CLASSICAL MECHANICS

Problems with Solutions

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Part I
Newtonian Mechanics

1 Single-particle problems

1.1 Overdamped harmonic oscillator

Using the general solution of the equation of motion for the free harmonic oscillator

$$\ddot{x}+2\Gamma\dot{x}+\omega_0^2 x=0, \tag{1}$$

obtain the explicitly real solution in the overdamped case $\Gamma<\omega_0$.

Solution. The general solution has the form

$$x(t)=C_+e^{i\Omega_+t}+C_-e^{i\Omega_-t}, \tag{2}$$

where

$$\Omega_\pm=i\Gamma\pm\tilde{\omega}_0,\qquad \tilde{\omega}_0\equiv\sqrt{\omega_0^2-\Gamma^2}=i\sqrt{\Gamma^2-\omega_0^2}. \tag{3}$$

Thus the explicitly real solution has the form

$$x(t)=C_+\exp\left[\left(-\Gamma-\sqrt{\Gamma^2-\omega_0^2}\right)t\right]+C_-\exp\left[\left(-\Gamma+\sqrt{\Gamma^2-\omega_0^2}\right)t\right]. \tag{4}$$

1.2 Power absorbed by harmonic oscillator

A harmonic force is applied to a harmonic oscillator at rest at $t=0$:

$$\ddot{x}+2\Gamma\dot{x}+\omega_0^2 x=f(t)=f_0\sin(\omega t). \tag{5}$$

Calculate the average absorbed power using

$$P_{\text{abs}}=\frac{1}{t}\int_0^t dt'\dot{x}(t')f(t') \tag{6}$$

at all times that are multiples of the period $T=2\pi/\omega$. Consider short-time and long-time limits for a weakly-damped oscillator, $\Gamma\ll\omega_0$. Use the solution of the equation of motion from the book keeping only resonant terms near resonance. Analyze the solution at long times and at $\Gamma=0$.

Solution. Since at $t = 0$ the oscillator was at rest, there is no free solution. The forced solution is given by

$$x(t) = -\frac{f_0}{4\omega_0}\frac{e^{i\omega t} - e^{(-\Gamma + i\omega_0)t}}{\Gamma - i\omega_0 + i\omega} + c.c., \tag{7}$$

where in the case of weak damping we set $\tilde{\omega}_0 \Rightarrow \omega_0$. It is convenient to first integrate the absorbed power by parts to simplify subsequent integration,

$$P_{\text{abs}} = \frac{1}{t}x(t)f(t) - \frac{1}{t}\int_0^t dt' x(t')\dot{f}(t'). \tag{8}$$

First,

$$\begin{aligned}
\frac{1}{t}x(t)f(t) &= -\frac{f_0^2}{4\omega_0 t}\frac{e^{i\omega t} - e^{(-\Gamma+i\omega_0)t}}{\Gamma - i\omega_0 + i\omega}\frac{e^{i\omega t} - e^{-i\omega t}}{2i} + c.c. \\
&= i\frac{f_0^2}{8\omega_0 t}\frac{e^{2i\omega t} - 1 - e^{(-\Gamma+i\omega_0+i\omega)t} + e^{(-\Gamma+i\omega_0-i\omega)t}}{\Gamma - i\omega_0 + i\omega} + c.c.
\end{aligned} \tag{9}$$

Near resonance $\omega_0 + \omega \cong 2\omega$, then $e^{2i\omega t} = 1$ because $t = n2\pi/\omega$. Thus one obtains

$$\frac{1}{t}x(t)f(t) = i\frac{f_0^2}{8\omega_0 t}\frac{e^{-\Gamma t}\left(-1 + e^{(i\omega_0 - i\omega)t}\right)}{\Gamma - i\omega_0 + i\omega} + c.c. \tag{10}$$

and further

$$\frac{1}{t}x(t)f(t) = i\frac{f_0^2}{8\omega_0 t}\frac{e^{-\Gamma t}\left\{-1 + \cos\left[(\omega - \omega_0)t\right] + i\sin\left[(\omega - \omega_0)t\right]\right\}(\Gamma - i\omega_0 + i\omega)}{(\omega - \omega_0)^2 + \Gamma^2} + c.c. \tag{11}$$

and

$$\frac{1}{t}x(t)f(t) = \frac{f_0^2}{4\omega_0 t}\frac{\left\{-1 + \cos\left[(\omega - \omega_0)t\right]\right\}(\omega_0 - \omega) - \Gamma\sin\left[(\omega - \omega_0)t\right]}{(\omega - \omega_0)^2 + \Gamma^2}. \tag{12}$$

This term vanishes at resonance and thus it can be neglected.

Further, using

$$\dot{f}(t) = \omega f_0 \cos(\omega t) = \frac{\omega}{2}f_0\left(e^{i\omega t} + e^{-i\omega t}\right), \tag{13}$$

one calculates

$$\begin{aligned}
P_{\text{abs}} &= -\frac{1}{t}\int_0^t dt' x(t')\dot{f}(t') \\
&= \frac{f_0^2\omega}{8\omega_0}\frac{1}{t}\int_0^t dt'\left[\frac{e^{i\omega t'} - e^{(-\Gamma+i\omega_0)t'}}{\Gamma - i\omega_0 + i\omega}\left(e^{i\omega t'} + e^{-i\omega t'}\right) + c.c.\right] \\
&= \frac{f_0^2\omega}{8\omega_0}\frac{1}{t}\int_0^t dt'\left[\frac{e^{2i\omega t'} + 1 - e^{(-\Gamma+i\omega_0+i\omega)t'} - e^{(-\Gamma+i\omega_0-i\omega)t'}}{\Gamma - i\omega_0 + i\omega} + c.c.\right] \\
&= \frac{f_0^2\omega}{8\omega_0 t}\frac{1}{\Gamma - i\omega_0 + i\omega}\left[\frac{e^{2i\omega t} - 1}{2i\omega} + t + \frac{e^{(-\Gamma+i\omega_0+i\omega)t} - 1}{\Gamma - i\omega_0 - i\omega} + \frac{e^{(-\Gamma+i\omega_0-i\omega)t} - 1}{\Gamma - i\omega_0 + i\omega}\right] + c.c.
\end{aligned} \tag{14}$$

Here the less-singular term containing $\omega + \omega_0$ has to be dropped in the resonance approximation. The first term vanishes since t is a multiple of the period. What remains can be split into two parts,

$$P_{\text{abs}} = P_{\text{abs}}^{(1)} + P_{\text{abs}}^{(2)}, \tag{15}$$

where

$$P_{\rm abs}^{(1)} = \frac{f_0^2}{8}\frac{1}{\Gamma - i\omega_0 + i\omega} + c.c. = \frac{f_0^2}{4}\frac{\Gamma}{(\omega-\omega_0)^2+\Gamma^2} \tag{16}$$

and

$$P_{\rm abs}^{(2)} = \frac{f_0^2}{8t}\frac{e^{(-\Gamma+i\omega_0-i\omega)t}-1}{(\Gamma - i\omega_0 + i\omega)^2} + c.c. \tag{17}$$

and we set $\omega \Rightarrow \omega_0$ in the prefactor near resonance. The first term, $P_{\rm abs}^{(1)}$, does not depend of time and is the stationary absorption at large times $t \gtrsim 1/\Gamma$. It peaks at resonance, $\omega = \omega_0$. The integral absorption is

$$\int_{-\infty}^{\infty} d\omega P_{\rm abs}^{(1)}(\omega) = \frac{f_0^2}{4}\int_{-\infty}^{\infty} d\omega \frac{\Gamma}{(\omega-\omega_0)^2+\Gamma^2} = \frac{f_0^2}{4}\pi. \tag{18}$$

In the limit $\Gamma \to 0$ it becomes a δ-function,

$$P_{\rm abs}^{(1)} = \frac{f_0^2}{4}\pi\delta(\omega-\omega_0). \tag{19}$$

Since, by definition of the δ-function,

$$\int_{-\infty}^{\infty} d\omega\delta(\omega-\omega_0) = 1, \tag{20}$$

approximating the peak by δ-function preserves integral absorption.

It can be shown that $P_{\rm abs}^{(2)}$ is important at short times $t \lesssim 1/\Gamma$. Thus, to simplify the calculation, we set $\Gamma \Rightarrow 0$ to obtain

$$P_{\rm abs}^{(2)} = \frac{f_0^2}{4t}\frac{1-\cos[(\omega-\omega_0)t]}{(\omega-\omega_0)^2}. \tag{21}$$

One can scale this contribution as

$$P_{\rm abs}^{(2)} = \frac{f_0^2}{4}t\frac{1-\cos[(\omega-\omega_0)t]}{[(\omega-\omega_0)t]^2} \tag{22}$$

from which one can see that the peak width is $\Delta\omega \propto 1/t$ and the peak height is $\propto t$. Thus, this peak is very broad at the beginning and becomes narrower in the course of time. The integral absorption is given by

$$\int_{-\infty}^{\infty} d\omega P_{\rm abs}^{(2)}(\omega) = \frac{f_0^2}{4}\int_{-\infty}^{\infty} dx \frac{1-\cos x}{x^2} = \frac{f_0^2}{4}\sqrt{2\pi} \tag{23}$$

and is time independent.

The general picture is the following. At short times $t \lesssim 1/\Gamma$ there is a time-dependent broadening peak $P_{\rm abs}^{(2)}$. At long times $t \gtrsim 1/\Gamma$ the width of the peak $P_{\rm abs}^{(2)}$ falls below that of $P_{\rm abs}^{(1)}$, so that the latter remains. To prove that there is only one peak at any time rather then two peaks of different widths, one has to use the complete expression for $P_{\rm abs}^{(2)}$ that contains Γ.

1.3 Energy loss by a free weakly-damped harmonic oscillator

Consider a free weakly-damped harmonic oscillator, $\Gamma \ll \omega_0$. Using the general solution for the free oscillator, calculate the time dependence of its energy and the relative energy loss during one period. Calculate the energy loss per period perturbatively using the solution for a free undamped oscillator and the formula for the time derivative of the total energy due to viscous drag.

Solution. The general solution has the form

$$x(t) \cong C_1 e^{-\Gamma t}\cos\omega_0 t + C_2 e^{-\Gamma t}\sin\omega_0 t, \tag{24}$$

where we have set $\tilde{\omega}_0 \Rightarrow \omega_0$ in the weak-damping limit. The total energy of the oscillator is given by

$$E = \frac{m\dot{x}^2}{2} + \frac{kx^2}{2} = \frac{m}{2}\left(\dot{x}^2 + \omega_0^2 x^2\right). \tag{25}$$

In the weak-damping limit

$$\dot{x}(t) \cong -C_1 e^{-\Gamma t}\omega_0 \sin\omega_0 t + C_2 e^{-\Gamma t}\omega_0 \cos\omega_0 t. \tag{26}$$

Substituting x and $\dot{x}$ into the energy, one obtains

$$E \cong \frac{m\omega_0^2}{2} e^{-2\Gamma t}\left[(C_1\cos\omega_0 t + C_2 \sin\omega_0 t)^2 + (-C_1 \sin\omega_0 t + C_2\cos\omega_0 t)^2\right] \tag{27}$$

and, finally,

$$E \cong \frac{m\omega_0^2}{2} e^{-2\Gamma t}\left(C_1^2 + C_2^2\right). \tag{28}$$

The relative energy loss is defined by

$$\eta \equiv \frac{\Delta E}{E} = \frac{e^{-2\Gamma t} - e^{-2\Gamma(t+T)}}{e^{-2\Gamma t}} = 1 - e^{-2\Gamma T} \cong 2\Gamma T = \frac{4\pi\Gamma}{\omega_0} \ll 1. \tag{29}$$

Let us now consider the perturbative solution of the problem. For the undamped motion one has

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t \tag{30}$$

and

$$v(t) = \dot{x}(t) = -C_1\omega_0 \sin\omega_0 t + C_2\omega_0\cos\omega_0 t. \tag{31}$$

The time derivative of the total energy is given by

$$\dot{E} = -\alpha v^2 = -2m\Gamma v^2 = 2m\Gamma v^2\omega_0^2\left(-C_1\sin\omega_0 t + C_2\cos\omega_0 t\right)^2. \tag{32}$$

Integrating this, one obtains

$$\Delta E = -\int_0^T \dot{E}dt = m\Gamma v^2\omega_0^2\left(C_1^2 + C_2^2\right)T. \tag{33}$$

Now

$$\eta \equiv \frac{\Delta E}{E} = \frac{m\Gamma v^2\omega_0^2\left(C_1^2 + C_2^2\right)T}{\frac{m\omega_0^2}{2}\left(C_1^2 + C_2^2\right)} = 2\Gamma T = \frac{4\pi\Gamma}{\omega_0}, \tag{34}$$

same as above.

1.4 Motion of a charged particle in mutually perpendicular electric and magnetic fields

Integrate equation of motion of a charged particle in mutually perpendicular electric and magnetic fields,

$$m\dot{\mathbf{v}} = q\left[\mathbf{v}\times\mathbf{B}\right] + q\mathbf{E}, \tag{35}$$

$\mathbf{E}\perp\mathbf{B}$. For simplicity, consider the case of the velocity perpendicular to $\mathbf{B}$.

Solution. Choose z axis along $\mathbf{B}$ and x axis along $\mathbf{E}$, than the particle will be moving in xy plane. The equation of motion in components has the form

$$\begin{aligned} \dot{v}_x &= \omega_c v_y + a \\ \dot{v}_y &= -\omega_c v_x, \end{aligned} \tag{36}$$

where $a \equiv qE/m$. In terms of the new variable

$$\tilde{v}_y = v_y + a/\omega_c \tag{37}$$

the system of equations becomes uniform,

$$\begin{aligned} \dot{v}_x &= \omega_c \tilde{v}_y \\ \dot{\tilde{v}}_y &= -\omega_c v_x. \end{aligned} \tag{38}$$

Its solution can be found in the main text,

$$\begin{aligned} v_x(t) &= v_{x0} \cos \omega_c t + \tilde{v}_{y0} \sin \omega_c t \\ \tilde{v}_y(t) &= \tilde{v}_{y0} \cos \omega_c t - v_{x0} \sin \omega_c t. \end{aligned} \tag{39}$$

Returning to the original variables, one obtains

$$\begin{aligned} v_x(t) &= v_{x0} \cos \omega_c t + (v_{y0} + a/\omega_c) \sin \omega_c t \\ v_y(t) &= a/\omega_c + (v_{y0} + a/\omega_c) \cos \omega_c t - v_{x0} \sin \omega_c t. \end{aligned} \tag{40}$$

One can see that electric field creates a drift (average velocity) in the y directions, that is, in the direction of $\mathbf{E} \times \mathbf{B}$. The drift velocity in the natural form reads

$$\bar{\mathbf{v}} = \frac{qE}{m\omega_c} \frac{\mathbf{B} \times \mathbf{E}}{BE} = \frac{\mathbf{B} \times \mathbf{E}}{B^2}. \tag{41}$$

Integrating the velocity, one obtains the trajectory having the form of a cycloid.

2 Motion in one dimension

2.1 Period of the perturbed harmonic oscillator

Find the dependence of the frequency of an anharmonic oscillator with

$$U(x) = \frac{kx^2}{2} + \alpha x^4 \tag{42}$$

on the energy E considering the quartic term as perturbation. What is the applicability condition of the perturbative method?

Solution. Use the formula for the period

$$T(E) = 2\sqrt{2m} \int_0^{x_m} \frac{dx}{\sqrt{E - U(x)}}, \tag{43}$$

where the turning point x_m satisfies the equation

$$\frac{kx^2}{2} + \alpha x^4 = E. \tag{44}$$

It is convenient to scale x with x_m to do the perturbation theory in the integrand only. Choosing the new variable $y \equiv x/x_m$, one rewrites the period as

$$T = 4\sqrt{\frac{m}{k}} \int_0^1 \frac{dy}{\sqrt{1 + \frac{2\alpha}{k} x_m^2 - y^2 - \frac{2\alpha}{k} x_m^2 y^4}}. \tag{45}$$

With $\sqrt{k/m} = \omega_0$ and

$$\frac{2\alpha}{k} x_m^2 \equiv \beta \tag{46}$$

the period can be rewritten as

$$T = \frac{4}{\omega_0} \int\limits_0^1 \frac{dy}{\sqrt{1 - y^2 + \beta(1 - y^4)}}. \tag{47}$$

Expanding the integrand up to the first order in $\beta \ll 1$, one obtains

$$T = \frac{4}{\omega_0} \left(\int\limits_0^1 \frac{dy}{\sqrt{1-y^2}} - \frac{\beta}{2} \int\limits_0^1 dy \frac{1+y^2}{\sqrt{1-y^2}} \right) = \frac{4}{\omega_0} \left(\frac{\pi}{2} - \frac{\beta}{2} \frac{3\pi}{4} \right) \tag{48}$$

and then

$$T(E) = T_0 \left(1 - \frac{3}{4}\beta \right) = T_0 \left(1 - \frac{3\alpha}{2k} x_m^2 \right). \tag{49}$$

As the correction term already contains α, it is sufficient to use the lowest-order result x_m^2 obtained by neglecting α in Eq. (44), $x_m^2 = 2E/k$. The final result for the period reads

$$T(E) = T_0 \left(1 - \frac{3}{4}\beta \right) = T_0 \left(1 - \frac{3\alpha}{k^2} E \right). \tag{50}$$

It is recommended to write down the result for the frequency

$$\omega_0(E) = \omega_0 \left(1 + \frac{3\alpha}{k^2} E \right), \tag{51}$$

because its qualitative dependence is easier to memorize: Supersquare $U(x)$ — $\alpha > 0$ — frequency increases with energy.

