponding integral. The velocity of CM is related to the total momentum,

$$\mathbf{V} = \frac{1}{M} \sum_i m_i \mathbf{v}_i = \frac{\mathbf{P}}{M}. \tag{128}$$

This formula evokes an image of a system considered as one body of mass M moving with the velocity $\mathbf{V}$. Dynamics of CM is due to the external forces,

$$M\dot{\mathbf{V}} = \dot{\mathbf{P}} = \sum_i \mathbf{F}_i^{\mathrm{ext}}, \tag{129}$$

where Eq. (94) was used. CM of an isolated system is moving with a constant velocity.

In many cases it is convenient to put the origin of the coordinate system at the center of mass since it leads to simplifications. In particular, in the CM frame the system is at rest as the whole and there is only internal motion. Let the primed frame in Eq. (102) be CM frame, so that

$$\mathbf{r}_i = \mathbf{r}_i' + \mathbf{R} \tag{130}$$

and

$$\mathbf{v}_i = \mathbf{v}_i' + \mathbf{V}. \tag{131}$$

Position of the center of mass in the CM frame is zero,

$$\mathbf{R}' = \frac{1}{M} \sum_i m_i \mathbf{r}_i' = \frac{1}{M} \sum_i m_i \left(\mathbf{r}_i - \mathbf{R}\right) = \mathbf{R} - \mathbf{R} = 0. \tag{132}$$

Total momentum in the CM frame is zero, too,

$$\mathbf{P}' = \sum_i m_i \mathbf{v}_i' = \sum_i m_i \left(\mathbf{v}_i - \mathbf{V}\right) = \mathbf{P} - \mathbf{P} = 0. \tag{133}$$

Angular momentum defined by Eq. (97) in the CM frame becomes

$$\begin{aligned}
\mathbf{L}' &= \sum_i \left[\mathbf{r}_i' \times \mathbf{p}_i'\right] = \sum_i \left[\left(\mathbf{r}_i - \mathbf{R}\right) \times m_i \left(\mathbf{v}_i - \mathbf{V}\right)\right] \\
&= \mathbf{L} + \mathbf{R} \times \mathbf{P} - \mathbf{R} \times \sum_i m_i \mathbf{v}_i - \sum m_i \mathbf{r}_i \times \mathbf{V} \\
&= \mathbf{L} + \mathbf{R} \times \mathbf{P} - \mathbf{R} \times \mathbf{P} - \mathbf{R} \times \mathbf{P}
\end{aligned} \tag{134}$$

that finally yields

$$\mathbf{L} = \mathbf{L}' + \mathbf{R} \times \mathbf{P}. \tag{135}$$

This means that the total angular momentum $\mathbf{L}$ consists of the internal angular momentum $\mathbf{L}'$ and the angular momentum $\mathbf{R} \times \mathbf{P}$ corresponding to the motion of the system as the whole.

Kinetic energy of a system of particles can be transformed as

$$E_k = \frac{1}{2}\sum_i m_i \left(\mathbf{v}_i' + \mathbf{V}\right)^2 = \frac{1}{2}\sum_i m_i \mathbf{v}_i'^2 + \frac{M\mathbf{V}^2}{2} + \mathbf{P}' \cdot \mathbf{V} = E_k' + \frac{M\mathbf{V}^2}{2} \tag{136}$$

since $\mathbf{P}' = 0$. Thus also kinetic energy consists of the internal kinetic energy and the kinetic energy corresponding to the motion of the system as the whole.

An isolated system of two interacting masses can be described as one so-called *reduced mass* moving around the CM of the system. Choosing a coordinate system with the origin at the CM, one obtains the constraint on the positions of the bodies

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0. \tag{137}$$

As the single dynamical variable one can choose the position-difference vector

$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2. \tag{138}$$

Using these two equations, one can express the individual positions as

$$\mathbf{r}_1 = \frac{m_2}{m_1 + m_2}\mathbf{r}, \qquad \mathbf{r}_2 = -\frac{m_1}{m_1 + m_2}\mathbf{r}. \tag{139}$$

The total energy of the system

$$E = \frac{m_1\mathbf{v}_1^2}{2} + \frac{m_2\mathbf{v}_2^2}{2} + U\left(|\mathbf{r}_1 - \mathbf{r}_2|\right) \tag{140}$$

becomes

$$E = \frac{m\mathbf{v}^2}{2} + U\left(|\mathbf{r}|\right), \tag{141}$$

where $\mathbf{v} = \dot{\mathbf{r}}$ and

$$m \equiv \frac{m_1 m_2}{m_1 + m_2} \tag{142}$$

is the reduced mass. The equation of motion of the system can be obtained from either of the two equations for the individual bodies, e.g.,

$$m_1\ddot{\mathbf{r}}_1 = m\ddot{\mathbf{r}} = -\frac{\partial U}{d\mathbf{r}_1} = -\frac{\partial U}{d\mathbf{r}}. \tag{143}$$

The resulting equation of motion for the reduced mass

$$m\ddot{\mathbf{r}} = -\frac{\partial U}{d\mathbf{r}} \tag{144}$$

can be obtained from the equation of motion for the second body as well.

6 One-dimensional conservative motion

For one-dimensional conservative systems conservation of the total energy can be used to reduce the second-order differential equation to a first-order differential equation that can be straightforwardly integrated. The basic form of a one-dimensional system, a particle, has the energy

$$E = \frac{mv^2}{2} + U(x). \tag{145}$$

In addition, there are systems that become one-dimensional because of a constrait. For instance, a mass on a light rod (pendulum) making a motion along a circle in the xy plane can be described by a single dynamical variable, the angle φ, that makes it effectively one-dimensional. Solving the above equation for $v = \dot{x}$, one arrives at the first-order ODE

$$\dot{x} = \sqrt{\frac{2\left[E - U(x)\right]}{m}}. \tag{146}$$

Here the integral of motion E plays the role of an integration constant. Further integration of Eq. (146) is done as follows

$$t = \int dt = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - U(x)}}. \tag{147}$$

This defines $x(t)$ implicitly. Another integration constant comes from the indefinite integral.