2.2 Runaway in the tilted washboard potential with damping

Consider a particle of mass m and viscous damping α moving in the washboard potential $U(x) = U_0 [1 - \cos(ax)]$. (a) In the limit of small α, considering damping as a perturbation, find the total energy loss for the particle sliding from one top of the potential to the neighboring one (e.g., from $ax = -\pi$ to $ax = \pi$). (b) Find the critical value of the applied force F at which unbound motion sets in.

Solution. (a) The energy change per unit of time due to the viscous damping is given by $\dot{E} = -\alpha v^2$. In the limit of small α one can integrate this over the time of the motion considering the motion as unaffected by damping. In this lowest-order approximation the total energy loss reads

$$\Delta E = \alpha \int\limits_0^{t_{\max}} v^2 dt = \alpha \int\limits_{-x_m}^{x_m} v dx. \tag{52}$$

The unperturbed total energy is given by

$$E = U_0 [1 - \cos(ax_m)] = \frac{mv^2}{2} + U_0 [1 - \cos(ax)]. \tag{53}$$

From here one obtains

$$v = \sqrt{\frac{2U_0}{m} [\cos(ax) - \cos(ax_m)]}. \tag{54}$$

Substituting this into the formula above yields

$$\Delta E = 2\alpha\sqrt{\frac{2U_0}{m}}\int\limits_0^{x_m}\sqrt{\cos(ax) - \cos(ax_m)}dx. \tag{55}$$

Working out this integral is similar to the calculation of the period of motion in the washboard potential. First, one writes

$$\Delta E = 4\alpha\sqrt{\frac{U_0}{m}}\int\limits_0^{x_m}\sqrt{\sin^2(ax_m/2) - \sin^2(ax/2)}dx. \tag{56}$$

Then, substitution

$$\begin{aligned} \sin(ax/2) &= \sin(ax_m/2)\sin\xi \\ (a/2)\cos(ax/2)\,dx &= \sin(ax_m/2)\cos\xi d\xi \end{aligned} \tag{57}$$

yields

$$\Delta E = 4\alpha\sqrt{\frac{U_0}{m}}\int\limits_0^{\pi/2}\sin(ax_m/2)\sqrt{1-\sin^2\xi}\frac{\sin(ax_m/2)\cos\xi}{(a/2)\cos(ax/2)}d\xi \tag{58}$$

and then

$$\Delta E = \frac{8\alpha}{a}\sqrt{\frac{U_0}{m}}\sin^2\left(\frac{ax_m}{2}\right)\int\limits_0^{\pi/2}\frac{\cos^2\xi}{\sqrt{1-\sin^2(ax_m/2)\sin^2\xi}}d\xi. \tag{59}$$

Fortunately, we have $\sin(ax_m/2) = 1$ and the integral simplifies to

$$\Delta E = \frac{8\alpha}{a}\sqrt{\frac{U_0}{m}}\int\limits_0^{\pi/2}\cos\xi d\xi = \frac{8\alpha}{a}\sqrt{\frac{U_0}{m}}. \tag{60}$$

(b) In the absence of the applied force, the particle sliding from the top of the barrier will not overcome another barrier and turn back, gradually losing energy and ending up at the energy minimum. This is the bound regime. The unbound or runaway regime sets in if the energy gained by sliding in the direction of the applied force exceeds the energy lost due to the dissipation, that is,

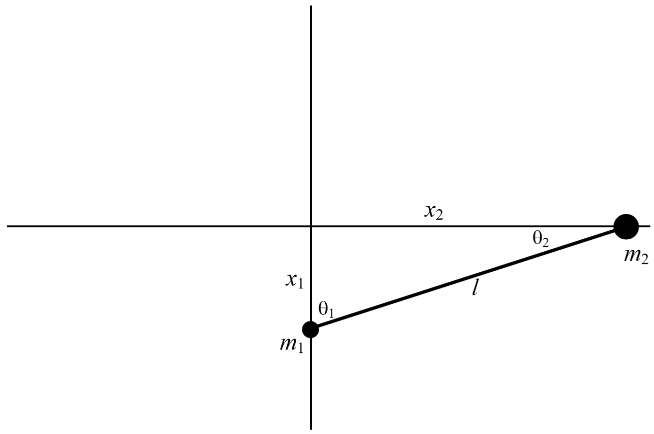
$$F\frac{2\pi}{a} > \Delta E. \tag{61}$$

With ΔE above, this yields the criterion for the runaway regime

$$F > \frac{4\alpha}{\pi}\sqrt{\frac{U_0}{m}}. \tag{62}$$

2.3 Two connected masses on perpendicular rods

Two masses, m_1 amd m_2 can slide without friction along two perpendicular rods and are connected by a light rod of length l. There is no gravity. Find the period of the motion of the system, assuming the initial state $\theta_2 = 0$ and $v_1 = v_0 < 0$.



Solution. Let us consider θ_2 as the dynamic variable, then

$$x_1 = l\sin\theta_2, \qquad x_2 = l\cos\theta_2. \tag{63}$$

For $v_1(0) < 0$ and $\theta_2(0) = 0$ the angle θ_2 will increase with time. The kinetic energy has the form

$$E_k = \frac{m_1\dot{x}_1^2}{2} + \frac{m_2\dot{x}_2^2}{2} = \frac{1}{2}\left(m_1\cos^2\theta_2 + m_2\sin^2\theta_2\right)\dot{\theta}_2^2. \tag{64}$$

Since E_k is conserved, $E_k = E =$ const, one can find the first integral of the equations of motion as

$$\dot{\theta}_2 = \sqrt{\frac{2E}{m_1\cos^2\theta_2 + m_2\sin^2\theta_2}} = \sqrt{\frac{2E}{m_1}}\frac{1}{\sqrt{1+\frac{m_2-m_1}{m_1}\sin^2\theta_2}}. \tag{65}$$

Integrating this equation, one obtains

$$\int\limits_0^{\theta_2} d\theta\sqrt{1+\frac{m_2-m_1}{m_1}\sin^2\theta} = \sqrt{\frac{2E}{m_1}}t. \tag{66}$$

Changing of θ_2 from 0 to $\pi/4$ corresponds to one quarter of the period, thus the period is equal to

$$T = 4\sqrt{\frac{m_1}{2E}}\int\limits_0^{\pi/2} d\theta\sqrt{1-\frac{m_1-m_2}{m_1}\sin^2\theta} = 4\sqrt{\frac{m_1}{2E}}E\left(\frac{m_1-m_2}{m_1}\right), \tag{67}$$

where $E(m)$ is elliptic integral of the second kind. The result above is valid for $m_1 > m_2$. In the opposite case on has to interchange m_1 and m_2. For $m_1 = m_2 = m$ the result simplifies to

$$T = \pi\sqrt{\frac{2m}{E}} = \frac{2\pi}{v_0}. \tag{68}$$

3 Systems with constraints and special coordinates

3.1 Mass on a conical surface

Write down dynamical variables (energy, angular momentum) and equation of motion for a mass moving on the surface of a cone with angle θ. Add gravity force in the direction of cone's apex.

Solution. Placing the origin of the coordinate system at the cone's apex, one can use spherical coordinates with $\theta =$ const and r changing together with φ. Here, to take into account the constraint, one has to project all vectors on the direction of motion $\mathbf{e}_r$, $\mathbf{e}_\varphi$, ignoring acceleration and forces in the direction $\mathbf{e}_\theta$. Writing $\mathbf{r} = r\mathbf{e}_r$ and using formulas in the main text, one obtains the velocity

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\frac{\partial \mathbf{e}_r}{\partial \varphi}\dot{\varphi} = \mathbf{e}_r\dot{r} + \mathbf{e}_\varphi r\sin\theta\dot{\varphi}. \tag{69}$$

Now kinetic energy has the form

$$E_k = \frac{m\mathbf{v}^2}{2} = \frac{m}{2}\left(\dot{r}^2 + r^2\sin^2\theta\dot{\varphi}^2\right). \tag{70}$$

Angular momentum has the form

$$\mathbf{l} = \mathbf{r}\times m\mathbf{v} = mr\mathbf{e}_r \times (\mathbf{e}_r\dot{r} + \mathbf{e}_\varphi r\sin\theta\dot{\varphi}) = -mr^2\sin\theta\dot{\varphi}\mathbf{e}_\theta. \tag{71}$$

Its projection onto the symmetry axis z reads

$$l_z = \mathbf{l}\cdot\mathbf{e}_z = mr^2\sin^2\theta\dot{\varphi}. \tag{72}$$

Acceleration has the form

$$\begin{aligned}\dot{\mathbf{v}} &= \mathbf{e}_r\ddot{r} + \frac{\partial \mathbf{e}_r}{\partial \varphi}\dot{\varphi}\dot{r} + \mathbf{e}_\varphi\dot{r}\sin\theta\dot{\varphi} + \mathbf{e}_\varphi r\sin\theta\ddot{\varphi} + \frac{\partial \mathbf{e}_\varphi}{\partial \varphi}\dot{\varphi}r\sin\theta\dot{\varphi}\\ &= \mathbf{e}_r\ddot{r} + \mathbf{e}_\varphi r\sin\theta\ddot{\varphi} + \mathbf{e}_\varphi 2\sin\theta\dot{\varphi}\dot{r} - \mathbf{e}_r r\sin^2\theta\dot{\varphi}^2\end{aligned} \tag{73}$$

or, finally,

$$\dot{\mathbf{v}} = \mathbf{e}_r\left(\ddot{r} - r\sin^2\theta\dot{\varphi}^2\right) + \mathbf{e}_\varphi\left(r\sin\theta\ddot{\varphi} + 2\sin\theta\dot{\varphi}\dot{r}\right). \tag{74}$$

Thus, the equation of motion of body confined to the cone is

$$\ddot{r} - r\sin^2\theta\dot{\varphi}^2 = \frac{1}{ma}F_r \tag{75}$$

$$r\sin\theta\ddot{\varphi} + 2\sin\theta\dot{\varphi}\dot{r} = \frac{1}{ma}F_\varphi. \tag{76}$$

The lhs of the second equation is proportional to $\dot{l}_z$. In the case $F_\varphi = 0$ one has $l_z = \text{const}$ and the first equation becomes autonomous,

$$\ddot{r} = \frac{l_z^2}{m^2r^3\sin^2\theta} + \frac{1}{ma}F_r. \tag{77}$$

In the case of gravity force directed toward the apex of the cone,

$$\mathbf{F} = -mg\mathbf{e}_z, \tag{78}$$

potential energy has the form

$$U = mgz = mgr\cos\theta. \tag{79}$$

Thus, ignoring θ-component, one obtains

$$\mathbf{F} = -\frac{\partial U}{\partial \mathbf{r}} = -\mathbf{e}_r\frac{\partial U}{\partial r} - \mathbf{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi} = -\mathbf{e}_r mg\cos\theta, \tag{80}$$

that is,

$$F_r = -mg\cos\theta, \qquad F_\varphi = 0. \tag{81}$$

3.2 Bead on an elliptic trajectory

A bead considered as a point mass m is moving along an elliptic ring in the absence of external forces and friction. Using the equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{82}$$

find the period and frequency of the motion as function of the energy. Consider limiting cases.

Solution. The period can be defined as

$$T = 4 \int_0^a \frac{dt}{dx} dx = 4 \int_0^a \frac{dx}{\dot{x}}, \tag{83}$$

while $\dot{x}$ can be found from the energy

$$E = \frac{m\mathbf{v}^2}{2} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) \tag{84}$$

and the ellipse equation. Differentiating the latter, one obtains

$$\frac{x\dot{x}}{a^2} + \frac{y\dot{y}}{b^2} = 0. \tag{85}$$

Eliminating $\dot{y}$, one obtains

$$E = \frac{m}{2}\dot{x}^2 \left[1 + \left(\frac{b}{a}\right)^4 \frac{x^2}{y^2}\right]. \tag{86}$$

Here y can be eliminated using the ellipse equation that yields

$$E = \frac{m}{2}\dot{x}^2 \left[1 + \left(\frac{b}{a}\right)^4 \frac{x^2}{b^2\left(1 - x^2/a^2\right)}\right] = \frac{m}{2}\dot{x}^2 \left[1 + \left(\frac{b}{a}\right)^2 \frac{x^2/a^2}{\left(1 - x^2/a^2\right)}\right] = \frac{m}{2}\dot{x}^2 \frac{1 - u^2\epsilon^2}{1 - u^2}, \tag{87}$$

where

$$u \equiv x/a, \qquad \epsilon \equiv \sqrt{1 - b^2/a^2}. \tag{88}$$

Resolving for $\dot{x}$, one obtains

$$T = 4a \int_0^1 \frac{du}{\dot{x}} = \frac{4a}{v} \int_0^1 du \sqrt{\frac{1 - u^2\epsilon^2}{1 - u^2}}, \tag{89}$$

where

$$v = \sqrt{\frac{2E}{m}} \tag{90}$$

is the constant speed of the bead. Note that this solution works for $b < a$. With the substitution

$$u = \sin\xi, \qquad \frac{du}{\sqrt{1 - u^2}} = d\xi \tag{91}$$

the integral reduces to the complete second-kind elliptic integral

$$E(m) \equiv \int_0^{\pi/2} d\xi \sqrt{1 - m\sin^2\xi} \tag{92}$$

having particular cases

$$E(0) = \frac{\pi}{2}, \qquad E(1) = 1. \tag{93}$$

Thus

$$T = \frac{4a}{v} E\left(1 - \frac{b^2}{a^2}\right), \qquad b \leq a. \tag{94}$$

In the other case the result has the form

$$T = \frac{4b}{v} E\left(1 - \frac{a^2}{b^2}\right), \qquad a \leq b. \tag{95}$$

The limiting cases are the circle, $a = b$, where $T = 2\pi a/v$, and a piece of a straight line, $b = 0$, where $T = 4a/v$.

4 Motion in a central field

4.1 Non-closed orbits

Orbits are closed only in the cases $U(r) \propto r^2$ (spatial harmonic oscillator) and $U(r) \propto 1/r$ (gravity, Coulomb interaction). Consider the case of gravity with an additional small potential energy $\delta U(r) = -\beta/r^n$. Treating δU as a perturbation, calculate the additional angle $\delta\varphi$ by which the body is rotating during one unperturbed cycle.