Motion of a mechanical system can take place only in the region $E > U$, where kinetic energy is positive. Regions $E < U$ are inaccessible for the system and are called *barrier regions*. The system hits barriers and changes direction of its motion at *turning points*, where $E = U$ and thus $v = 0$. Positions of turning points depends on the energy. If the motion of the system is limited by left and right turning ponts x_1 and x_2 being the two roots of the equation $E = U(x)$, the particle performs oscillations between these points with the period

$$T(E) = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}}. \tag{148}$$

In general, the period depends on the energy E. If E is close to the minimum of $U(x)$ where the latter is parabolic,

$$U(x) \cong U_0 + \frac{k}{2}\left(x - x_0\right)^2, \tag{149}$$

the system becomes a harmonic oscillator with the period independent of the energy. The motion of the harmonic oscillator has been determined above by solving its equation of motion. Within the present approach, setting $U_0 = x_0 = 0$, one finds $x_{1,2} = \pm x_m$, where $x_m = \sqrt{2E/k}$. Then one can integrate Eq. (147) as follows

$$t = \sqrt{\frac{m}{k}} \int \frac{dx}{\sqrt{x_m^2 - x^2}} = \frac{1}{\omega_0}\left(\arcsin\frac{x}{x_m} + \varphi_0\right), \tag{150}$$

where $\omega_0 \equiv \sqrt{k/m}$ is the oscillator frequency and φ_0 is an integration constant. Resolving this formula for x, one finally obtains

$$x(t) = x_m \sin\left(\omega_0 t - \varphi_0\right), \tag{151}$$

a form of Eq. (40).

Square potential energy near the minimum is a borderline. If $U(x)$ is subsquare (grows slower than x^2), the frequency of oscillations decreases with the amplitude. For a supersquare $U(x)$ oscillation frequency increases with the amplitude. For instance, for the potential energy $U(x) = A\left|x\right|^n$ turning points are $x_{1,2} = \pm x_m$, where $x_m = (E/A)^{1/n}$. The period given by Eq. (148) becomes

$$T(E) = 2\sqrt{\frac{2m}{A}} \int_0^{x_m} \frac{dx}{\sqrt{x_m^n - x^n}}. \tag{152}$$

Changing integration variable to $y \equiv x/x_m$, one obtains

$$T(E) = 2\sqrt{\frac{2m}{A}} x_m^{1-n/2} \int_0^1 \frac{dy}{\sqrt{1 - y^n}} = 2\sqrt{\frac{2m}{A}} \left(\frac{E}{A}\right)^{1/n-1/2} \int_0^1 \frac{dy}{\sqrt{1 - y^n}}, \tag{153}$$

where the integral is just a number. For the frequency one has

$$\omega_0(E) \propto 1/T \propto E^{(n-2)/(2n)} \tag{154}$$

that illustrates the above statement.

Consider a particle in a washboard potential

$$U(x) = U_0 \left[1 - \cos(ax)\right]. \tag{155}$$

Near the minima at $ax = 2\pi n$, $n = 0, \pm 1, \pm 2$, etc., one has $U(x) \cong kx^2/2$ with $k = U_0 a^2$ and behaves similarly to the harmonic oscillator. The whole problem is mathematically equivalent to that of pendulum. At $ax = \pi + 2\pi n$ potential energy has maxima, $U_{\max} = 2U_0$. Let us calculate the period of oscillations, considering the region around the minimum at $x = 0$. Eq. (148) becomes

$$T = 2\sqrt{\frac{2m}{U_0}} \int_0^{x_m} \frac{dx}{\sqrt{\cos(ax) - \cos(ax_m)}}, \tag{156}$$

where x_m is the right turning point satisfying $E = U_0 \left[1 - \cos(ax_m)\right]$. The integral can be transformed into the form similar to those above,

$$T = 2\sqrt{\frac{m}{U_0}} \int_0^{x_m} \frac{dx}{\sqrt{\sin^2(ax_m/2) - \sin^2(ax/2)}}. \tag{157}$$

Next, one can employ the variable change

$$\begin{aligned} \sin(ax/2) &= \sin(ax_m/2)\sin\xi \\ (a/2)\cos(ax/2)\,dx &= \sin(ax_m/2)\cos\xi d\xi \end{aligned} \tag{158}$$

that yields

$$T = \frac{4}{a}\sqrt{\frac{m}{U_0}} \int_0^{\pi/2} \frac{1}{\cos(ax/2)} \frac{\cos\xi d\xi}{\sqrt{1 - \sin^2\xi}} = \frac{4}{a}\sqrt{\frac{m}{U_0}} \int_0^{\pi/2} \frac{d\xi}{\cos(ax/2)}. \tag{159}$$

Now, eliminating x in the integrand, one obtains

$$T = \frac{4}{a}\sqrt{\frac{m}{U_0}} K\left[\sin^2(ax_m/2)\right],$$

where $K(m)$ is the elliptic integral of the first kind,

$$K(m) = \int_0^{\pi/2} \frac{d\xi}{\sqrt{1 - m\sin^2\xi}}. \tag{160}$$

Let us consider limiting cases of this formula. In the case of small oscillations near the minimum $ax_m \ll 1$ and one can use

$$K(m) \cong \int_0^{\pi/2} d\xi \left(1 + \frac{1}{2} m \sin^2\xi\right) = \frac{\pi}{2}\left(1 + \frac{1}{4}m\right). \tag{161}$$

Then the period becomes

$$T = T_0 \left[1 + \frac{1}{16}(ax_m)^2 + \ldots\right], \qquad T_0 = \frac{2\pi}{a}\sqrt{\frac{m}{U_0}}. \tag{162}$$

If turning points approach the maxima of $U(x)$, then $m = \sin^2(ax_m/2) \to 1$. At this point the elliptic integral logarithmically diverges and so does the period. This is physically understandable because the particle is spending a lot of time near turning points when the latter approach the maxima.

If the energy of the particle exceeds the maxima of potential energy, its motion becomes unbounded and unidirectional, so that there is no period any longer. Different regimes can be conveniently represented on the two-dimensional phase space of the particle (x, v). Small oscillations near minima make elliptic trajectories in the phase space. Unbounded motion makes infinite curved trajectories. In the limit of a very high energy trajectories become straight lines $v = \sqrt{2E/m} = \text{const}$. Special trajectories are those corresponding to $E = U_{\max}$ and separate bound and unbound trajectories. They are called separatrices.

7 Systems with constraints and special coordinates

Some mechanical systems include *constraints* that are restricting motion of its parts. Simplest examples are mass on the incline, two masses connected by a light rod and a mathematical pendulum (mass on light rod, the other point of the rod fixed at the pivot point or fulcrum). Constaints that can be eliminated leading to decreasing of the number of dynamical variables of the system (its degrees of freedom) are called *holonomic*. More rare *non-holonomic* constrains cannot be eliminated and they have to be added to the equations of motion via reaction forces. A cylinder rolling on a plane withoud slipping is a system with a holonomic constraint, as the rotation angle of the cylinder can be expressed via displacement of its CM or vice versa. However, a disk or a sphere rolling on a plane without slipping can make a more complicated motion than back and forth, so that the constraint cannot be eliminated and is non-holonomic. Removing friction makes the sphere on a plane a holonomic system while the disc remains non-holonomic. Finally, any rigid body can be considered as a collection of point masses with lots of constraints, althought this point of view is not practically significant. In some important cases constrants can be resolved by choosing a special coordinate system.

7.1 Polar coordinate system; pendulum

Consider, as an example, a mathematical pendulum (simply pendulum), a mass m on a light rod of length l, moving in xy plane. It is convenient to change to polar coordinates with the center at the fulcrum

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \tag{163}$$

where the constraint has the simple form $r = l$. For the position vector one obtains the expression

$$\mathbf{r} = l\left(\mathbf{e}_x \cos\varphi + \mathbf{e}_y \sin\varphi\right), \tag{164}$$

One can see that, under the constraint, the two-component vector $\mathbf{r} = (x, y)$ has been reduced to a single variable φ.

The velocity of the pendulum has the form

$$\mathbf{v} = \dot{\mathbf{r}} = l\left(-\mathbf{e}_x \sin\varphi + \mathbf{e}_y \cos\varphi\right)\dot{\varphi}. \tag{165}$$

One can check $\mathbf{v} \cdot \mathbf{r} = 0$, that is, both vectors are perpendicular. This means that velocity is tangential with respect to the circular trajectory the pendulum is making. Kinetic enegy reads

$$E_k = \frac{m\mathbf{v}^2}{2} = \frac{ml^2\omega^2}{2}, \tag{166}$$

where

$$\omega \equiv \dot{\varphi} \tag{167}$$

is the angular velocity.

Acceleration is given by

$$\dot{\mathbf{v}} = l\left(-\mathbf{e}_x \sin\varphi + \mathbf{e}_y \cos\varphi\right)\ddot{\varphi} - l\left(\mathbf{e}_x \cos\varphi + \mathbf{e}_y \sin\varphi\right)\dot{\varphi}^2. \tag{168}$$

One can see that the first term in $\dot{\mathbf{v}}$ is tangential and collinear with velocity, while the second term is collinear with $\mathbf{r}$ and directed toward the center (fulcrum). This is centripetal acceleration. Introducing angular velocityone can write centripetal acceleration in the form

$$\mathbf{a}_c = -\omega^2 \mathbf{r}. \tag{169}$$

It is convenient to project everything on the (local) direction of the circular trajectory and the perpendicular (radial) direction. For this, one can introduce orthogonal unit vectors

$$\begin{aligned} \mathbf{e}_r &= \mathbf{r}/r = \mathbf{e}_x \cos\varphi + \mathbf{e}_y \sin\varphi \\ \mathbf{e}_\varphi &= -\mathbf{e}_x \sin\varphi + \mathbf{e}_y \cos\varphi \end{aligned} \tag{170}$$

that satisfy

$$\frac{\partial \mathbf{e}_r}{\partial \varphi} = \mathbf{e}_\varphi, \qquad \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\mathbf{e}_r. \tag{171}$$

Note also the expression for the gradient in the polar coordinate system

$$\frac{\partial}{\partial \mathbf{r}} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\varphi \frac{1}{r}\frac{\partial}{\partial \varphi}. \tag{172}$$

Eqs. (170) can be used to rewrite the above formulas as

$$\begin{aligned} \mathbf{r} &= l\mathbf{e}_r \\ \mathbf{v} &= l\omega \mathbf{e}_\varphi \\ \dot{\mathbf{v}} &= l\dot{\omega}\mathbf{e}_\varphi - l\omega^2 \mathbf{e}_r, \end{aligned} \tag{173}$$

that is,

$$\begin{aligned} v_r &= 0, \qquad v_\varphi = l\omega \\ \dot{v}_r &= -l\omega^2, \qquad \dot{v}_\varphi = l\dot{\omega}. \end{aligned} \tag{174}$$

Note that $\dot{v}_r \neq \partial_t v_r = 0$. From Eq. (170) one obtains

Constraint causes an additional reaction force

$$\mathbf{N} = f_N \mathbf{e}_r \tag{175}$$

that helps to keep the body on the trajectory. Let us project the equation of motion

$$m\dot{\mathbf{v}} = \mathbf{F} + \mathbf{N}, \tag{176}$$

where $\mathbf{F}$ is the external force, onto $\mathbf{e}_r$ and $\mathbf{e}_\varphi$. One obtains

$$\begin{aligned} m\dot{\mathbf{v}} \cdot \mathbf{e}_r &= -ml\omega^2 = (\mathbf{F} + \mathbf{N}) \cdot \mathbf{e}_r = F_r + f_N \\ m\dot{\mathbf{v}} \cdot \mathbf{e}_\varphi &= ml\dot{\omega} = (\mathbf{F} + \mathbf{N}) \cdot \mathbf{e}_\varphi = F_\varphi. \end{aligned} \tag{177}$$

Here the second equation yields the equation of motion for the pendulum,

$$\dot{\omega} = F_\varphi / (ml). \tag{178}$$

The first equation yields the reaction force,

$$f_N = -ml\omega^2 - F_r. \tag{179}$$

The latter may be important in the problem of mechanical stability of a real-life rod but is irrelevant in the idealized consideration of constraints. Thus one can just project the Newton's second law onto the direction of the trajectory enforced by the constraint, that leads to Eq. (178).