Solution: The rotation angle corresponding to the period of motion is given by

$$\Delta\varphi = 2\pi + \delta\varphi = 2\int_{r_1}^{r_2} dr \frac{l/r^2}{\sqrt{2m\left[E - \frac{l^2}{2mr^2} + \frac{\alpha}{r} - \delta U(r)\right]}}. \tag{96}$$

Here $\delta U = 0$ corresponds to $\delta\varphi = 0$. To find $\delta\varphi$, one can try to expand the integral in powers of δU. However, differentiation of the integrand with respect to δU yields a diverging integral. In fact, this divergence is fictitious, since one also has to expand turning points r_1 and r_2 in powers of δU, and if everything is properly taken into account, divergencies should cancel. This procedure is, of course, pretty cumbersome. Fortunately, one can use a trick to avoid these complications and make the calculation elegant, writing the formula above in the form

$$\Delta\varphi = 2\pi + \delta\varphi = -\frac{\partial}{\partial l} 2\int_{r_1}^{r_2} dr \sqrt{2m\left[E - \frac{l^2}{2mr^2} + \frac{\alpha}{r} - \delta U(r)\right]}. \tag{97}$$

Now expansion up to the linear term in δU does not lead to divergence and there is no contribution from differentiation of the turning points. Moreover, the resulting expression

$$\delta\varphi = \frac{\partial}{\partial l}\int_{r_1}^{r_2} dr \frac{2m\delta U(r)}{\sqrt{2m\left[E - \frac{l^2}{2mr^2} + \frac{\alpha}{r}\right]}} \tag{98}$$

is similar to the initial unperturbed formula for $\Delta\varphi$. Here one can change from integration over r to integration over φ using the unperturbed trajectory $r(\varphi)$. With the help of

$$\frac{d\varphi}{dr} = \frac{l}{r^2} \frac{1}{\sqrt{2m\left[E - \frac{l^2}{2mr^2} + \frac{\alpha}{r}\right]}} \tag{99}$$

one obtains

$$\delta\varphi = \frac{\partial}{\partial l}\left[\frac{2m}{l}\int_0^{\pi} d\varphi r^2(\varphi)\delta U(r(\varphi))\right]. \tag{100}$$

For our particular form of δU this expression becomes

$$\delta\varphi = \frac{\partial}{\partial l}\left[\frac{2m\beta}{l}\int\limits_0^\pi \frac{d\varphi}{r^{n-2}(\varphi)}\right] = \frac{\partial}{\partial l}\left[\frac{2m\beta}{l r_0^{n-2}}\int\limits_0^\pi d\varphi\,(1+\epsilon\cos\varphi)^{n-2}\right], \tag{101}$$

where $r_0 = l^2/(\alpha m)$ and

$$\epsilon \equiv \sqrt{1+\frac{E}{|E_{\min}|}} \tag{102}$$

is independent of l. Thus the result has the form

$$\delta\varphi = \frac{\partial}{\partial l}\left[\frac{2m\beta\,(\alpha m)^{n-2}}{l^{2n-3}}\int\limits_0^\pi d\varphi\,(1+\epsilon\cos\varphi)^{n-2}\right] \tag{103}$$

and, finally,

$$\delta\varphi = -\frac{(2n-3)\,2m\beta\,(\alpha m)^{n-2}}{l^{2n-2}}\int\limits_0^\pi d\varphi\,(1+\epsilon\cos\varphi)^{n-2}. \tag{104}$$

In particular, one obtains

$$\delta\varphi = -\frac{2\pi m\beta}{l^2}, \qquad n=2 \tag{105}$$

and

$$\delta\varphi = -\frac{6\pi m^2\alpha\beta}{l^4}, \qquad n=3. \tag{106}$$

4.2 Small-angle scattering

Find scattering angle χ for the central potential $U(r) = \beta/(a^2+r^2)$ in the limit of fast-moving particle. What is the maximal value of χ? Plot differential scattering cross-section.

Solution. The fast-moving particle will be scattered at small angles χ for any target distance ρ, because the potential energy is not diverging at $r \to 0$. One can use the small-scattering angle formula

$$\chi = -\frac{1}{mv_\infty^2}\int\limits_{-\infty}^{\infty} dx\,\left.\frac{\partial U(x,y)}{\partial y}\right|_{y=\rho} \tag{107}$$

that becomes

$$\chi = \frac{2\beta\rho}{mv_\infty^2}\int\limits_{-\infty}^{\infty}\frac{dx}{(a^2+\rho^2+x^2)^2} = \frac{\pi\beta}{mv_\infty^2}\frac{\rho}{(a^2+\rho^2)^{3/2}}. \tag{108}$$

One can see that

$$\chi \propto \begin{cases}\rho, & \rho \ll a\\ 1/\rho^2, & \rho \gg a.\end{cases} \tag{109}$$

The maximum of χ is defined by

$$\frac{\partial\chi}{\partial\rho} = \frac{\pi\beta}{mv_\infty^2}\frac{a^2-2\rho^2}{(a^2+\rho^2)^{5/2}} = 0 \tag{110}$$

that yields maximal scattering at $\rho = a/\sqrt{2}$,

$$\chi_{\max} = \frac{\pi\beta}{mv_\infty^2}\frac{2}{3^{3/2}a^2}. \tag{111}$$

Differential scattering cross-section is defined by

$$d\sigma = 2\pi\rho(\chi)\left|\frac{d\rho(\chi)}{d\chi}\right| d\chi. \tag{112}$$

Thus one has to plot the function $d\sigma/d\chi$ vs χ. This can be conveniently done in the parametric form using ρ as a parameter,

$$\begin{aligned}\frac{d\sigma}{d\chi} &= 2\pi\rho\left(\frac{\partial\chi}{\partial\rho}\right)^{-1} = \frac{2mv_\infty^2}{\beta}\frac{\rho\left(a^2+\rho^2\right)^{5/2}}{|a^2-2\rho^2|} \\ \chi &= \frac{\pi\beta}{mv_\infty^2}\frac{\rho}{\left(a^2+\rho^2\right)^{3/2}}.\end{aligned} \tag{113}$$

For plotting it is convenient to use $\tilde{\chi}$ defined by

$$\tilde{\chi} \equiv \left(\frac{\pi\beta}{mv_\infty^2}\right)^2 \chi = \frac{\rho}{\left(a^2+\rho^2\right)^{3/2}}. \tag{114}$$

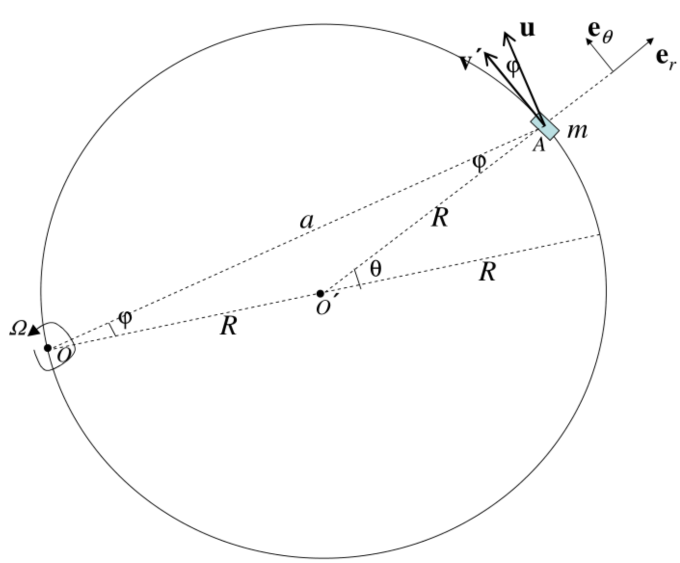
In terms of $\tilde{\chi}$ parametric representation of differential scattering cross-section becomes

$$\begin{aligned}\frac{d\sigma}{d\tilde{\chi}} &= \frac{2\pi\rho\left(a^2+\rho^2\right)^{5/2}}{|a^2-2\rho^2|} \\ \tilde{\chi} &= \frac{\rho}{\left(a^2+\rho^2\right)^{3/2}}.\end{aligned} \tag{115}$$

Part II
Lagrangian mechanics

5 Lagrange function and Lagrange equations

5.1 Bead on a rotating ring



A ring of radius R is rotating in its plane with the constant angular velocity Ω around a point O. A bead of mass m is sliding along the ring without friction. Describing the position of the bead on the ring with the angle θ: a) Construct Lagrange function and obtain equations of motion. b) Identify effective kinetic, potential, and total energies.

Solution. a) In this problem the potential energy is absent, thus the Lagrange function has the form

$$\mathcal{L} = \frac{m\mathbf{v}^2}{2}, \tag{116}$$

where $\mathbf{v}$ is the bead's velocity that consists of two contribution, sliding of the bead and rotating of the ring, respectively,

$$\mathbf{v} = \mathbf{u} + \mathbf{v}'. \tag{117}$$

Thus

$$\mathcal{L} = \frac{m}{2}\left(u^2 + 2\mathbf{u}\cdot\mathbf{v}' + v'^2\right). \tag{118}$$

Here

$$v' = R\dot{\theta} \tag{119}$$

and, from the triangles,

$$u = a\Omega = 2R\cos\varphi\Omega = 2R\cos\frac{\theta}{2}\Omega. \tag{120}$$

The angle between $\mathbf{v}'$ and $\mathbf{u}$ is $\varphi = \theta/2$, so that the Lagrange function becomes

$$\begin{aligned} \mathcal{L} &= \frac{m}{2}\left(u^2 + 2uv'\cos\frac{\theta}{2} + v'^2\right) \\ &= \frac{mR^2}{2}\left(4\cos^2\frac{\theta}{2}\Omega^2 + 4\cos^2\frac{\theta}{2}\Omega\dot{\theta} + \dot{\theta}^2\right) \\ &= mR^2\left[\Omega^2(1+\cos\theta) + \Omega\dot{\theta}(1+\cos\theta) + \frac{1}{2}\dot{\theta}^2\right] \\ &= mR^2\left[\Omega^2(1+\cos\theta) + \frac{1}{2}\dot{\theta}^2\right]. \end{aligned} \tag{121}$$

The cross-term term in the above expression has been dropped since it is a full time derivative:

$$\dot{\theta}(1+\cos\theta) = \frac{d}{dt}(\theta + \sin\theta). \tag{122}$$

that does not make a contribution into the Lagrange equation that can be checked directly. The Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \tag{123}$$

has the form

$$\ddot{\theta} + \Omega^2\sin\theta = 0, \tag{124}$$

the equation of motion for the pendulum.

b) Already from the final expression for the Lagrangian, it is clear that the problem is equivalent to that of a pendulum and the effective kinetic and potential energies are given by

$$E_{k,\mathrm{eff}} = \frac{1}{2}mR^2\dot{\theta}^2, \qquad U_{\mathrm{eff}} = -mR^2\Omega^2(1+\cos\theta). \tag{125}$$

The total effective energy

$$E_{\mathrm{eff}} = \frac{1}{2}mR^2\dot{\theta}^2 - mR^2\Omega^2(1+\cos\theta) \tag{126}$$

is conserved. Note that the true total energy is just $\mathcal{L}$ and it is not conserved.

5.2 Two masses on circles with a spring

Mass m_1 can slide without friction on a ring of radius R_1 in the plane $z = 0$ with its center at $(x, y) = (0, 0)$. Another mass m_2 is restricted to the circle of radius R_2 with its center at $(x, y) = (0, a)$. The two masses are connected by a spring so that their interaction energy is $U = \frac{1}{2}kd^2$, where d is the distance between the particles. Write down the Lagrangian and equations of motion. What integral of motion arises in the case $a = 0$? For $m_1 = m_2 = m$, $R_1 = R_2 = R$, and $a \leq 2R$ find the frequencies of small oscillations.

Solution. Use polar coordinates for each mass,

$$x_1 = R_1 \cos\varphi_1, \qquad y_1 = R_1 \sin\varphi_1 \tag{127}$$

and

$$x_2 = R_2 \cos\varphi_2, \qquad y_2 = a + R_2 \sin\varphi_2. \tag{128}$$

The distance d between the particles is given by

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (R_1 \cos\varphi_1 - R_2 \cos\varphi_2)^2 + (R_1 \sin\varphi_1 - a - R_2 \sin\varphi_2)^2 \tag{129}$$

and simplifies to

$$d^2 = R_1^2 + R_2^2 - 2R_1R_2 \cos(\varphi_1 - \varphi_2) + a^2 - 2a(R_1 \sin\varphi_1 - R_2 \sin\varphi_2). \tag{130}$$

Kinetic energy of the system is given by

$$E_k = \frac{1}{2} m_1 R_1^2 \dot\varphi_1^2 + \frac{1}{2} m_2 R_2^2 \dot\varphi_2^2, \tag{131}$$

so that the Lagrangian reads

$$\mathcal{L} = E_k - U = \frac{1}{2} m_1 R_1^2 \dot\varphi_1^2 + \frac{1}{2} m_2 R_2^2 \dot\varphi_2^2 + k[R_1R_2 \cos(\varphi_1 - \varphi_2) + a(R_1 \sin\varphi_1 - R_2 \sin\varphi_2)]. \tag{132}$$

In the case $a = 0$ one can change to the sum and difference angles as dynamic variables,

$$\varphi_+ = \varphi_1 + \varphi_2, \qquad \varphi_- = \varphi_1 - \varphi_2. \tag{133}$$

The Lagrangian depends only on φ_-, thus φ_+ is a cyclic variable and the corresponding generalized momentum is conserved. To find the latter, rewrite the kinetic energy using

$$\varphi_1 = \frac{1}{2}(\varphi_+ + \varphi_-), \qquad \varphi_2 = \frac{1}{2}(\varphi_+ - \varphi_-). \tag{134}$$

One obtains

$$\begin{aligned}\mathcal{L} &= \frac{1}{8} m_1 R_1^2 (\dot\varphi_+ + \dot\varphi_-)^2 + \frac{1}{8} m_2 R_2^2 (\dot\varphi_+ - \dot\varphi_-)^2 + kR_1R_2 \cos\varphi_- \\ &= \frac{1}{8}\left(m_1R_1^2 + m_2R_2^2\right)\left(\dot\varphi_+^2 + \dot\varphi_-^2\right) + \frac{1}{4}\left(m_1R_1^2 - m_2R_2^2\right)\dot\varphi_+\dot\varphi_- + kR_1R_2 \cos\varphi_-.\end{aligned} \tag{135}$$

The integral of motion is

$$\begin{aligned}p_+ &= \frac{\partial \mathcal{L}}{\partial \dot\varphi_+} = \frac{1}{4}\left(m_1R_1^2 + m_2R_2^2\right)\dot\varphi_+ + \frac{1}{4}\left(m_1R_1^2 - m_2R_2^2\right)\dot\varphi_- \\ &= \frac{1}{4}\left(m_1R_1^2 + m_2R_2^2\right)(\dot\varphi_1 + \dot\varphi_2) + \frac{1}{4}\left(m_1R_1^2 - m_2R_2^2\right)(\dot\varphi_1 - \dot\varphi_2) \\ &= \frac{1}{2} m_1 R_1^2 \dot\varphi_1 + \frac{1}{2} m_2 R_2^2 \dot\varphi_2.\end{aligned} \tag{136}$$

This is nothing else then half the *angular momentum* of the system. This is an expected result, because the external torque *with respect to the common center* acting on this system is zero. For $a \neq 0$ there is no common center and the angular momentum is not conserved.