If $\mathbf{F}$ is a potential force,

$$F_\varphi = \mathbf{F}\cdot\mathbf{e}_\varphi = -\frac{\partial U}{\partial \mathbf{r}}\cdot\mathbf{e}_\varphi. \tag{180}$$

Using Eq. (172), one obtains

$$F_\varphi = -\frac{1}{r}\frac{\partial U}{\partial \varphi} = -\frac{1}{l}\frac{\partial U}{\partial \varphi}. \tag{181}$$

For the gravity force acting on the pendulum,

$$\mathbf{F} = mg\mathbf{e}_x, \qquad U(\mathbf{r}) = -mg\mathbf{r}\cdot\mathbf{e}_x, \tag{182}$$

one obtains

$$F_\varphi = \mathbf{F}\cdot\mathbf{e}_\varphi = mg\mathbf{e}_x\cdot\mathbf{e}_\varphi = -mg\sin\varphi. \tag{183}$$

Potential energy expressed via φ has the form

$$U(\varphi) = -mg\mathbf{r}\cdot\mathbf{e}_x = -mgl\left(\mathbf{e}_x\cos\varphi + \mathbf{e}_y\sin\varphi\right)\cdot\mathbf{e}_x = -mgl\cos\varphi, \tag{184}$$

so that the total kinetic energy becomes

$$E = \frac{ml^2\omega^2}{2} - mgl\cos\varphi. \tag{185}$$

From Eq. (181) one obtains

$$F_\varphi = -\frac{1}{l}\frac{\partial U}{\partial \varphi} = -mg\sin\varphi, \tag{186}$$

same as above. This, the equation of motion for the pendulum, Eq. (178), takes the final form

$$\ddot{\varphi} + \omega_0^2\sin\varphi = 0, \qquad \omega_0 = \sqrt{\frac{g}{l}}. \tag{187}$$

Here ω_0 is the frequency of pendulum's oscillations in the limit of small amplitude, where $\sin\varphi \cong \varphi$. Note that because of the constraint the problem has become non-linear.

7.2 Spherical coordinate system; Spherical pendulum

Consider a mass m on a light rod of length a, fixed at a fulcrum at the other side, the system now being able to move on a sphere of radius a. To resolve the constraint, one can choose the spherical coordinate system

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta, \tag{188}$$

where θ is azimuthal angle and φ is polar angle. In this system the constraint has the simple form $r = a$. One can introduce the local orthogonal frame defined by

$$\begin{aligned}
\mathbf{e}_r &= \mathbf{r}/r = \mathbf{e}_x\sin\theta\cos\varphi + \mathbf{e}_y\sin\theta\sin\varphi + \mathbf{e}_z\cos\theta \\
\mathbf{e}_\theta &= \mathbf{e}_x\cos\theta\cos\varphi + \mathbf{e}_y\cos\theta\sin\varphi - \mathbf{e}_z\sin\theta \\
\mathbf{e}_\varphi &= -\mathbf{e}_x\sin\varphi + \mathbf{e}_y\cos\varphi.
\end{aligned} \tag{189}$$

These vectors satisfy

$$\begin{aligned} \frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta, \qquad \frac{\partial \mathbf{e}_r}{\partial \varphi} = \mathbf{e}_\varphi \sin\theta && (190)\\ \frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -\mathbf{e}_r, \qquad \frac{\partial \mathbf{e}_\theta}{\partial \varphi} = \mathbf{e}_\varphi \cos\theta && (191)\\ \frac{\partial \mathbf{e}_\varphi}{\partial \theta} &= 0, \qquad \frac{\partial \mathbf{e}_\varphi}{\partial \varphi} = -\mathbf{e}_x \cos\varphi - \mathbf{e}_y \sin\varphi = -\mathbf{e}_r \sin\theta - \mathbf{e}_\theta \cos\theta, && (192) \end{aligned}$$

as well as

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\varphi \tag{193}$$

plus cyclic permutations. The gradient in the spherical system has the form

$$\frac{\partial}{\partial \mathbf{r}} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \mathbf{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}. \tag{194}$$

The velocity of the spherical pendulum can be obtained by differentiating the first line of Eq. (189) using Eq. (190),

$$\mathbf{v} = \dot{\mathbf{r}} = a\left(\frac{\partial \mathbf{e}_r}{\partial \theta}\dot{\theta} + \frac{\partial \mathbf{e}_r}{\partial \varphi}\dot{\varphi}\right) = a\mathbf{e}_\theta \dot{\theta} + a\mathbf{e}_\varphi \sin\theta\dot{\varphi}. \tag{195}$$

As was done for the polar coordinate system above, to obtain the equation of motion in the spherical system, it is sufficient to project components of Newton's second law onto the local direction of the plane, i.e., discarding centripetal terms. In vector form of this projected equation of motion reads

$$m\dot{\mathbf{v}} - \mathbf{F} - \left((m\dot{\mathbf{v}} - \mathbf{F}) \cdot \mathbf{e}_r\right) \mathbf{e}_r = 0 \tag{196}$$