The Lagrange equations have the form

$$
\begin{aligned}
0 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} - \frac{\partial \mathcal{L}}{\partial \varphi_1} = m_1 R_1^2 \ddot{\varphi}_1 + kR_1 R_2 \sin(\varphi_1 - \varphi_2) - aR_1 \cos\varphi_1 \\
0 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} - \frac{\partial \mathcal{L}}{\partial \varphi_2} = m_2 R_2^2 \ddot{\varphi}_2 + kR_1 R_2 \sin(\varphi_2 - \varphi_1) - aR_2 \cos\varphi_2 .
\end{aligned} \tag{137}
$$

Now consider small oscillations around the point of the minimal potential energy, that is, the distance between the masses. In the case $R_1 = R_2 = R$, and $a \leq 2R$ the minimum of the distance corresponds to the intersection of the two circles at

$$
\varphi_1 = \varphi_0 = \arcsin\frac{a}{2R}, \qquad \varphi_1 = -\varphi_0, \tag{138}
$$

as well as at $\varphi_1 = \pi - \varphi_0$ and $\varphi_1 = \pi + \varphi_0$. We will consider the former case using

$$
\varphi_1 = \varphi_0 + \delta\varphi_1, \qquad \varphi_2 = -\varphi_0 + \delta\varphi_2 \tag{139}
$$

and expand the Lagrangian

$$
\mathcal{L} = \frac{1}{2} m R^2 \left(\dot{\varphi}_1^2 + \dot{\varphi}_2^2\right) + kR\left[R\cos(\varphi_1 - \varphi_2) + a\left(\sin\varphi_1 - \sin\varphi_2\right)\right] \tag{140}
$$

that is,

$$
\mathcal{L} = \frac{1}{2} m R^2 \left(\delta\dot{\varphi}_1^2 + \delta\dot{\varphi}_2^2\right) + kR\left[R\cos(2\varphi_0 + \delta\varphi_1 - \delta\varphi_2) + a\left(\sin(\varphi_0 + \delta\varphi_1) - \sin(-\varphi_0 + \delta\varphi_2)\right)\right] \tag{141}
$$

in small $\delta\varphi_1$ and $\delta\varphi_2$ up to second order. First, the constant in the potential energy has to be discarded. Second, one has to check that at first order the result disappears, as it should be at the energy minimum. One has

$$
\begin{aligned}
& R\cos(2\varphi_0 + \delta\varphi_1 - \delta\varphi_2) + a\left(\sin(\varphi_0 + \delta\varphi_1) + \sin(\varphi_0 - \delta\varphi_2)\right) \\
\Rightarrow\ & -R\sin(2\varphi_0)(\delta\varphi_1 - \delta\varphi_2) + a\cos(\varphi_0)(\delta\varphi_1 - \delta\varphi_2) \\
=\ & \cos(\varphi_0)(\delta\varphi_1 - \delta\varphi_2)(-2R\sin\varphi_0 + a) = 0,
\end{aligned} \tag{142}
$$

OK. Now calculate quadratic terms:

$$
\begin{aligned}
& R\cos(2\varphi_0 + \delta\varphi_1 - \delta\varphi_2) + a\left(\sin(\varphi_0 + \delta\varphi_1) + \sin(\varphi_0 - \delta\varphi_2)\right) \\
\Rightarrow\ & -\frac{1}{2}R\cos(2\varphi_0)(\delta\varphi_1 - \delta\varphi_2)^2 - \frac{1}{2}a\sin(\varphi_0)\left(\delta\varphi_1^2 + \delta\varphi_2^2\right) \\
=\ & -\frac{1}{2}R\left(1 - 2\sin^2(\varphi_0)\right)(\delta\varphi_1 - \delta\varphi_2)^2 - \frac{1}{2}a\sin(\varphi_0)\left(\delta\varphi_1^2 + \delta\varphi_2^2\right) \\
=\ & -\frac{1}{2}R\left(1 - 2\left(\frac{a}{2R}\right)^2\right)(\delta\varphi_1 - \delta\varphi_2)^2 - \frac{1}{2}a\frac{a}{2R}\left(\delta\varphi_1^2 + \delta\varphi_2^2\right) \\
=\ & -\frac{1}{2}R(\delta\varphi_1 - \delta\varphi_2)^2 + \frac{a^2}{4R}\left[(\delta\varphi_1 - \delta\varphi_2)^2 - \delta\varphi_1^2 - \delta\varphi_2^2\right] \\
=\ & -\frac{1}{2}R(\delta\varphi_1 - \delta\varphi_2)^2 - \frac{a^2}{2R}\delta\varphi_1\delta\varphi_2 \\
=\ & -\frac{1}{2}R\left(\delta\varphi_1^2 + \delta\varphi_2^2\right) + \left(R - \frac{a^2}{2R}\right)\delta\varphi_1\delta\varphi_2 .
\end{aligned} \tag{143}
$$

Now, to second order in small deviations, one has

$$\mathcal{L} = \frac{1}{2} m R^2 \left(\delta\dot{\varphi}_1^2 + \delta\dot{\varphi}_2^2 \right) + kR \left[-\frac{1}{2} R \left(\delta\varphi_1^2 + \delta\varphi_2^2 \right) + \left(R - \frac{a^2}{2R} \right) \delta\varphi_1 \delta\varphi_2 \right]. \tag{144}$$

In matrix form this reads

$$\mathcal{L} = \frac{1}{2} \left(\delta\dot{\boldsymbol{\varphi}}^{\boldsymbol{T}} \cdot \mathbb{M} \cdot \delta\dot{\boldsymbol{\varphi}} - \delta\boldsymbol{\varphi}^{\boldsymbol{T}} \cdot \mathbb{K} \cdot \delta\boldsymbol{\varphi} \right), \tag{145}$$

where

$$\mathbb{M} = mR^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{K} = kR^2 \begin{pmatrix} 1 & -1 + \frac{a^2}{2R^2} \\ -1 + \frac{a^2}{2R^2} & 1 \end{pmatrix}. \tag{146}$$

The eigenvalue equation has the form

$$\left| -\omega^2 \mathbb{M} + \mathbb{K} \right| = 0, \tag{147}$$

that is,

$$\begin{vmatrix} -\omega^2 + \omega_0^2 & \left(-1 + \frac{a^2}{2R^2} \right) \omega_0^2 \\ \left(-1 + \frac{a^2}{2R^2} \right) \omega_0^2 & -\omega^2 + \omega_0^2 \end{vmatrix} = 0, \tag{148}$$

where $\omega_0^2 = k/m$. Further one obtains

$$\begin{aligned} 0 &= \left(\omega^2 - \omega_0^2 \right)^2 - \left(1 - \frac{a^2}{2R^2} \right)^2 \omega_0^4 = \left[\omega^2 - \omega_0^2 + \left(1 - \frac{a^2}{2R^2} \right) \omega_0^2 \right] \left[\omega^2 - \omega_0^2 - \left(1 - \frac{a^2}{2R^2} \right) \omega_0^2 \right] \\ &= \left(\omega^2 - \frac{a^2}{2R^2} \omega_0^2 \right) \left(\omega^2 - \left(2 - \frac{a^2}{2R^2} \right) \omega_0^2 \right). \end{aligned} \tag{149}$$

Thus the two oscillation modes are

$$\omega_1 = \frac{a}{\sqrt{2}R} \omega_0, \qquad \omega_2 = \sqrt{2} \sqrt{1 - \left(\frac{a}{2R} \right)^2} \omega_0 \tag{150}$$

In the limit $a \to 0$ the first mode softens that is related to the emergence of the integral of motion, the angular momentum. It is obvious that the first mode is the in-phase mode. The second mode softens in the limit $a \to 2R$ where the intersection of the two circles disappears. In the region $a > 2R$ one has to perform a separate calculation that is simpler than above since $\varphi_0 = \pi/2$. There are also two oscillation modes, one of which softens when a approaches $2R$ from above.

5.3 Mass on a plane connected to another mass hanging through the hole

A particle of mass m_1 sits on a smooth horizontal table and is connected by a light rope of length a to another particle of mass m_2 through a hole through which the rope is threaded. The second particle hangs straight beneath the hole. Write down the Lagrangian, identify cyclic variables and conserved quantities, set up equations of motion.

Solution. Use polar coordinate system for the first particle and denote its distance from the hole as r. The kinetic energy of the system is

$$E_k = \frac{m_1}{2} \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) + \frac{m_2}{2} \dot{r}^2 \tag{151}$$

while the potential energy is given by

$$U = m_2 g r + \text{const}. \tag{152}$$

The rope length a is irrelevant. The Lagrangian has the form

$$\mathcal{L} = E_k - U = \frac{m_1}{2} \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) + \frac{m_2}{2} \dot{r}^2 - m_2 g r. \tag{153}$$

The cyclic variable is φ. The corresponding generalized momentum is angular momentum,

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m_1 r^2 \dot{\varphi} = l = \text{const.} \tag{154}$$

The Lagrange equation for r has the form

$$0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = (m_1 + m_2)\,\ddot{r} - m_1 r \dot{\varphi}^2 + m_2 g. \tag{155}$$

Eliminating $\dot{\varphi}$, one obtains a closed differential equation for r

$$(m_1 + m_2)\,\ddot{r} - \frac{l^2}{m_1 r^3} + m_2 g = 0. \tag{156}$$

This equation can be integrated by multiplying by the integrating factor $\dot{r}$ and obtaining the energy integral of motion:

$$(m_1 + m_2)\,\ddot{r}\dot{r} - \frac{l^2}{m_1 r^3}\dot{r} + m_2 g \dot{r} = \frac{d}{dt}\left(\frac{m_1 + m_2}{2}\dot{r}^2 + \frac{l^2}{2 m_1 r^2} + m_2 g r\right) = 0 \tag{157}$$

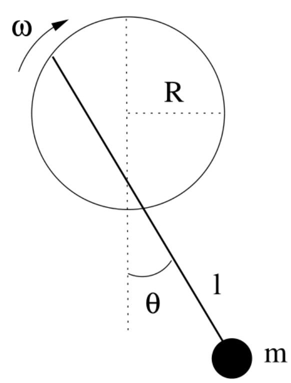
thus

$$\frac{m_1 + m_2}{2}\dot{r}^2 + \frac{l^2}{2 m_1 r^2} + m_2 g r = E = \text{const.} \tag{158}$$

For m_1 this is a motion in a central field with $U \propto r$. The orbits are non-closed. As usual, non-zero angular momentum prevents m_1 from falling onto the hole.

5.4 Pendulum on a wheel

The pivot of a simple pendulum is attached to a disc of radius R, which rotates in the plane of the pendulum with angular velocity ω. (See the diagram below). Write down the Lagrangian and derive the equations of motion for dynamical variable θ.



Solution. In terms of θ, the coordinates of the mass m read

$$\begin{aligned} x &= \quad l\sin\theta + R\sin(\omega t) \\ y &= -\quad l\cos\theta + R\cos(\omega t) \end{aligned} \tag{159}$$

(at $t = 0$ the pendulum support is at the top of the circle). Differentiating this one obtains

$$\begin{aligned} \dot{x} &= l\cos\theta\dot{\theta} + R\omega\cos(\omega t) \\ \dot{y} &= l\sin\theta\dot{\theta} - R\omega\sin(\omega t)\,. \end{aligned} \tag{160}$$

Now, $E_k = m\left(\dot{x}^2 + \dot{y}^2\right)/2$ and $U = mgy$, so that the Lagrange function becomes

$$\mathcal{L} = \frac{m}{2}\left[l^2\dot{\theta}^2 + 2lR\omega\left(\cos\theta\cos\left(\omega t\right) - \sin\theta\sin\left(\omega t\right)\right)\dot{\theta}\right] + mgl\cos\theta \tag{161}$$

or

$$\mathcal{L} = \frac{m}{2}\left[l^2\dot{\theta}^2 + 2lR\omega\cos\left(\theta + \omega t\right)\dot{\theta}\right] + mgl\cos\theta \tag{162}$$

up to irrelevant terms. The Lagrange equation has the form

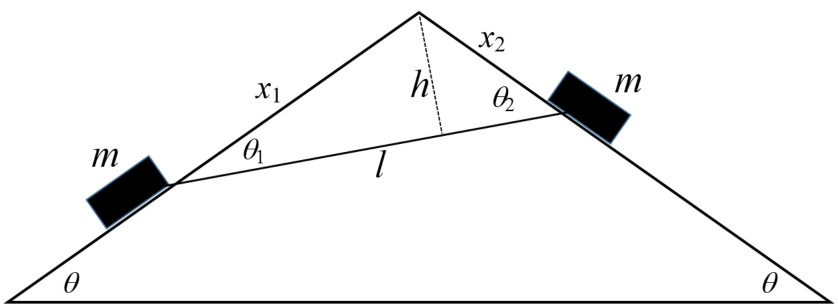
$$0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = ml^2\ddot{\theta} - mlR\omega^2\sin\left(\theta + \omega t\right) + mgl\sin\theta \tag{163}$$

and, finally,

$$\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{R}{l}\omega^2\sin\left(\theta + \omega t\right). \tag{164}$$

This is an equation of motion for a pendulum with an external force depending on pendulum's angle.

5.5 Two connected masses on a double incline



Two masses m on different sides of a double incline with the angle θ are connected by a light rod of length l. Find the Lagrangian, equations of motion and the frequency of small oscillations around the equilibrium configuration. *Tip*: As the dynamic variable choose the angle θ_1.

Solution. With respect to the top of the incline potential energy has the form

$$U = -mg\sin\theta\left(x_1 + x_2\right). \tag{165}$$

Let us express x_1 and x_2 via the angles θ_1 and θ_2. From the cosine theorem one obtains

$$l^2 = x_1^2 + x_2^2 - 2x_1x_2\cos\left(\pi - 2\theta\right) = x_1^2 + x_2^2 + 2x_1x_2\cos\left(2\theta\right). \tag{166}$$

Expressing the triangle height h in both ways yields another equation

$$x_1\sin\theta_1 = x_2\sin\theta_2. \tag{167}$$

From here one finds

$$x_1 = \frac{l\sin\theta_2}{\sqrt{\sin^2\theta_1 + \sin^2\theta_2 + 2\sin\theta_1\sin\theta_2\cos\left(2\theta\right)}} \tag{168}$$

and a similar result for x_2. Taking into account the sum rule

$$\theta_1 + \theta_2 = 2\theta, \tag{169}$$

one can prove that

$$\sin^2\theta_1 + \sin^2\theta_2 + 2\sin\theta_1\sin\theta_2\cos\left(2\theta\right) = \sin^2\left(2\theta\right). \tag{170}$$

After this the final result becomes

$$x_1 = l \frac{\sin \theta_2}{\sin (2\theta)}, \qquad x_2 = l \frac{\sin \theta_1}{\sin (2\theta)}. \tag{171}$$

Now the potential energy takes the form

$$U = -mgl \sin \theta \frac{\sin \theta_1 + \sin \theta_2}{\sin (2\theta)}. \tag{172}$$

Eliminating θ_2, after simplifications one finally obtains

$$U = -mgl \tan \theta \cos (\theta - \theta_1). \tag{173}$$

The angle θ_1 changes within the interval $(0, 2\theta)$, and at the boundaries of this interval one has $U = -mgl \sin \theta$, as it should be. The minimum of U is at $\theta_1 = \theta$, as could be expected on symmetry grounds.

Kinetic energy of the system can be obtained from Eq. (171):

$$E_k = \frac{m}{2} \left(\dot{x}_1^2 + \dot{x}_2^2 \right) = \frac{ml^2}{2} \frac{\left[\cos^2 (2\theta - \theta_1) + \cos^2 \theta_1 \right] \dot{\theta}_1^2}{\sin^2 (2\theta)} \tag{174}$$

After simplification one obtains

$$E_k = \frac{ml^2}{2} \frac{\left[1 + \cos (2\theta) \cos (2\theta - 2\theta_1) \right] \dot{\theta}_1^2}{\sin^2 (2\theta)}. \tag{175}$$

Thus the Lagrangian has the form

$$\mathcal{L} = E_k - U = \frac{ml^2}{2} \frac{\left[1 + \cos (2\theta) \cos (2\theta - 2\theta_1) \right] \dot{\theta}_1^2}{\sin^2 (2\theta)} + mgl \tan \theta \cos (\theta - \theta_1). \tag{176}$$

The Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0 \tag{177}$$

becomes

$$\frac{ml^2}{\sin^2 (2\theta)} \left\{ \left[1 + \cos (2\theta) \cos (2\theta - 2\theta_1) \right] \ddot{\theta}_1 + 2 \cos (2\theta) \sin (2\theta - 2\theta_1) \dot{\theta}_1^2 \right\} - mgl \tan \theta \sin (\theta - \theta_1) = 0 \tag{178}$$

or

$$\left[1 + \cos (2\theta) \cos (2\theta - 2\theta_1) \right] \ddot{\theta}_1 + 2 \cos (2\theta) \sin (2\theta - 2\theta_1) \dot{\theta}_1^2 - \frac{g}{l} \tan \theta \sin (\theta - \theta_1) \sin^2 (2\theta) = 0. \tag{179}$$

Near equilibrium one can expand the equation of motion in small $\delta\theta_1$ setting $\theta_1 = \theta + \delta\theta_1$. Keeping only linear terms, one obtains

$$\left[1 + \cos (2\theta) \right] \delta\ddot{\theta}_1 + \frac{g}{l} \tan \theta \sin^2 (2\theta) \, \delta\theta_1 = 0 \tag{180}$$

or

$$\delta\ddot{\theta}_1 + 2 \tan \theta \sin^2 \theta \frac{g}{l} \delta\theta_1 = 0. \tag{181}$$

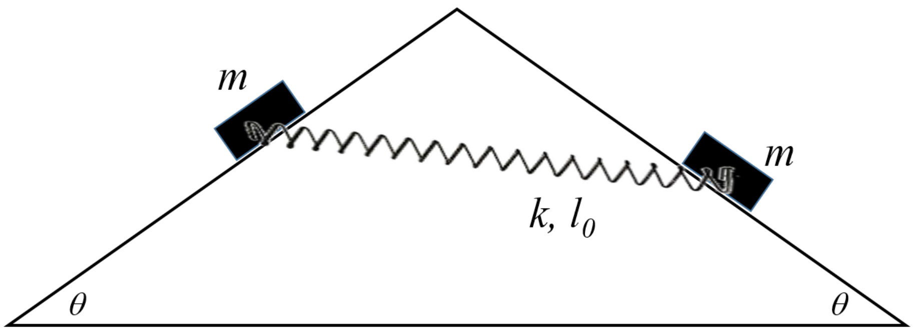
Thus the oscillation frequency is given by

$$\omega_0 = \sqrt{2 \tan \theta \sin^2 \theta \frac{g}{l}}. \tag{182}$$

For $\theta \ll 1$ the frequency becomes small, $\omega_0 \propto \theta^{3/2}$.

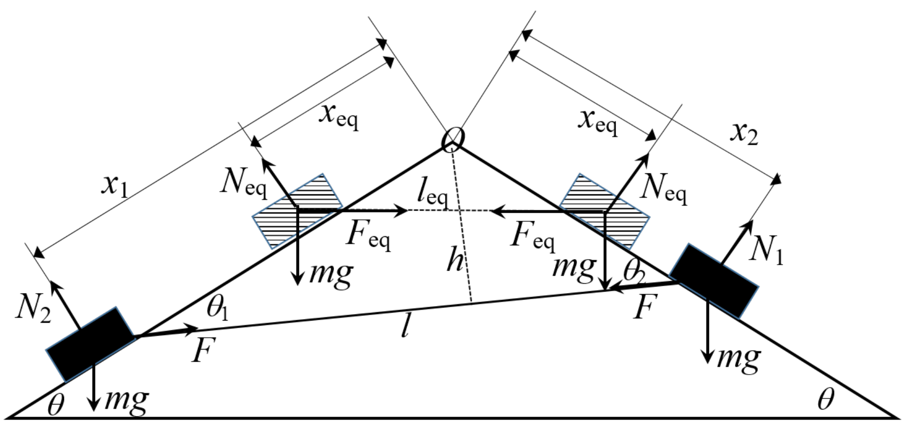
Part III
Oscillations

5.6 Two masses on a double incline connected by a spring



Two masses m on a double incline with the angle θ are connected by a spring of the equilibrium length l_0 and stiffness k. Find oscillation frequencies. Consider the limit $k \to \infty$.

Solution. Let us take the top of the double incline for the origin O and measure the position of the masses on the incline from this point, as shown in the figure.



Newtonian solution below contains some involved geometry. The equilibrium position of the masses should be symmetric. From the balance of horizontal and vertical forces one obtains

$$N_{\rm eq} \cos\theta = mg, \qquad N_{\rm eq} \sin\theta = F_{\rm eq} = k(l_{\rm eq} - l_0). \tag{183}$$

From this one finds

$$F_{\rm eq} = k(l_{\rm eq} - l_0) = mg \tan\theta. \tag{184}$$

Also one can project the forces onto the direction of motion along the slopes and the direction perpendicular to it. In this case the normal force N becomes irrelevant and one obtains

$$mg \sin\theta = F_{\rm eq} \cos\theta \tag{185}$$

that leads to the same result. Equilibruim positions of the masses are characterized by

$$x_{\rm eq} = l_{\rm eq}/(2\cos\theta). \tag{186}$$

Equations of motion for the masses have the form (see figure)

$$\begin{aligned} m\ddot{x}_1 &= mg \sin\theta - F \cos\theta_1 \\ m\ddot{x}_2 &= mg \sin\theta - F \cos\theta_2, \end{aligned} \tag{187}$$

where $F = k(l - l_0)$ and

$$l = \sqrt{x_1^2 + x_2^2 - 2x_1x_2\cos(\pi - 2\theta)} = \sqrt{x_1^2 + x_2^2 + 2x_1x_2\cos(2\theta)}. \tag{188}$$

The angles θ_1 and θ_2 satisfy the two equations,

$$\theta_1 + \theta_2 = 2\theta \tag{189}$$

and

$$x_1 \sin\theta_1 = x_2 \sin\theta_2 \tag{190}$$

that is the triangle height h expressed in two ways.

One can see that the problem is nonlinear, and even to obtain solvable equations of motion one has first to solve the algebraic problem of finding θ_1 and θ_2 in terms of x_1 and x_2. Since we are interested in small oscillations around the equilibrium configuration, we expand everything up to the linear order in small deviations δx_1 and δx_2 defined by

$$x_1 = x_{\rm eq} + \delta x_1, \qquad x_2 = x_{\rm eq} + \delta x_2. \tag{191}$$

Correspondingly,

$$\theta_1 = \theta + \delta\theta_1, \qquad \theta_2 = \theta + \delta\theta_2. \tag{192}$$

Now

$$\begin{aligned}
l &\cong \sqrt{x_{\rm eq}^2 + 2x_{\rm eq}\delta x_1 + x_{\rm eq}^2 + 2x_{\rm eq}\delta x_2 + 2x_{\rm eq}^2\cos(2\theta) + 2x_{\rm eq}(\delta x_1 + \delta x_2)\cos(2\theta)} \\
&= x_{\rm eq}\sqrt{2[1+\cos(2\theta)] + \frac{\delta x_1 + \delta x_2}{x_{\rm eq}} 2[1+\cos(2\theta)]} \\
&= 2x_{\rm eq}\cos\theta\sqrt{1 + \frac{\delta x_1 + \delta x_2}{x_{\rm eq}}} = l_{\rm eq}\sqrt{1 + \frac{\delta x_1 + \delta x_2}{x_{\rm eq}}} \\
&\cong l_{\rm eq} + \cos\theta\,(\delta x_1 + \delta x_2) \equiv l_{\rm eq} + \delta l,
\end{aligned}$$

where Eq. (186) was used. Linearized Eqs. (189) and (190) become

$$\begin{aligned}
\delta\theta_1 + \delta\theta_2 &= 0 \\
(\delta x_1 - \delta x_2)\sin\theta &= -\; x_{\rm eq}\cos\theta\,(\delta\theta_1 - \delta\theta_2)\,.
\end{aligned} \tag{193}$$

Solving these equations, one obtains

$$\delta\theta_1 = -\delta\theta_2 = -\frac{\delta x_1 - \delta x_2}{2x_{\rm eq}}\tan\theta = -\frac{\delta x_1 - \delta x_2}{l_{\rm eq}}\sin\theta. \tag{194}$$

After that Eqs. (187) one expands

$$\begin{aligned}
F\cos\theta_1 &= (F_{\rm eq} + k\delta l)\cos(\theta + \delta\theta_1) \\
&\cong F_{\rm eq}\cos\theta + k\delta l\cos\theta - F_{\rm eq}\sin\theta\delta\theta_1 \\
&= mg\sin\theta + k\cos^2\theta\,(\delta x_1 + \delta x_2) + mg\sin^2\theta\tan\theta\frac{\delta x_1 - \delta x_2}{l_{\rm eq}} \\
&= mg\sin\theta + \left(k\cos^2\theta + \frac{mg\sin^2\theta\tan\theta}{l_{\rm eq}}\right)\delta x_1 + \left(k\cos^2\theta - \frac{mg\sin^2\theta\tan\theta}{l_{\rm eq}}\right)\delta x_2,
\end{aligned} \tag{195}$$

where Eq. (184) was used. Similarly one obtains

$$F\cos\theta_2 \cong mg\sin\theta + \left(k\cos^2\theta - \frac{mg\sin^2\theta\tan\theta}{l_{\rm eq}}\right)\delta x_1 + \left(k\cos^2\theta + \frac{mg\sin^2\theta\tan\theta}{l_{\rm eq}}\right)\delta x_2. \tag{196}$$

Now the equations of motion become

$$\begin{aligned} m\delta\ddot{x}_1 + A_+\delta x_1 + A_-\delta x_2 &= 0 \\ m\delta\ddot{x}_2 + A_-\delta x_1 + A_+\delta x_2 &= 0, \end{aligned} \tag{197}$$

where

$$A_\pm \equiv k\cos^2\theta \pm \frac{mg\sin^2\theta\tan\theta}{l_{\mathrm{eq}}}. \tag{198}$$

The eingenfrequencies follow from the secular equation

$$\left(-m\omega^2 + A_+\right)^2 - A_-^2 = 0. \tag{199}$$

The two solutions are

$$\omega_\pm^2 = \frac{1}{m}\left(A_+ \pm A_-\right). \tag{200}$$

Explicitly this becomes

$$\omega_+^2 = 2\cos^2\theta\frac{k}{m}, \qquad \omega_-^2 = 2\sin^2\theta\tan\theta\frac{g}{l_{\mathrm{eq}}}. \tag{201}$$

Thus, there are two decoupled modes. One mode is due to the stiffness of the spring and another is due to the gravity. In the stiffness mode the masses are moving anti-phase while in the gravity mode they are moving in-phase with $l = l_{\mathrm{eq}} = \mathrm{const.}$ In the limit $k \to \infty$ the stiffness mode disappeares and only the gravity mode remains. In this case the system has only one degree of freedom.

Solution using Lagrangial mechanics below is more straightforward. Kinetic energy of the system is given by

$$E_k = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right). \tag{202}$$

The potential energy reads

$$U = -mg\sin\theta\left(x_1 + x_2\right) + \frac{1}{2}k\left(l - l_0\right)^2, \tag{203}$$

where

$$l = \sqrt{x_1^2 + x_2^2 - 2x_1x_2\cos\left(\pi - 2\theta\right)} = \sqrt{x_1^2 + x_2^2 + 2x_1x_2\cos\left(2\theta\right)}. \tag{204}$$

Thus the Langrange function becomes

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_2^2\right) + mg\sin\theta\left(x_1 + x_2\right) - \frac{1}{2}k\left(l - l_0\right)^2. \tag{205}$$

The Lagrange equations have the form

$$0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_i} - \frac{\partial\mathcal{L}}{\partial x_i} = m\ddot{x}_i - mg\sin\theta + k\left(1 - \frac{l_0}{l}\right)\left[x_i + x_{3-i}\cos\left(2\theta\right)\right]. \tag{206}$$

Next one has to expand the elastic term in small deviations from the equilibrium corresponding to the minimum of the potential energy. The latter can be searched for under the constraint $x_1 = x_2 = x$, so that

$$U = -2mg\sin\theta x + \frac{1}{2}k\left(x\sqrt{2\left[1 + \cos\left(2\theta\right)\right]} - l_0\right)^2 \tag{207}$$

or

$$U = -2mg\sin\theta x + \frac{1}{2}k\left(2x\cos\theta - l_0\right)^2 \tag{208}$$

and the equilibrium value $x = x_{\mathrm{eq}}$ is defined by

$$0 = \frac{\partial U}{\partial x} = -2mg\sin\theta + k\left(2x\cos\theta - l_0\right)2\cos\theta, \tag{209}$$

so that

$$x_{\rm eq} = \frac{1}{2\cos\theta}\left(\frac{mg}{k}\tan\theta + l_0\right). \tag{210}$$

Now one has to expand the elastic term up to linear terms in small deviations from the equilibrium:

$$x_1 = x_{\rm eq} + \delta x_1, \qquad x_2 = x_{\rm eq} + \delta x_2. \tag{211}$$

One has

$$\begin{aligned} l &\cong \sqrt{x_{\rm eq}^2 + 2x_{\rm eq}\delta x_1 + x_{\rm eq}^2 + 2x_{\rm eq}\delta x_2 + 2x_{\rm eq}^2\cos(2\theta) + 2x_{\rm eq}(\delta x_1 + \delta x_2)\cos(2\theta)} \\ &= x_{\rm eq}\sqrt{2\left[1+\cos(2\theta)\right] + \frac{\delta x_1 + \delta x_2}{x_{\rm eq}} 2\left[1+\cos(2\theta)\right]} \\ &= 2x_{\rm eq}\cos\theta\sqrt{1+\frac{\delta x_1+\delta x_2}{x_{\rm eq}}} = l_{\rm eq}\sqrt{1+\frac{\delta x_1+\delta x_2}{x_{\rm eq}}} \\ &\cong l_{\rm eq} + \cos\theta\left(\delta x_1 + \delta x_2\right), \end{aligned} \tag{212}$$

where $l_{\rm eq} = 2x_{\rm eq}\cos\theta$. Now

$$1 - \frac{l_0}{l} \cong 1 - \frac{l_0}{l_{\rm eq} + \cos\theta(\delta x_1 + \delta x_2)} \cong 1 - \frac{l_0}{l_{\rm eq}}\left(1 - \frac{\cos\theta(\delta x_1 + \delta x_2)}{l_{\rm eq}}\right) \tag{213}$$

and the Lagrange equations become

$$0 = m\delta\ddot{x}_i - mg\sin\theta + k\left[1 - \frac{l_0}{l_{\rm eq}}\left(1 - \frac{\cos\theta(\delta x_1 + \delta x_2)}{l_{\rm eq}}\right)\right]\left[x_{\rm eq}(1+\cos(2\theta)) + \delta x_i + \delta x_{3-i}\cos(2\theta)\right]. \tag{214}$$

The zero-order part of these equations,

$$-mg\sin\theta + k\left(1 - \frac{l_0}{l_{\rm eq}}\right)x_{\rm eq}(1+\cos(2\theta)) = 0, \tag{215}$$

because of Eq. (210), so that at linear order one has

$$0 = m\delta\ddot{x}_i + k\left(1 - \frac{l_0}{l_{\rm eq}}\right)(\delta x_i + \delta x_{3-i}\cos(2\theta)) + k\frac{l_0}{l_{\rm eq}}\frac{\cos\theta(\delta x_1 + \delta x_2)}{l_{\rm eq}}x_{\rm eq}(1+\cos(2\theta)), \tag{216}$$

or

$$0 = m\delta\ddot{x}_i + k\left(1 - \frac{l_0}{l_{\rm eq}}\right)(\delta x_i + \delta x_{3-i}\cos(2\theta)) + k\frac{l_0}{l_{\rm eq}}\frac{\delta x_1 + \delta x_2}{2}(1+\cos(2\theta)). \tag{217}$$

This becomes

$$\begin{aligned} m\delta\ddot{x}_1 + k\left(1 - \frac{l_0}{l_{\rm eq}}\frac{1-\cos(2\theta)}{2}\right)\delta x_1 + k\left(\cos(2\theta) + \frac{l_0}{l_{\rm eq}}\frac{1-\cos(2\theta)}{2}\right)\delta x_2 &= 0 \\ m\delta\ddot{x}_2 + k\left(\cos(2\theta) + \frac{l_0}{l_{\rm eq}}\frac{1-\cos(2\theta)}{2}\right)\delta x_1 + k\left(1 - \frac{l_0}{l_{\rm eq}}\frac{1-\cos(2\theta)}{2}\right)\delta x_2 &= 0. \end{aligned} \tag{218}$$

Eigenfrequencies of this system of equations satisfy the equation

$$\left[-m\omega^2 + k\left(1 - \frac{l_0}{l_{\rm eq}}\frac{1-\cos(2\theta)}{2}\right)\right]^2 - k^2\left(\cos(2\theta) + \frac{l_0}{l_{\rm eq}}\frac{1-\cos(2\theta)}{2}\right)^2 = 0 \tag{219}$$

or

$$\left[-m\omega^2 + k(1+\cos(2\theta))\right]\left[-m\omega^2 + k\left(1 - \frac{l_0}{l_{\rm eq}}\right)(1-\cos(2\theta))\right] = 0 \tag{220}$$

or

$$\left[-m\omega^2 + 2k\cos^2\theta\right]\left[-m\omega^2 + 2k\left(1 - \frac{l_0}{l_{\rm eq}}\right)\sin^2\theta\right] = 0. \tag{221}$$

Thus the two eigenfrequencies are given by

$$\omega_1^2 = 2\cos^2\theta\frac{k}{m}, \qquad \omega_2^2 = 2\sin^2\theta\frac{k}{m}\left(1 - \frac{l_0}{l_{\rm eq}}\right). \tag{222}$$

One can see that the first of the oscillation modes is purely elastic. The second oscillation frequency with the help of Eq. (215) can be rewritten as

$$\omega_2^2 = 2\sin^2\theta\tan\theta\frac{g}{l_{\rm eq}}, \tag{223}$$

that is, the second oscillation mode is a pure gravity mode. This mode survives in the limit $k \to \infty$.

Part IV
Rotational motion of rigid bodies

6 Kinetic energy, moments of inertia

6.1 Moments of inertia

Calculate tensors of inertia with respect to the principal axes of the following bodies:

1. Hollow sphere of mass M and radius R.
2. Cone of the height h and radius of the base R, both with respect to the apex and to the center of mass.
3. Body of a box shape with sides a, b, and c. Consider the limit of a thin rod.