or

$$(m\dot{\mathbf{v}} - \mathbf{F}) \times \mathbf{e}_r = 0. \tag{197}$$

In terms of the spherical components of the vectors this becomes

$$m\dot{v}_\theta = F_\theta, \qquad m\dot{v}_\varphi = F_\varphi. \tag{198}$$

For projected acceleration one obtains

$$\begin{aligned} \dot{\mathbf{v}}/a &= \mathbf{e}_\theta \ddot{\theta} + \mathbf{e}_\varphi \sin\theta\ddot{\varphi} + \mathbf{e}_\varphi \cos\theta\dot{\theta}\dot{\varphi} + \dot{\mathbf{e}}_\theta \dot{\theta} + \dot{\mathbf{e}}_\varphi \sin\theta\dot{\varphi}\\ &= \mathbf{e}_\theta \ddot{\theta} + \mathbf{e}_\varphi \sin\theta\ddot{\varphi} + \mathbf{e}_\varphi \cos\theta\dot{\theta}\dot{\varphi} + \left(\frac{\partial \mathbf{e}_\theta}{\partial \theta}\dot{\theta} + \frac{\partial \mathbf{e}_\theta}{\partial \varphi}\dot{\varphi}\right)\dot{\theta} + \left(\frac{\partial \mathbf{e}_\varphi}{\partial \theta}\dot{\theta} + \frac{\partial \mathbf{e}_\varphi}{\partial \varphi}\dot{\varphi}\right)\sin\theta\dot{\varphi}\\ &= \mathbf{e}_\theta \ddot{\theta} + \mathbf{e}_\varphi \sin\theta\ddot{\varphi} + \mathbf{e}_\varphi \cos\theta\dot{\theta}\dot{\varphi} + \mathbf{e}_\varphi \cos\theta\dot{\varphi}\dot{\theta} - \mathbf{e}_\theta \cos\theta\dot{\varphi}\sin\theta\dot{\varphi}\\ &= \mathbf{e}_\theta \ddot{\theta} + \mathbf{e}_\varphi \sin\theta\ddot{\varphi} + \mathbf{e}_\varphi 2\cos\theta\dot{\theta}\dot{\varphi} - \mathbf{e}_\theta \sin\theta\cos\theta\dot{\varphi}^2 \end{aligned} \tag{199}$$

or, finally,

$$\dot{\mathbf{v}}/a = \mathbf{e}_\theta\left(\ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2\right) + \mathbf{e}_\varphi\left(\sin\theta\ddot{\varphi} + 2\cos\theta\dot{\theta}\dot{\varphi}\right) \tag{200}$$

with centripetal terms dropped. Thus, the equation of motion of body confined to a sphere is Eqs. (198) in the form

$$\begin{aligned} \ddot{\theta} - \sin\theta\cos\theta\dot{\varphi}^2 &= \frac{1}{ma}F_\theta && (201)\\ \sin\theta\ddot{\varphi} + 2\cos\theta\dot{\theta}\dot{\varphi} &= \frac{1}{ma}F_\varphi && (202) \end{aligned}$$

that is strongly nonlinear.

To obtain components of the force, one has to express the potential energy via θ and φ and use Eq. (194). This yields

$$F_\theta = -\frac{1}{r}\frac{\partial U}{\partial \theta}, \qquad F_\varphi = -\frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi} \tag{203}$$

with $r \Rightarrow l$. If gravity force is applied,

$$U(\mathbf{r}) = -mg\mathbf{r}\cdot\mathbf{e}_z = -mga\cos\theta, \tag{204}$$

then

$$F_\theta = -mg\sin\theta, \qquad F_\varphi = 0. \tag{205}$$

Consider angular momentum $\mathbf{l} = \mathbf{r}\times m\mathbf{v}$ of a body confined to a sphere. Expressing it in the spherical coordinate system and using Eq. (193), one obtains

$$\mathbf{l} = a\mathbf{e}_r \times ma\left(\mathbf{e}_\theta\dot{\theta} + \mathbf{e}_\varphi\sin\theta\dot{\varphi}\right) = ma^2\left(\mathbf{e}_\varphi\dot{\theta} - \mathbf{e}_\theta\sin\theta\dot{\varphi}\right). \tag{206}$$

The torque is defined by

$$\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F} = a\mathbf{e}_r \times (\mathbf{e}_\theta F_\theta + \mathbf{e}_\varphi F_\varphi) = a\mathbf{e}_\varphi F_\theta - a\mathbf{e}_\theta F_\varphi. \tag{207}$$

In the case of gravity force directed along z axis, as defined above, the component of the torque $\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$ along z axis is zero: $\tau_z = \boldsymbol{\tau}\cdot\mathbf{e}_r = a\mathbf{e}_\varphi F_\theta\cdot\mathbf{e}_z = 0$. This means that z component of the angular momentum is conserved. Using Eq. (189), one obtains

$$l_z = \mathbf{l}\cdot\mathbf{e}_z = -ma^2\sin\theta\dot{\varphi}\mathbf{e}_\theta\cdot\mathbf{e}_z = ma^2\sin^2\theta\dot{\varphi} = \text{const.} \tag{208}$$

A similar result can be obtained from Eq. (202) in the case $F_\varphi = 0$ by multiplying by the integrating factor $\sin\theta$ as follows:

$$\left(\sin\theta\ddot{\varphi} + 2\cos\theta\dot{\theta}\dot{\varphi}\right)\sin\theta = \partial_t\sin^2\theta\dot{\varphi} = 0. \tag{209}$$

Integrating this, one obtains Eq. (208). Now, eliminating $\dot{\varphi}$in Eq. (201), one obtains an autonomous equation of motion for θ,

$$\ddot{\theta} = \frac{l_z^2}{(ma^2)^2}\frac{\cos\theta}{\sin^3\theta} + \frac{F_\theta}{ma}. \tag{210}$$

The total energy of a particle on a sphere can be obtained using Eq. (195),

$$E = \frac{ma^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right) + U(\theta,\varphi). \tag{211}$$

In the axially-symmetric case $U = U(\theta)$ one has $l_z = \text{const}$, and the energy can be expressed as

$$E = \frac{ma^2}{2}\left(\dot{\theta}^2 + \frac{l_z^2}{(ma^2)^2}\frac{1}{\sin^2\theta}\right) + U(\theta). \tag{212}$$

This is an effectively one-dimensional motion that can be integrated after resolving for $\dot{\theta}$. Eq. (212) itself can be obtained as an integral of motion of Eq. (210) multiplying by the integrating factor $\dot{\theta}$, similarly to what was done in the main text. Moreover, one can obtain Eq. (211) by integrating Eqs. (201) and (202).

8 Motion in a central field

Consider motion of a body in a central field, $U = U(r)$, so that $\mathbf{F} = -\nabla U = -(dU/dr)\,\mathbf{e}_r$ is directed radially. In this case the angular momentum is conserved, $\mathbf{l} = \mathbf{r} \times m\mathbf{v} = \text{const}$, see discussion after Eq. (96). Since both $\mathbf{r}$ and $\mathbf{v}$ are perpendicular to $\mathbf{l}$, the body is moving in the plane perpendicular to $\mathbf{l}$. It is convenient to use polar coordinate system to describe this motion. Using

$$\mathbf{r} = \mathbf{e}_r r, \qquad \mathbf{v} = \mathbf{e}_\varphi \dot{\varphi} r + \mathbf{e}_r \dot{r} \tag{213}$$

(see Sec. 7.1), one obtains

$$l = mr^2\dot{\varphi} = \text{const.} \tag{214}$$

As $r > 0$, one can see that φ is changing monotonically and $\dot{\varphi}$ is not changing the sign. This equation can be rewritten in terms of the sectorial velocity $\dot{S}$ as

$$\dot{S} \equiv \frac{1}{2} r^2 \dot{\varphi} = \frac{l}{2m} = \text{const.} \tag{215}$$

This is Kepler's second law.

Substituting Eq. (214) into the energy, one obtains

$$E = \frac{m\mathbf{v}^2}{2} + U(r) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + U(r) = \frac{m}{2}\left(\dot{r}^2 + \frac{l^2}{m^2 r^2}\right) + U(r) \tag{216}$$

or

$$E = \frac{m\dot{r}^2}{2} + U_{\text{eff}}(r), \tag{217}$$

where

$$U_{\text{eff}}(r) = U(r) + \frac{l^2}{2mr^2}. \tag{218}$$

The last term here is the so-called centrifugal energy. Then one can proceed as in the case of one-dimensional motion. First, one resolves the formula above for $\dot{r}$,

$$\dot{r} = \sqrt{\frac{2}{m}\left[E - U_{\text{eff}}(r)\right]}. \tag{219}$$

Integrating this, one obtains the dependence $r(t)$ implicitly,

$$t = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}}. \tag{220}$$

The trajectory of the body is defined in the polar system by the dependence $r(\varphi)$. One can find the inverse function $\varphi(r)$ by writing $\dot{\varphi} = \partial_r \varphi \dot{r}$ and using Eqs. (214) and (219). One obtains

$$\frac{d\varphi}{dr} = \frac{\dot{\varphi}}{\dot{r}} = \sqrt{\frac{m}{2}} \frac{l/(mr^2)}{\sqrt{E - U_{\text{eff}}(r)}} \tag{221}$$

and, integrating this,

$$\varphi = \int \frac{dr}{r^2} \frac{l}{\sqrt{2m\left[E - U_{\text{eff}}(r)\right]}}. \tag{222}$$

Because of the centrifugal energy, the body cannot fall into the center for typical attractive forces ($U \propto 1/r$), because the positive centrifugal energy is growing faster. As in the case of one-dimensional motion, the body is moving within the classically accessible region $U_{\text{eff}}(r) < E$. If the radial motion is between two turning points, $r_1 < r < r_2$, this is a bound state and the body is orbiting around the center. The state with only one turning point is a *scattering* state. The body is coming from infinity and then goes away after scattering on attracting or repelling center. If one chooses a turning point as the origin for φ, the traejectory $r(\varphi)$ will be symmetric with respect to this turning point, $r(\varphi) = r(-\varphi)$.

The period of motion in bound states is defined as time needed for the body to go from one turning point to the other and back

$$T = \sqrt{2m} \int_{r_1}^{r_2} \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}}. \tag{223}$$

The rotation angle corresponding to the period of motion is given by

$$\Delta\varphi = l\sqrt{\frac{2}{m}} \int_{r_1}^{r_2} \frac{dr}{r^2} \frac{1}{\sqrt{E - U_{\text{eff}}(r)}}. \tag{224}$$

Closed orbits correspond to $\Delta\varphi = 2\pi m/n$, where m and n are natural numbers. Closed orbit is a special case, since for an arbitrary $U(r)$ the rotating angle $\Delta\varphi$ is arbitrary. However, there are two important particular cases, in which orbits are closed. First, this is gravitation, $U \propto 1/r$. Second, this is harmonic oscillator, $U \propto r^2$.

If two turning points are close to each other, $r_1 \cong r_2 \cong R$, then the orbit is nearly circular. In this case, in the integral for $\Delta\varphi$ one can approximate $1/r^2 \Rightarrow 1/R^2$. After that the imtegral becomes the same as for the period and one obtains the relation $T = mR^2\Delta\varphi/l$. With $\Delta\varphi = 2\pi$ and $l = mrv$ one obtains $T = 2\pi R/v$, as it should be. This is the case of rotation of the Earth around the Sun.

8.1 Kepler's problem

Consider the important case $U \propto 1/r$ that corresponds to gravitational interaction between stars and planets, as well as to the Coulomb interaction of charged particles. The effective energy of Eq. (218) has the form

$$U_{\text{eff}}(r) = -\frac{\alpha}{r} + \frac{l^2}{2mr^2}, \qquad \alpha \equiv GMm, \tag{225}$$

where G is gravitational constant, M and m are masses of the two gravitating bodies. If $M \gg m$ (The Sun and the Earth), one can consider the light body m rotating around the heavy body M put in the center. This will be our main case. If the masses are comparable, they will be both moving but the problem can be reduced to a one-body problem using the reduced mass (see Sec. 5). This effective energy has the minimum $U_{\text{eff}}(r_0) = E_{\min}$ with

$$r_0 = \frac{l^2}{\alpha m}, \qquad E_{\min} = -\frac{\alpha^2 m}{2l^2}. \tag{226}$$

For $E_{\min} < E < 0$ there are two turning points and the motion is bounded (orbiting). For $E > 0$ there is only one turning point and the motion is unbounded (scattering). The trajectory can be found from Eq. (222). With the new variable $u = 1/r$, $du = -dr/r^2$ one obtains

$$\begin{aligned} \varphi &= -l \int \frac{du}{\sqrt{2m\left(E + \alpha u - \frac{l^2}{2m}u^2\right)}} = -\int \frac{ds}{\sqrt{\frac{2mE}{l^2} + \frac{2m\alpha}{l^2}u - u^2}} \\ &= -\int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{m^2\alpha^2}{l^4} - \left(u - \frac{m\alpha}{l^2}\right)^2}} = -\arccos \frac{u - \frac{m\alpha}{l^2}}{\sqrt{\frac{2mE}{l^2} + \frac{m^2\alpha^2}{l^4}}}. \end{aligned} \tag{227}$$

This can be rewritten as

$$\frac{r_0}{r} = 1 + \epsilon \cos\varphi, \tag{228}$$

where

$$\epsilon \equiv \sqrt{1 + \frac{E}{|E_{\min}|}}. \tag{229}$$

For $E<0$ this trajectory is an ellipse (Kepler's first law) with *excentricity* ϵ and axes

$$a=\frac{r_0}{1-\epsilon^2}=\frac{\alpha}{2|E|},\qquad b=\frac{r_0}{\sqrt{1-\epsilon^2}}=\frac{l}{\sqrt{2m|E|}}. \tag{230}$$

In the case $E=E_{\min}$ one has $\epsilon=0$ and the ellipse degenerates into a circle $r=r_0=\frac{l^2}{m\alpha}=\frac{m^2r^2v^2}{m\alpha}$. From this one obtains $r=\alpha/(mv^2)$.