Solution. (1) We calculate tensor of inertia with respect to the center of mass that is licated in the geometrical center. Tensor of inertia is diaginal and all its diagonal elements are the same, $I_{xx} = I_{yy} = I_{zz} = I$. Let us calculate, for instance, I as I_{zz}. Introducing the surface mass density

$$\sigma \equiv \frac{M}{S} = \frac{M}{4\pi R^2}, \tag{224}$$

one defines $I = I_{zz}$ as

$$I = \sigma\int_S dS\left(x^2 + y^2\right) = \sigma R^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi} d\varphi\left(x^2 + y^2\right). \tag{225}$$

In this and all similar cases, it is convenient to use $u = \cos\theta$ as an integration variable. Using $x^2 + y^2 = R^2\sin^2\theta = R^2\left(1 - u^2\right)$, one obtains

$$I = \sigma R^2 2\pi R^2\int_{-1}^1 du\left(1 - u^2\right) = 4\pi R^2\sigma R^2\int_0^1 du\left(1 - u^2\right) = MR^2\left(1 - \frac{1}{3}\right) = \frac{2}{3}MR^2. \tag{226}$$

One can calculate moments of inertia using the symmetry, $I_{xx} = I_{yy} = I_{zz}$ and writing

$$I = \frac{1}{3}\left(I_{xx} + I_{yy} + I_{zz}\right) = \frac{1}{3}\sigma\int_S dS\left(y^2 + z^2 + z^2 + x^2 + x^2 + y^2\right) = \frac{2}{3}\sigma R^2\int_S dS = \frac{2}{3}MR^2. \tag{227}$$

In this calculation one can replace integration by summation, then it becomes suitable for a non-calculus-based physics course.

(2) Let us first calculate moments of inertia of the cone with respect to its apex and then obtain those with respect to CM with the help of Steiner theorem. The volume of the cone is

$$V = \int_0^h dz S(z) = \int_0^h dz \pi r^2(z) = \int_0^h dz \pi \left(R\frac{z}{h}\right)^2 = \frac{1}{3}\pi R^2 h. \tag{228}$$

Introducing the mass density

$$\rho \equiv \frac{M}{V}, \tag{229}$$

one proceeds

$$I_{zz} = \rho \int_V dV \left(x^2 + y^2\right) = \rho \int_0^h dz \int_0^{Rz/h} 2\pi r dr\, r^2 = \rho \int_0^h dz \frac{\pi}{2} \left(R\frac{z}{h}\right)^4 = \frac{1}{10}\rho\pi R^4 h = \frac{3}{10} M R^2. \tag{230}$$

Similarly

$$I'_{xx} = \rho \int_V dV \left(y^2 + z^2\right), \qquad I'_{yy} = \rho \int_V dV \left(z^2 + x^2\right). \tag{231}$$

Since, by symmetry, $I'_{xx} = I'_{yy} \equiv I_\perp$, one can simplify the calculation,

$$I_\perp = \frac{1}{2}\left(I'_{xx} + I'_{yy}\right) = \frac{1}{2}\rho \int_V dV \left(x^2 + y^2 + 2z^2\right) = \frac{1}{2} I_{zz} + I_\perp", \tag{232}$$

where

$$I_\perp" \equiv \rho \int_V dV z^2 = \rho \int_0^h dz \pi \left(R\frac{z}{h}\right)^2 z^2 = \frac{1}{5}\rho\pi R^2 h^3 = \frac{3}{5} M h^2. \tag{233}$$

Thus

$$I'_\perp = \frac{3}{20} M R^2 + \frac{3}{5} M h^2. \tag{234}$$

According to the Steiner theorem, transverse moment of inertia with respect to CM is given by

$$I_\perp = I'_\perp - M a^2, \tag{235}$$

where a is the distance from the apex to CM. The latter can be calculated as

$$a = \frac{1}{M}\rho \int_V dV z = \frac{1}{M}\rho \int_0^h dz \pi \left(R\frac{z}{h}\right)^2 z = \frac{1}{M}\rho \frac{1}{4}\pi R^2 h^2 = \frac{3}{4} h. \tag{236}$$

Using this, one obtains

$$I_\perp = \frac{3}{20} M R^2 + \frac{3}{80} M h^2. \tag{237}$$

I_{zz} with respect to CM is the same.

(3) Calculate moments of inertia with respect to CM.

$$I_{zz} = \frac{M}{abc} \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dx \left(x^2 + y^2\right) = \frac{M}{abc}\left(\frac{1}{3}a^3 b + \frac{1}{3}ab^3\right) c = \frac{1}{3} M \left(a^2 + b^2\right). \tag{238}$$

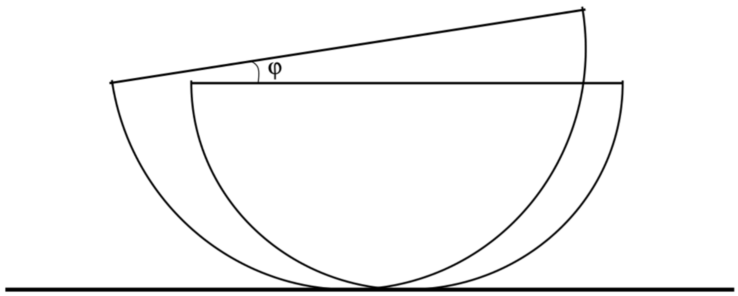
Similarly one obtains

$$I_{xx} = \frac{1}{3} M \left(b^2 + c^2 \right), \qquad I_{yy} = \frac{1}{3} M \left(c^2 + a^2 \right). \tag{239}$$

The limit of a thin rod can be obtained from this by setting $a = b = 0$ and $c = L$. This yields

$$I_{xx} = I_{yy} = \frac{1}{3} M L^2, \qquad I_{zz} = 0. \tag{240}$$

6.2 Rolling half-cylinder



Consider a half-cylinder of mass M and radius R on a horizontal plane.

1. Find the position of its center of mass (CM) and the moment of inertia with respect to CM.
2. Write down the Lagrange function in terms of the angle φ (see Figure)
3. Find the frequency of cylinder's oscillations in the linear regime, $\varphi \ll 1$.

Solution. (1) Position of CM with respect to the center of the cylinder is given by the integral over cross-section

$$a = \frac{1}{M} \frac{2M}{\pi R^2} \int_S y dS = \frac{2}{\pi R^2} \int_0^\pi d\phi \int_0^R r dr \, r \sin \phi = \frac{2}{\pi R^2} 2 \frac{R^3}{3} = \frac{4}{3\pi} R. \tag{241}$$

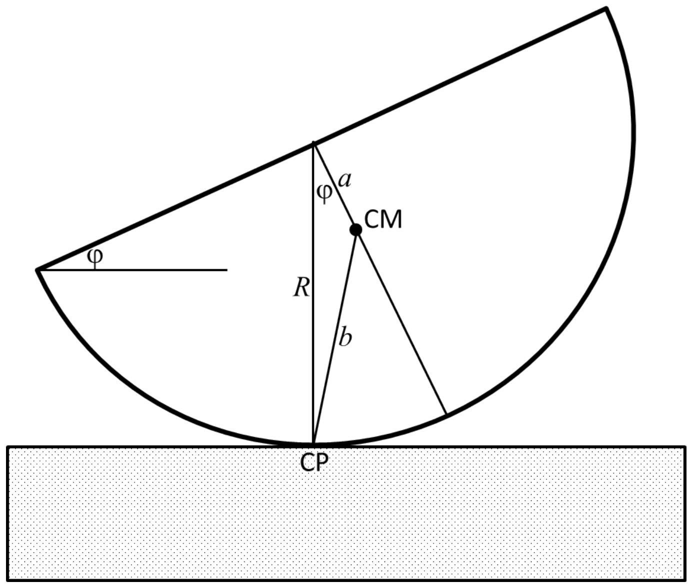
The moment of inertia of the half-cylinder with respect to the geometrical center of the cylinder is, obviously, the same as that of the cylinder,

$$I' \equiv I'_{zz} = \frac{1}{2} M R^2. \tag{242}$$

The moment of inertia with respect to CM can be obtained with the help of the Steiner theorem,

$$I = I' - M a^2 = \left(\frac{1}{2} - \frac{16}{9\pi^2} \right) M R^2 \approx 0.32 M R^2. \tag{243}$$

(2)



The easiest way to write down the kinetic energy is to consider rotation around the instantaneous axis going through the contact point (CP). The moment of inertia with respect to CP I" is given by the Steiner theorem,

$$I" = I + Mb^2 = I + M\left(R^2 + a^2 - 2Ra\cos\varphi\right), \tag{244}$$

where the cosine rule was used. Substituting a, one obtains

$$I" = \left(\frac{3}{2} - \frac{8}{3\pi}\cos\varphi\right) MR^2. \tag{245}$$

Potential energy is similar to that of the pendulum, $U = -Mga\cos\varphi$. Finally the Lagrangian becomes

$$\mathcal{L} = E_k - U = \frac{1}{2} I" \dot{\varphi}^2 + Mga\cos\varphi \tag{246}$$

that becomes

$$\mathcal{L} = \frac{1}{2}\left(\frac{3}{2} - \frac{8}{3\pi}\cos\varphi\right) MR^2\dot{\varphi}^2 + \frac{4}{3\pi} MgR\cos\varphi. \tag{247}$$

The system is similar to the pendulum but the coefficient in the kinetic energy is variable.

(3) In the case of small oscillations around equilibrium one can replace $\cos\varphi \Rightarrow 1$ in the kinetic energy, since there is already a small quantity $\dot{\varphi}^2$. After this the problem becomes mathematically equivalent to the pendulum. One obtains

$$\mathcal{L} = \frac{1}{2}\left(\frac{3}{2} - \frac{8}{3\pi}\right) MR^2\dot{\varphi}^2 + \frac{4}{3\pi} MgR\left(1 - \frac{1}{2}\varphi^2\right), \tag{248}$$

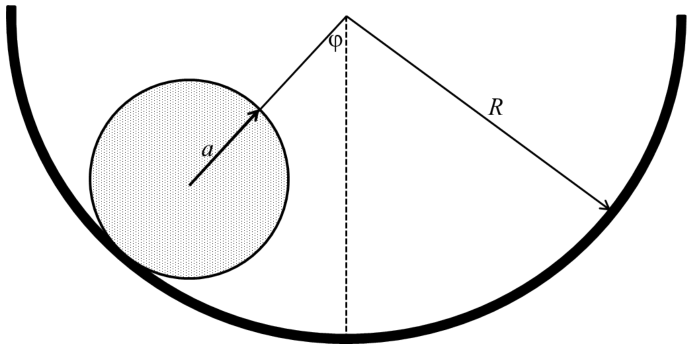
equation of motion becomes

$$\left(\frac{3}{2} - \frac{8}{3\pi}\right) MR^2\ddot{\varphi} + \frac{4}{3\pi} MgR\varphi = 0, \tag{249}$$

and the oscillation frequency is given by

$$\omega_0^2 = \frac{\frac{4}{3\pi}}{\frac{3}{2} - \frac{8}{3\pi}} \frac{g}{R} = \frac{1/2}{9\pi/16 - 1} \frac{g}{R}. \tag{250}$$

6.3 Cylinder rolling inside a cylinder



A cylinder of mass M and radius a is rolling without slipping inside a stationary cylinder of radius $R > a$. Find its kinetic energy.

Solution. The speed of CM of the cylinder is

$$V = (R - a)\,\dot{\varphi}. \tag{251}$$

From the no-slipping condition $V = a\omega$ one obtains the angular velocity of the cylinder

$$\omega = \frac{R - a}{a}\dot{\varphi}. \tag{252}$$

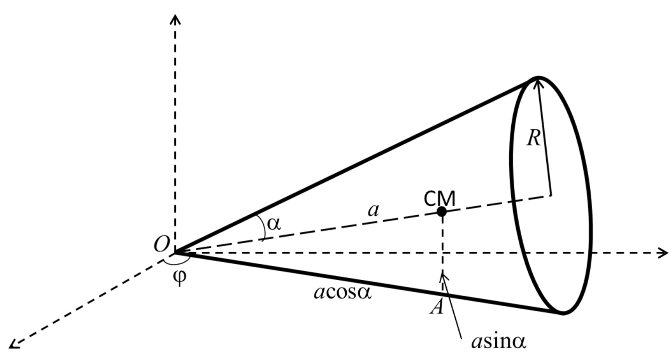
Now the kinetic energy can be written as the sum of the kinetic energy of CM and rotation around CM as

$$E_k = \frac{MV^2}{2} + \frac{I\omega^2}{2} = \frac{1}{2}\left(M + \frac{I}{a^2}\right)(R - a)^2\,\dot{\varphi}^2. \tag{253}$$

Substituting $I = MR^2/2$ for the cylinder, one obtains

$$E_k = \frac{3}{4}M\,(R - a)^2\,\dot{\varphi}^2. \tag{254}$$

6.4 Cone rolling on a plane



Find kinetic energy of a cone rolling on a plane without slipping. The height of the cone is h, the apex angle is 2α. The mass is M.

Solution. The cone is rolling in such a way that its apex is stationary. We chose a frame such that the apex is at its origin O. The orientation of the cone is described by the angle φ that the contact line OA makes with one of the horizontal axes. CM of the cone is at the distance a from the apex. At any moment of time the cone is rotating around the instantaneous axis OA. The angular velocity can be found via the speed of CM V as

$$\omega = \frac{V}{a\sin\alpha} = \frac{a\cos\alpha\dot{\varphi}}{a\sin\alpha} = \cot\alpha\dot{\varphi}. \tag{255}$$

Here a canceled, thus one could use any point on the cone's symmetry axis instead of CM. Kinetic energy can be found by projecting $\boldsymbol{\omega}$ onto the 3 and perpendicular axes of the cone

$$E_k = \frac{1}{2} I_3 (\omega \cos\alpha)^2 + \frac{1}{2} I'_\perp (\omega \sin\alpha)^2 . \tag{256}$$

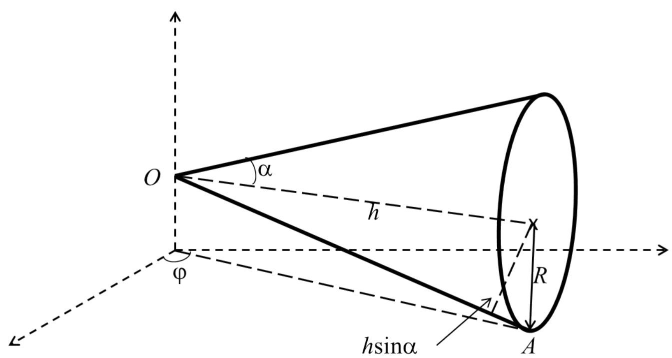
Using moments of inertia with respect to the apex from the problem above,

$$I_3 = \frac{3}{10} MR^2 = \frac{3}{10} Mh^2 \tan^2\alpha, \qquad I'_\perp = \frac{3}{20} MR^2 + \frac{3}{5} Mh^2 = \frac{3}{20} Mh^2 \tan^2\alpha + \frac{3}{5} Mh^2, \tag{257}$$

one obtains

$$E_k = \frac{1}{2} Mh^2 \left[\frac{3}{10} \cos^2\alpha + \frac{3}{20} \sin^2\alpha + \frac{3}{5} \cos^2\alpha \right] \dot{\varphi}^2 = \frac{3}{40} Mh^2 \left(1 + 5\cos^2\alpha\right) \dot{\varphi}^2. \tag{258}$$

6.5 Rolling cone with a horizontal axis



Find kinetic energy of a cone rolling on a plane and having the symmetry axis horizontal (see Figure).

Solution. In this case the instantaneous axis of rotation is the line OA. To find ω, one can use, say, the center of the cone's base. The speed of this point is $V = h\dot{\varphi}$, so that

$$\omega = \frac{V}{h \sin\alpha} = \frac{h\dot{\varphi}}{h \sin\alpha} = \frac{\dot{\varphi}}{\sin\alpha}. \tag{259}$$

Kinetic energy is given by

$$E_k = \frac{1}{2} I_3 (\omega \cos\alpha)^2 + \frac{1}{2} I'_\perp (\omega \sin\alpha)^2 . \tag{260}$$

Using moments of inertia with respect to the apex from the problem above,

$$I_3 = \frac{3}{10} MR^2 = \frac{3}{10} Mh^2 \tan^2\alpha, \qquad I'_\perp = \frac{3}{20} MR^2 + \frac{3}{5} Mh^2 = \frac{3}{20} Mh^2 \tan^2\alpha + \frac{3}{5} Mh^2, \tag{261}$$

one obtains

$$E_k = \frac{1}{2} Mh^2 \left[\frac{3}{10} + \frac{3}{20} \tan^2\alpha + \frac{3}{5} \right] \dot{\varphi}^2 = \frac{3}{40} Mh^2 \left(\frac{1}{\cos^2\alpha} + 5 \right) \dot{\varphi}^2. \tag{262}$$

7 Rotational dynamics

7.1 Rod on the axis

Thin rod of length l and mass M is mounted on an axis at its center.

a) If the angle θ between the rod and the axis is fixed and the rod rotates with the angular velocity $\omega = \dot{\varphi}$ around the axis, what is the (i) kinetic energy of the rod; (ii) breaking torque acting from the rod on the axis?

b) Set up Lagrange equations for the rod in the case where both θ and φ can freely change. Find integrals of motion and the effective potential energy. If you have access to mathematical software, you can try to produce numerical solutions with particular initial conditions such as $\theta(0) = \theta_0$, $\dot\theta(0) = 0$, $\varphi(0) = 0$, $\dot\varphi(0) = \omega_0$.

c) Consider the motion of this system confined to the vicinity of $\theta = \pi/2$ and try to integrate Lagrange equations analytically.