The area of the ellipse is

$$S=\pi ab=\frac{\pi\alpha l}{\sqrt{(2|E|)^3\,m}}. \tag{231}$$

This area is being covered during the period of motion. With the help of Eq. (215) one obtains $S=\dot{S}T$ and

$$T=\frac{S}{\dot{S}}=\frac{\pi\alpha l}{\sqrt{(2|E|)^3\,m}}\frac{2m}{l}=\pi\alpha\sqrt{\frac{m}{2|E|^3}}, \tag{232}$$

depending only on the energy and diverging for $E\to 0$. One can express the period via the linear size of the orbit a using Eq. (230). The result is

$$T=2\pi\sqrt{\frac{m}{\alpha}}a^{3/2}. \tag{233}$$

The relation $T^2\propto a^3$ is Kepler's third law.

Time dependence of the motion can be conveniently represented in parametric form. For this, one can rewrite Eq. (220) as

$$t=\sqrt{\frac{m}{2|E|}}\int\frac{rdr}{\sqrt{-r^2+\frac{\alpha}{|E|}r-\frac{l^2}{2m|E|}}}=\sqrt{\frac{ma}{\alpha}}\int\frac{rdr}{\sqrt{-(r-a)^2+a^2\epsilon^2}}. \tag{234}$$

With the substitution

$$r=a\left(1-\epsilon\cos\xi\right) \tag{235}$$

the integral can be calculated as

$$t=\sqrt{\frac{ma}{\alpha}}\int\frac{a\left(1-\epsilon\cos\xi\right)a\epsilon\sin\xi d\xi}{\sqrt{a^2\epsilon^2\left(1-\cos^2\xi\right)}}=\sqrt{\frac{ma^3}{\alpha}}\int\left(1-\epsilon\cos\xi\right)d\xi=\sqrt{\frac{m}{\alpha}}a^{3/2}\left(\xi-\epsilon\sin\xi\right). \tag{236}$$

These two formulas provide the dependence $r(t)$ parametrically. Period of motion corresponds to changing of ξ by 2π: r returns to the same value while t increases by T given by Eq. (233).

The time dependence of the angle φ is now defined by Eq. (228). Moreover, one can express Descartes coordinates (x,y) parametrically, too. First, one finds

$$x=r\cos\varphi=\frac{r_0-r}{\epsilon}=\frac{a\left(1-\epsilon^2\right)-a\left(1-\epsilon\cos\xi\right)}{\epsilon}=a\left(\cos\xi-\epsilon\right). \tag{237}$$

Then one finds

$$y=\sqrt{r^2-x^2}=a\sqrt{1-\epsilon^2}\sin\xi. \tag{238}$$

Cases of unbound motion, $E>0$, and repelling potential $U(r)=\alpha/r$ (unbound motion for any energy) can be considered in a similar way.

8.2 Scattering by a central field

Consider an unbound motion of a body or a particle in a central field. The particle is coming from infinity, experiences the action of the central force, and then goes to infinity again, however, having changed its direction of motion. Change of direction of a body as the result of interaction with other bodies is called *scattering*. As was mentioned above, the traejectory is symmetric, $r(\varphi) = r(-\varphi)$, if one chooses the turning point r_1 as the origin of φ. Thus the change of direction (deflection angle) χ as the result of scattering can be obtained from Eq. (222) as

$$\chi = \pi - 2\varphi_0, \tag{239}$$

where

$$\varphi_0 = \int_{r_1}^{\infty} \frac{dr}{r^2} \frac{l}{\sqrt{2m\left[E - U_{\mathrm{eff}}(r)\right]}} \tag{240}$$

is half of the angle between the initial and final parts of the trajectory. In the following it is convenient, instead of integrals of motion E and l, use the speed at infinity v_∞ and the target distance ρ. The latter is the minimal distance from the center, corresponding to the straight trajectory in the absence of the central force. Using

$$E = \frac{mv_\infty^2}{2}, \qquad l = \rho m v_\infty, \tag{241}$$

one can rewrite Eq. (240) in the form

$$\chi = \pi - 2\int_{r_1}^{\infty} \frac{dr}{r^2} \frac{\rho}{\sqrt{1 - \frac{2U_{\mathrm{eff}}(r)}{mv_\infty^2}}}, \qquad \frac{2U_{\mathrm{eff}}(r)}{mv_\infty^2} = \frac{\rho^2}{r^2} + \frac{2U(r)}{mv_\infty^2}. \tag{242}$$

In experiments there is usually a beam of particles with different ρ that are being scattered. It is convenient to introduce the differential scattering cross-section $d\sigma$ defined by

$$d\sigma\left(\chi\right) = dN/n, \tag{243}$$

where dN is the number of particles scattered during a unit of time within the angular interval $d\chi$ around χ and n is the number of particles crossing a unit of area of the beam during a unit of time. It is assumed that n is uniform. One can see that $d\sigma$ has the unit of area. Since $\chi = \chi(\rho)$, particles scattered within $d\chi$ are those within target distance interval $d\rho = |d\rho/d\chi|\, d\chi$ that corresponds to $d\chi$. The number of such particles scattered during a unit of time is $dN = 2\pi\rho d\rho n$. Thus finally one obtains

$$d\sigma\left(\chi\right) = 2\pi\rho\left(\chi\right)\left|\frac{d\rho}{d\chi}\right| d\chi, \tag{244}$$

where $\rho\left(\chi\right)$ and $d\rho/d\chi$ can be found from Eq. (242).

If the particle is being scattered on another particle that is at rest before the collision, one can change into in the center-of-mass frame and consider scattering of a particle with the reduced mass on the force center located at the CM, using the formulas above. After that one has to change back into the laboratory frame that results in redefinition of angles and transformation of the results that requires some algebra.

Consider scattering of a charged particle by the Coulomb field, $U(r) = \alpha/r$, as in celestial mechanics. Calculating the integral in Eq. (242) as was done in Eq. (227), one arrives at

$$\chi = \pi - 2\arccos\frac{1}{\sqrt{1 + \rho^2/\rho_0^2}}, \tag{245}$$

where

$$\rho_0 \equiv \frac{\alpha}{mv_\infty^2} \tag{246}$$

is the characteristic distance. Resolving for ρ, one obtains

$$\rho^2 = \rho_0^2 \tan^2\left(\frac{\pi-\chi}{2}\right) = \rho_0^2 \cot^2\frac{\chi}{2}.$$

Now Eq. (244) yields

$$d\sigma = \pi\rho_0^2 \frac{\cos(\chi/2)}{\sin^3(\chi/2)} d\chi. \tag{247}$$

Instead of scattering within $d\chi$, one can consider scattering within the infinitesimal body angle

$$d\Omega = 2\pi \sin\chi d\chi. \tag{248}$$

This yields Rutherford formula

$$d\sigma = \left(\frac{\alpha}{2mv_\infty^2}\right)^2 \frac{d\Omega}{\sin^4(\chi/2)}. \tag{249}$$

In some cases one can define total scattering cross-section

$$\sigma = \int d\sigma = \int d\chi \frac{d\sigma}{d\chi}. \tag{250}$$

With the help of Eq. (244) it can be written as

$$\sigma = 2\pi \int \rho d\rho. \tag{251}$$

The image behind this formula is a scattering center in form of a circle of some finite radius a and the area $\sigma = \pi a^2$. If the target distance ρ satisfies $\rho < a$, the particle will hit the target and be scattered. Otherwise it will not be scattered at all. For all interactions that decrease gradually at infinity, particles with *all* ρ will be scattered. Correspondingly, the integral in Eq. (250) diverges at small χ. Only if the interaction has a cut-off, the total scattering cross-section is finite. The simplest example is elastic scattering on a rigid sphere of radius a. Here one immediately finds

$$\rho = a\sin\varphi_0 = a\sin\frac{\pi-\chi}{2} = a\cos\frac{\chi}{2}. \tag{252}$$

Substitution into Eq. (250) yields the differential cross-section

$$d\sigma = \frac{\pi a^2}{2}\sin\chi d\chi. \tag{253}$$

Integrating over χ yields

$$\sigma = \int d\sigma = \frac{\pi a^2}{2}\int_0^\pi \sin\chi d\chi = \pi a^2, \tag{254}$$

as it should be.

One can also define total scattering cross-section for specific events, for instance, for falling onto the attracting center $U = -\alpha/r^2$. In this case the effective potential energy of Eq. (218) reads

$$U_{\text{eff}}(r) = -\frac{\alpha}{r^2} + \frac{l^2}{2mr^2} = \left(-\alpha + \frac{l^2}{2m}\right)\frac{1}{r^2}. \tag{255}$$

Absorbed will be all particles for which the coefficient in $U_{\text{eff}}(r)$ is negative,

$$\frac{l^2}{2m} = \frac{(\rho m v_\infty)^2}{2m} < \alpha, \tag{256}$$

otherwise particles will escape to infinity. In terms of the target distance this condition has the form

$$\rho^2 < \rho_0^2 \equiv \frac{2\alpha}{mv_\infty^2}. \tag{257}$$

Total scattering scoss-section then becomes

$$\sigma = \pi\rho_0^2 = \frac{2\pi\alpha}{mv_\infty^2}. \tag{258}$$

Although in this case one can find σ easily, finding the differential cross-section requires a more serious calculation.

8.3 Small-angle scattering

If target distance ρ or particle's speed are large, the trajectory is nearly a straight line, say, along x axis. Passing the force center, the particle acquires a perpendicular momentum p_y as the result. Now the small scattering angle χ can be found as

$$\chi \cong \frac{p_y}{mv_\infty} \ll 1. \tag{259}$$

The momentum p_y can be found as

$$p_y = \int_{-\infty}^{\infty} dt F_y = -\int_{-\infty}^{\infty} dt \frac{\partial U}{\partial y} = -\int_{-\infty}^{\infty} dt \frac{dU}{dr}\frac{\partial r}{\partial y} = -\int_{-\infty}^{\infty} dt \frac{dU}{dr}\frac{y}{r}. \tag{260}$$

Considering, at the lowest order in the perturbation, the motion along x as undisturbed, one can use $dt = dx/v_\infty$ and also $y = \rho$. This yields

$$p_y = -\frac{\rho}{v_\infty}\int_{-\infty}^{\infty} \frac{dx}{r}\frac{dU}{dr}. \tag{261}$$

Changing to integration over r with the use of

$$x = \sqrt{r^2 - \rho^2}, \qquad dx = \frac{rdr}{\sqrt{r^2-\rho^2}}, \tag{262}$$

one obtains

$$p_y = -\frac{2\rho}{v_\infty}\int_{\rho}^{\infty} \frac{dU}{dr}\frac{dr}{\sqrt{r^2-\rho^2}} \tag{263}$$

and, finally, the deflection angle

$$\chi = -\frac{2\rho}{mv_\infty^2}\int_{\rho}^{\infty} \frac{dU}{dr}\frac{dr}{\sqrt{r^2-\rho^2}}. \tag{264}$$

Finding differential cross-section $d\sigma$ requires inverting the above formula that requires knowing the explicit form of $U(r)$.

The method used above is beautiful and instructive. However, the final integral formula is hardly simpler than Eqs. (239) and (246). The really new result is that for the arbitrary potential energy $U(x, y)$ (we set $z = 0$), obtained similarly,

$$\chi = -\frac{1}{mv_\infty^2}\int_{-\infty}^{\infty} dx \left.\frac{\partial U(x,y)}{\partial y}\right|_{y=\rho}. \tag{265}$$

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