Solution.

a) i) Introducing the linear mass density $\gamma \equiv M/l$, for the kinetic energy one obtains

$$E_k = \frac{\gamma}{2}\int_{l/2}^{l/2} du\,[\boldsymbol{\omega}\times\mathbf{r}]^2 = \frac{\gamma\omega^2\sin^2\theta}{2}\int_{l/2}^{l/2} du\,u^2 = \frac{\gamma\omega^2 l^3\sin^2\theta}{24} = \frac{M\omega^2 l^2\sin^2\theta}{24}. \tag{263}$$

This can be rewritten in the form

$$E_k = \frac{I_\theta\omega^2}{2}, \qquad I_\theta = \frac{Ml^2\sin^2\theta}{12}. \tag{264}$$

(ii) To calculate the breaking torque, one has first to work out the angular momentum $\mathbf{L}$ that lies in the plane spanned by $\boldsymbol{\omega}$ (z axis) and the axis of the rod and perpendicular to the latter:

$$\mathbf{L} = \gamma\int_{l/2}^{l/2} du\,[\mathbf{r}\times[\boldsymbol{\omega}\times\mathbf{r}]] = \gamma\int_{l/2}^{l/2} du\left\{\boldsymbol{\omega}r^2 - \mathbf{r}\,(\mathbf{r}\cdot\boldsymbol{\omega})\right\} = \gamma\omega\mathbf{e}_2\sin\theta\int_{l/2}^{l/2} du\,u^2 = I\omega\sin\theta\mathbf{e}_2, \tag{265}$$

where we use sliding embedded vectors, $\mathbf{e}_2 = \mathbf{e}_A$. As the rod is precessing around z axis, $\mathbf{e}_2$ is precessing as well. Thus the Newtonian equation for $\mathbf{L}$ has the form

$$\dot{\mathbf{L}} = I\omega\sin\theta\,[\boldsymbol{\omega}\times\mathbf{e}_2] = I\omega^2\sin\theta\cos\theta\mathbf{e}_1 = \boldsymbol{\tau}. \tag{266}$$

This yields the breaking torque $\boldsymbol{\tau}$, acting between the rod and the axis. τ reaches its maximum for $\theta = \pi/4$.

b) In the case when both θ and φ are free to change, the kinetic energy of the rod can be written as

$$E_k = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2. \tag{267}$$

With the sliding embedded vectors and $I_2 = I_1 = I$ one obtains

$$\mathcal{L} = E_k = \frac{1}{2}I\left(\dot\theta^2 + \sin^2\theta\dot\varphi^2\right). \tag{268}$$

Here φ is cyclic variable, and the corresponding integral of motion is the projection of the angular momentum on z axis,

$$l_z \equiv l = \frac{\partial\mathcal{L}}{\partial\dot\varphi} = I\sin^2\theta\dot\varphi. \tag{269}$$

Lagrange equation for θ reads

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} - \frac{\partial\mathcal{L}}{\partial\theta} = I\ddot\theta - I\sin\theta\cos\theta\dot\varphi^2 = 0. \tag{270}$$

With the account of angular-momentum conservation, one obtains the equation

$$I\ddot\theta - \frac{\cos\theta}{\sin^3\theta}\frac{l}{I}^2 = 0. \tag{271}$$

This can be written in the form

$$I\ddot{\theta} = -\frac{\partial U_{\rm eff}}{\partial \theta}, \qquad U_{\rm eff} = \frac{l}{2I}^2 \frac{1}{\sin^2\theta}. \tag{272}$$

$U_{\rm eff}(\theta)$ has a minimum at $\theta = \pi/2$.

c) Near the minimum of the effective potential energy, one can use

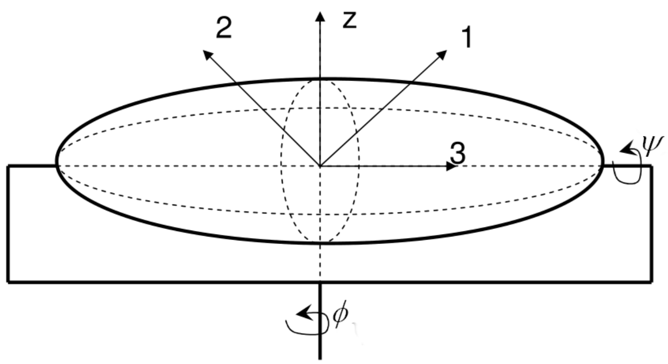
$$\theta = \frac{\pi}{2} + \delta\theta, \qquad \delta\theta \ll 1. \tag{273}$$

With $\sin\theta \cong 1$ and $\cos\theta \cong -\delta\theta$ the equation of motion becomes

$$\delta\ddot{\theta} + \omega_0^2 \delta\theta = 0, \qquad \omega_0 = \frac{l}{I}. \tag{274}$$

This is the equation of motion of a harmonic oscillator.

7.2 Asymmetric top with a $\theta = 0$ holder



Consider an asymmetric top with moments of inertia $I_1 < I_2$ supported by a holder that allows the top to freely rotate changing its Euler angles ϕ and ψ while preserving $\theta = \pi/2$, see figure. The axes of the holder cross at the center of mass of the top.

a) Set up Lagrange equations for this top, find integrals of motion;

b) Eliminate ϕ to obtain an effective energy for ψ. What kinds of motion for ψ are possible? Analyze the behavior of ψ near the minimum of the effective potential energy.

c) If you have access to mathematical software, you can try to produce numerical solutions with particular initial conditions.

Solution.

a) Angular velocity is due to the two types of rotational motion,

$$\boldsymbol{\omega} = \mathbf{e}_z\dot{\phi} + \mathbf{e}_3\dot{\psi}. \tag{275}$$

Projecting $\mathbf{e}_z$ onto the principal axes $\mathbf{e}_1$ and $\mathbf{e}_2$, one obtains kinetic energy and thus the Lagrangian:

$$\mathcal{L} = E_k = E = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}\left[I_1\sin^2\psi + I_2\cos^2\psi\right]\dot{\phi}^2 + \frac{1}{2}I_3\dot{\psi}^2. \tag{276}$$

At $\psi = 0$ one has $\mathbf{e}_2 = \mathbf{e}_z$. Lagrange equations have the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left\{\left[I_1\sin^2\psi + I_2\cos^2\psi\right]\dot{\phi}\right\} = 0 \tag{277}$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\psi}} - \frac{\partial \mathcal{L}}{\partial \psi} = I_3\ddot{\psi} - (I_2 - I_1)\sin\psi\cos\psi\dot{\phi}^2 = 0 \tag{278}$$

b) Here ϕ is cyclic variable and the corresponding generalized momentum is conserved. This is nothing else than vertical component of the angular momentum:

$$l_z \equiv l = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \left[I_1 \sin^2 \psi + I_2 \cos^2 \psi\right] \dot{\phi} = \text{const.} \tag{279}$$

Eliminating $\dot{\phi}$, one can rewrite energy as

$$E = \frac{1}{2} I_3 \dot{\psi}^2 + U_{\text{eff}}(\psi), \tag{280}$$

where effective energy is given by

$$U_{\text{eff}}(\psi) \equiv \frac{l^2}{2\left[I_1 \sin^2 \psi + I_2 \cos^2 \psi\right]} = \frac{l^2}{2\left[I_2 - (I_2 - I_1)\sin^2 \psi\right]} \tag{281}$$

Motion of the system now can be defined as in the case of one-dimensional motion – using energy conservation, resolving for $\dot{\psi}$ and then integrating. For $I_1 < I_2$ the minimum of effective potential energy is at $\psi = 0$, that corresponds to rotating around axis 2 having the greater moment of inertia. Near $\psi = 0$ one has

$$U_{\text{eff}}(\psi) \cong \frac{l^2}{2I_2}\left[1 + \left(1 - \frac{I_1}{I_2}\right)\sin^2 \psi\right] \cong \frac{l^2}{2I_2}\left(1 - \frac{I_1}{I_2}\right)\psi^2 + \text{const.} \tag{282}$$

Using the effective Newtonian equation

$$I_3 \ddot{\psi} + \frac{\partial U_{\text{eff}}(\psi)}{\partial \psi} = 0 \tag{283}$$

(that is equivalent to the second Lagrange equation), one obtains the harmonic-oscillator equation

$$\ddot{\psi} + \omega_0^2 \psi = 0, \qquad \omega_0 = \frac{l}{\sqrt{I_2 I_3}}\sqrt{1 - \frac{I_1}{I_2}}. \tag{284}$$

Part V
Hamiltonian dynamics

8 Hamiltonian equations

8.1 Double pendulum

Consider a double pendulum that consists of two massless rods of lengths $l_1 = l_2 = l$ with masses $m_1 = m_2 = m$ attached to their ends. The first pendulum is attached to a fixed point and can freely swing about it. The second pendulum is attached to the end of the first one and can freely swing, too. The motion of both pendulums is confined to a plane, so that it can be described in terms of their angles with respect to the vertical, $\theta_1$and θ_2.

a) Write down the Lagrange function for this system.

b) Introduce generalized momenta p_1 and p_2 and change to Hamiltonian description. Find the transformation matrix that yields the velocities $\theta_1$and θ_2 in terms of the momenta p_1 and p_2. Write down Hamiltonian function $\mathcal{H}(\theta_1, p_1, \theta_2, p_2)$ using the transformation matrix.

c) Obtain Hamiltonian equations.

Solution.

a) Describing the pendula by the angles θ_1 and θ_2, one writes

$$\begin{aligned} x_1 &= l_1 \sin\theta_1, \qquad x_2 = l\sin\theta_1 + l\sin\theta_2 \\ y_1 &= -l\cos\theta_1, \qquad y_2 = -l\cos\theta_1 - l\cos\theta_2. \end{aligned} \tag{285}$$

Potential energy reads

$$U = mg\left(y_1 + y_2\right) = -mgl\left(2\cos\theta_2 + \cos\theta_1\right) \tag{286}$$

and the kinetic energy is given by

$$\begin{aligned} E_k &= \frac{m}{2}\left[\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right] = \frac{ml^2}{2}\left[\dot{\theta}_1^2 + \left(\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2\right)^2 + \left(\sin\theta_1\dot{\theta}_1 + \sin\theta_2\dot{\theta}_2\right)^2\right] \\ &= \frac{ml^2}{2}\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\dot{\theta}_1\dot{\theta}_2\right] \\ &= \frac{ml^2}{2}\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos\left(\theta_1 - \theta_2\right)\dot{\theta}_1\dot{\theta}_2\right]. \end{aligned} \tag{287}$$

This kinetic energy can be written in the matrix form

$$E_k = \frac{1}{2}\boldsymbol{\theta}^T \cdot \mathbb{A} \cdot \dot{\boldsymbol{\theta}}, \tag{288}$$

where $\boldsymbol{\theta}^T = (\theta_1, \theta_2)$ and

$$\mathbb{A} = ml^2 \begin{pmatrix} 2 & \cos\left(\theta_1 - \theta_2\right) \\ \cos\left(\theta_1 - \theta_2\right) & 1 \end{pmatrix}. \tag{289}$$

Lagrange function reads

$$\mathcal{L} = E_k - U = \frac{ml^2}{2}\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos\left(\theta_1 - \theta_2\right)\dot{\theta}_1\dot{\theta}_2\right] + mgl\left(2\cos\theta_2 + \cos\theta_1\right). \tag{290}$$

b) Generalized momenta has the form

$$\begin{aligned} p_1 &= \frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = ml^2\left[2\dot{\theta}_1 + \cos\left(\theta_1 - \theta_2\right)\dot{\theta}_2\right] \\ p_2 &= \frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = ml^2\left[\dot{\theta}_2 + \cos\left(\theta_1 - \theta_2\right)\dot{\theta}_1\right]. \end{aligned} \tag{291}$$

This can be written in the matrix form as $\mathbf{p} = \mathbb{A} \cdot \dot{\boldsymbol{\theta}}$. Resolving these equations for $\dot{\theta}_1$ and $\dot{\theta}_2$, one obtains

$$\begin{aligned} \dot{\theta}_1 &= \frac{1}{ml^2}\frac{p_1 - \cos\left(\theta_1 - \theta_2\right)p_2}{2 - \cos^2\left(\theta_1 - \theta_2\right)} \\ \dot{\theta}_2 &= \frac{1}{ml^2}\frac{2p_2 - \cos\left(\theta_1 - \theta_2\right)p_1}{2 - \cos^2\left(\theta_1 - \theta_2\right)}. \end{aligned} \tag{292}$$

This can be written in the matrix form $\dot{\boldsymbol{\theta}} = \mathbb{A}^{-1} \cdot \mathbf{p}$, where

$$\mathbb{A}^{-1} = \frac{1}{ml^2}\frac{1}{2 - \cos^2\left(\theta_1 - \theta_2\right)}\begin{pmatrix} 1 & -\cos\left(\theta_1 - \theta_2\right) \\ -\cos\left(\theta_1 - \theta_2\right) & 2 \end{pmatrix}. \tag{293}$$

Now Hamilton function becomes

$$\mathcal{H} = \frac{1}{2}\mathbf{p}^T \cdot \mathbb{A}^{-1} \cdot \mathbf{p} + U = \frac{1}{2}\mathbf{p}^T \cdot \dot{\boldsymbol{\theta}} + U \tag{294}$$

or, in components,

$$\mathcal{H}=\frac{1}{2ml^2}\frac{p_1^2-2\cos\left(\theta_1-\theta_2\right)p_1p_2+2p_2^2}{2-\cos^2\left(\theta_1-\theta_2\right)}-mgl\left(2\cos\theta_2+\cos\theta_1\right). \tag{295}$$

c) Hamiltonian equations are the following.

$$\dot{\theta}_1 = \frac{\partial\mathcal{H}}{\partial p_1}=\frac{1}{ml^2}\frac{p_1-\cos\left(\theta_1-\theta_2\right)p_2}{2-\cos^2\left(\theta_1-\theta_2\right)} \tag{296}$$

$$\dot{\theta}_2 = \frac{\partial\mathcal{H}}{\partial p_2}=\frac{1}{ml^2}\frac{2p_2-\cos\left(\theta_1-\theta_2\right)p_1}{2-\cos^2\left(\theta_1-\theta_2\right)}. \tag{297}$$

These two equations can be found above. Other Hamiltonian equations are

$$\dot{p}_1 = -\frac{\partial\mathcal{H}}{\partial\theta_1}=-\frac{1}{ml^2}\frac{\sin\left(\theta_1-\theta_2\right)\left\{\left[2+\cos^2\left(\theta_1-\theta_2\right)\right]p_2-2p_1\cos\left(\theta_1-\theta_2\right)\right\}}{\left[2-\cos^2\left(\theta_1-\theta_2\right)\right]^2}-mgl\sin\theta_1$$

$$\dot{p}_2 = -\frac{\partial\mathcal{H}}{\partial\theta_2}=-\frac{1}{ml^2}\frac{\sin\left(\theta_2-\theta_1\right)\left\{\left[2+\cos^2\left(\theta_1-\theta_2\right)\right]p_1-2p_2\cos\left(\theta_1-\theta_2\right)\right\}}{\left[2-\cos^2\left(\theta_1-\theta_2\right)\right]^2}-2mgl\sin\theta_2.$$

8.2 Vortex dynamics via Hamiltonian formalism

Consider equations of motions describing vortices of strength γ_i at positions $r_i=(x_i,y_i)$ in the plane:

$$\dot{x}_i=-\sum_{j\neq i}\gamma_j\frac{y_i-y_j}{\left|\mathbf{r}_i-\mathbf{r}_j\right|^2},\qquad \dot{y}_i=\sum_{j\neq i}\gamma_j\frac{x_i-x_j}{\left|\mathbf{r}_i-\mathbf{r}_j\right|^2}. \tag{298}$$

Consider the Hamiltonian $\mathcal{H}$ and the following modified Poisson brackets:

$$\mathcal{H}=-\frac{1}{2}\sum_{j\neq i}\gamma_i\gamma_j\ln\left|\mathbf{r}_i-\mathbf{r}_j\right|,\qquad \{f,g\}\equiv\sum_i\frac{1}{\gamma_i}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}-\frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right). \tag{299}$$

a) Check that Hamiltonian equations

$$\dot{x}_i=\{x_i,\mathcal{H}\},\qquad \dot{y}_i=\{y_i,\mathcal{H}\} \tag{300}$$

reproduce the equations of motions.
b) Show that the following quantities are conserved:

$$P_x\equiv\sum_i\gamma_iy_i,\qquad P_y\equiv-\sum_i\gamma_ix_i \tag{301}$$

Solution.
a) Work out Poisson brackets:

$$\dot{x}_i = \{x_i,\mathcal{H}\}=\sum_j\frac{1}{\gamma_j}\left(\frac{\partial x_i}{\partial x_j}\frac{\partial\mathcal{H}}{\partial y_j}-\frac{\partial x_i}{\partial y_j}\frac{\partial\mathcal{H}}{\partial x_j}\right)=\frac{1}{\gamma_i}\frac{\partial\mathcal{H}}{\partial y_i} \tag{302}$$

$$\dot{y}_i = \{y_i,\mathcal{H}\}=\sum_j\frac{1}{\gamma_j}\left(\frac{\partial y_i}{\partial x_j}\frac{\partial\mathcal{H}}{\partial y_j}-\frac{\partial y_i}{\partial y_j}\frac{\partial\mathcal{H}}{\partial x_j}\right)=-\frac{1}{\gamma_i}\frac{\partial\mathcal{H}}{\partial x_i}. \tag{303}$$

This is the standard form of Hamiltonian equations, except of the factors $1/\gamma_i$. Next, one differentiates the Hamiltonian, taking into account that there are two occurrences of a summation index coinciding with i:

$j' = i$ and $j = i$. One proceeds as

$$\begin{aligned}
\dot{x}_i &= \frac{1}{\gamma_i}\frac{\partial \mathcal{H}}{\partial y_i} = \frac{1}{\gamma_i}\frac{\partial}{\partial y_i}\left(-\frac{1}{2}\sum_{j\neq j'}\gamma_{j'}\gamma_j \ln\left|\mathbf{r}_{j'} - \mathbf{r}_j\right|\right) \\
&= -\sum_{j\neq i}\gamma_j \frac{\partial}{\partial y_i}\ln\left|\mathbf{r}_i - \mathbf{r}_j\right| = -\sum_{j\neq i}\gamma_j \frac{1}{\left|\mathbf{r}_i - \mathbf{r}_j\right|}\frac{\partial\left|\mathbf{r}_i - \mathbf{r}_j\right|}{\partial y_i} = -\sum_{j\neq i}\gamma_j \frac{y_i - y_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^2} \qquad (304)
\end{aligned}$$

that is the first equation of motion. The second equation of motion can be obtained in a similar way.

b) Check conservation directly:

$$\dot{P}_x = \sum_i \gamma_i \dot{y}_i = \sum_i \gamma_i \sum_{j\neq i}\gamma_j \frac{x_i - x_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^2} = \sum_{i\neq j}\gamma_i\gamma_j \frac{x_i - x_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^2}. \qquad (305)$$

Interchanging summation indices $i \rightleftharpoons j$ changes the sign of this expression:

$$\sum_{i\neq j}\gamma_i\gamma_j \frac{x_i - x_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^2} = \sum_{j\neq i}\gamma_j\gamma_i \frac{x_j - x_i}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^2} = -\sum_{i\neq j}\gamma_i\gamma_j \frac{x_i - x_j}{\left|\mathbf{r}_i - \mathbf{r}_j\right|^2}. \qquad (306)$$

Thus it is zero. In a similar way, one obtains $\dot{P}_y = 0$.

9 Canonical transformations

9.1 A

a) Canonical transformation between the two sets of variables is

$$\begin{aligned}
Q &= \ln(1 + \sqrt{q}\cos p) \qquad (307) \\
P &= 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p. \qquad (308)
\end{aligned}$$

Show directly that this transformation is canonical. Show that

$$F_{pQ}(p, Q) = -\left(e^Q - 1\right)^2 \tan p \qquad (309)$$

is generating function of this transformation.

Solution. Since the transformation does not explicitly depend on time, its canonicity can be checked by the Poisson-brackets criterion. One calculates

$$\begin{aligned}
\{Q, P\} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\
&= \frac{1}{1 + \sqrt{q}\cos p}\frac{2\cos p}{2\sqrt{q}}\left[(1 + \sqrt{q}\cos p)\sqrt{q}\cos p - q\sin^2 p\right] \\
&- \frac{-2\sqrt{q}\sin p}{1 + \sqrt{q}\cos p}\left[\frac{\cos p}{2\sqrt{q}}\sqrt{q}\sin p + (1 + \sqrt{q}\cos p)\frac{\sin p}{2\sqrt{q}}\right] \\
&= \frac{1}{1 + \sqrt{q}\cos p}\frac{1}{\sqrt{q}}\left[\sqrt{q}\cos^2 p + q\cos p\left(\cos^2 p - \sin^2 p\right) + 2q\sin^2 p\cos p + \sqrt{q}\sin^2 p\right] \\
&= \frac{1}{1 + \sqrt{q}\cos p}\frac{1}{\sqrt{q}}\left[\sqrt{q} + q\cos p\right] = 1. \qquad (310)
\end{aligned}$$

Thus, the transformation is canonical. Next, the proposed generating function is obtained from the primary generating function $F_{qQ}(q, Q)$ by Legendre transformation

$$F_{pQ} = F_{qQ} - pq. \qquad (311)$$

Its differential reads

$$dF_{pQ} = dF_{qQ} - pdq - qdp = pdq - PdQ - pdq - qdp = -qdp - PdQ. \tag{312}$$

Thus, $F_{pQ} = F_{pQ}(p, Q)$ and our canonical transformation is defined by

$$q = -\frac{\partial F_{pQ}}{\partial p}, \qquad P = -\frac{\partial F_{pQ}}{\partial Q}. \tag{313}$$

Let us work out these formulas. One has

$$q = -\frac{\partial F_{pQ}}{\partial p} = \left(e^Q - 1\right)^2 \frac{1}{\cos^2 p}. \tag{314}$$

Resolving this for Q, one obtains Eq. (307). Next, one obtains

$$P = -\frac{\partial F_{pQ}}{\partial Q} = \frac{\partial}{\partial Q}\left(e^Q - 1\right)^2 \tan p = 2\left(e^Q - 1\right) e^Q \tan p. \tag{315}$$

Inserting Eq. (307) here, one obtains Eq. (308).

9.2 B

For what values of α and β do the equations

$$Q = q^\alpha \cos(\beta p), \qquad P = q^\alpha \sin(\beta p) \tag{316}$$

represent a canonical transformation? What is the form of the generating function $F_{pQ}(p, Q)$ in this case?

Solution. Let us calculate the Poisson bracket,

$$\begin{aligned} \{Q, P\} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\ &= \alpha q^{\alpha-1}\cos(\beta p) q^\alpha \beta \cos(\beta p) + q^\alpha \beta \sin(\beta p) \alpha q^{\alpha-1} \sin(\beta p) \\ &= \alpha\beta q^{2\alpha-1}. \end{aligned} \tag{317}$$

The result is 1 and thus the transformation is canonical for $\alpha = 1/2$ and $\beta = 2$.

Let us now find a generating function for the canonical transformation

$$Q = \sqrt{q}\cos(2p), \qquad P = \sqrt{q}\sin(2p). \tag{318}$$

To use Eq. (313), first one has to express the transformation in terms of p and Q. One obtains

$$Q^2 + P^2 = q, \tag{319}$$

thus

$$P = \sqrt{Q^2 + P^2}\sin(2p). \tag{320}$$

Resolving for P, one obtains

$$P = Q\tan(2p). \tag{321}$$

Now, integrating the second equation of (313) , one obtains

$$F_{pQ} = -\frac{1}{2}Q^2\tan(2p) + f(p). \tag{322}$$

Here $f(p)$ is integration constant. Now the first equation of (313) becomes

$$q = -\frac{\partial F_{pQ}}{\partial p} = Q^2\frac{1}{\cos^2(2p)} + f'(p). \tag{323}$$

Setting $f(p) = 0$ and resolving for Q, one obtains the first equation in (318). Thus Eq. (322) with $f(p) = 0$ is the generating function of our transformation.

10 Hamilton-Jacobi equation and separation of variables

10.1 Coulomb + uniform field via parabolic coordinates

Find the complete integral of Hamilton-Jacobi equations for the problem with potential energy

$$U = \frac{\alpha}{r} + Fz. \tag{324}$$

a) Write Hamilton-Jacobi equation in spherical coordinates and check whether variables separate.
b) Write Hamilton-Jacobi equation in cylindrical coordinates and check whether variables separate.
c) Use parabolic coordinates (ξ, η, φ) that are related to cylindrical coordinates (ρ, z, φ) as

$$z = \frac{1}{2}(\xi - \eta), \qquad \rho = \sqrt{\xi\eta}. \tag{325}$$

Show that variables separate and find the complete integral of Hamilton-Jacobi equations.

Solution.
a) In spherical coordinates the Hamiltonian has the form

$$\mathcal{H} = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + \frac{\alpha}{r} + Fr\cos\theta. \tag{326}$$

Solution of Hamilton-Jacobi equation

$$\frac{\partial \mathcal{S}}{\partial t} + \mathcal{H}\left(q, \frac{\partial \mathcal{S}}{\partial q}\right) = 0 \tag{327}$$

for this time-independent problem can be searched for in the form

$$\mathcal{S} = -Et + \mathcal{S}_0(q) \tag{328}$$

that yields

$$-E + \frac{1}{2m}\left[\left(\frac{\partial \mathcal{S}_0}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathcal{S}_0}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial \mathcal{S}_0}{\partial \varphi}\right)^2\right] + \frac{\alpha}{r} + Fr\cos\theta = 0. \tag{329}$$

Here φ is cyclic variable, thus the solution has the form

$$\mathcal{S}_0 = \mathcal{S}_0^{(r,\theta)} + l\varphi, \tag{330}$$

where l is the constant z component of the angular momentum. HJ equation for the remaining function $\mathcal{S}_0^{(r,\theta)}$ takes the form

$$-E + \frac{1}{2m}\left[\left(\frac{\partial \mathcal{S}_0^{(r,\theta)}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathcal{S}_0^{(r,\theta)}}{\partial \theta}\right)^2 + \frac{l^2}{r^2\sin^2\theta}\right] + \frac{\alpha}{r} + Fr\cos\theta = 0. \tag{331}$$

Here, because of the last term, one cannot split a fragment of this equation that depends only on θ (also only on r). Thus variables do not separate.
b) Cylindrical coordinates are defined as

$$x = \rho\cos\varphi, \qquad y = \rho\sin\varphi, \qquad z = z. \tag{332}$$

Kinetic energy has the form

$$E_k = \frac{m}{2}\left(\dot\rho^2 + \rho^2\dot\varphi^2 + \dot z^2\right). \tag{333}$$

Momenta are defined by

$$p_\rho = \frac{\partial E_k}{\partial \dot{\rho}} = m\dot{\rho}, \qquad p_\varphi = \frac{\partial E_k}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi}, \qquad p_z = m\dot{z}. \tag{334}$$

Thus the Hamiltonian has the form

$$\mathcal{H} = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) + \frac{\alpha}{\sqrt{\rho^2+z^2}} + Fz. \tag{335}$$

Here, again, φ is cyclic variable and the solution of HJ equation can be searched in the form

$$\mathcal{S} = -Et + l\varphi + \mathcal{S}_0^{(\rho,z)} \tag{336}$$

that yields the short HJ equation

$$-E + \frac{1}{2m}\left[\left(\frac{\partial \mathcal{S}_0^{(\rho,z)}}{\partial \rho}\right)^2 + \left(\frac{\partial \mathcal{S}_0^{(\rho,z)}}{\partial z}\right)^2 + \frac{l^2}{\rho^2}\right] + \frac{\alpha}{\sqrt{\rho^2+z^2}} + Fz = 0. \tag{337}$$

Here variables do not separate because of Coulomb-energy term.

c) Now use Eq. (325) and additional relations following from it:

$$r = \sqrt{\rho^2+z^2} = \frac{1}{2}(\xi+\eta) \tag{338}$$

and

$$\xi = r + z, \qquad \eta = r - z. \tag{339}$$

First, calculate $\dot{\rho}^2 + \dot{z}^2$ in Eq. (333):

$$\begin{aligned} \dot{\rho}^2 + \dot{z}^2 &= \left(\frac{1}{2}\sqrt{\frac{\eta}{\xi}}\dot{\xi} + \frac{1}{2}\sqrt{\frac{\xi}{\eta}}\dot{\eta}\right)^2 + \left(\frac{1}{2}\dot{\xi} - \frac{1}{2}\dot{\eta}\right)^2 \\ &= \frac{1}{4}\left(\frac{\eta}{\xi}\dot{\xi}^2 + 2\dot{\xi}\dot{\eta} + \frac{\xi}{\eta}\dot{\eta}^2 + \dot{\xi}^2 - 2\dot{\xi}\dot{\eta} + \dot{\eta}^2\right) = \frac{(\xi+\eta)}{4}\left(\frac{\dot{\xi}^2}{\xi} + \frac{\dot{\eta}^2}{\eta}\right). \end{aligned} \tag{340}$$

Now kinetic energy of Eq. (333) becomes

$$E_k = \frac{m}{2}\left[\frac{(\xi+\eta)}{4}\left(\frac{\dot{\xi}^2}{\xi} + \frac{\dot{\eta}^2}{\eta}\right) + \xi\eta\dot{\varphi}^2\right]. \tag{341}$$

Generalized momenta are given by

$$p_\xi = \frac{\partial E_k}{\partial \xi} = \frac{m}{4}(\xi+\eta)\frac{\dot{\xi}}{\xi}, \qquad p_\eta = \frac{\partial E_k}{\partial \eta} = \frac{m}{4}(\xi+\eta)\frac{\dot{\eta}}{\eta}, \qquad p_\varphi = m\xi\eta\dot{\varphi}. \tag{342}$$

Expressing velocities through momenta, inserting them into kinetic energy and adding potential energy yields the Hamiltonian

$$\mathcal{H} = \frac{2}{m}\frac{\xi p_\xi^2 + \eta p_\eta^2}{\xi+\eta} + \frac{p_\varphi^2}{2m\xi\eta} + U, \tag{343}$$

where

$$U = \frac{2\alpha}{\xi+\eta} + \frac{F}{2}(\xi-\eta) = \frac{4\alpha + F\left(\xi^2-\eta^2\right)}{2(\xi+\eta)}. \tag{344}$$

Now, searching for the action in the form

$$\mathcal{S} = -Et + l\varphi + \mathcal{S}_0^{(\xi,\eta)}, \tag{345}$$

one obtains HJ equation in the form

$$-E + \frac{2}{m(\xi+\eta)}\left[\xi\left(\frac{\partial\mathcal{S}_0^{(\xi,\eta)}}{\partial\xi}\right)^2 + \eta\left(\frac{\partial\mathcal{S}_0^{(\xi,\eta)}}{\partial\eta}\right)^2\right] + \frac{l^2}{2m\xi\eta} + \frac{4\alpha + F\left(\xi^2-\eta^2\right)}{2(\xi+\eta)} = 0. \tag{346}$$

After multiplying by $\xi+\eta$ variables separate:

$$-E(\xi+\eta) + \frac{2}{m}\left[\xi\left(\frac{\partial\mathcal{S}_0^{(\xi,\eta)}}{\partial\xi}\right)^2 + \eta\left(\frac{\partial\mathcal{S}_0^{(\xi,\eta)}}{\partial\eta}\right)^2\right] + \frac{l^2}{2m}\left(\frac{1}{\xi}+\frac{1}{\eta}\right) + \frac{4\alpha + F\left(\xi^2-\eta^2\right)}{2} = 0. \tag{347}$$

Solution of this equation can be searched for in the form

$$\mathcal{S}_0^{(\xi,\eta)}(\xi,\eta) = \mathcal{S}_0^{(\xi)}(\xi) + \mathcal{S}_0^{(\eta)}(\eta). \tag{348}$$

Equating the two parts of HJ equation to constants P and $-P$, one obtains two separate ODE's,

$$-E\xi + \frac{2}{m}\xi\left(\frac{\partial\mathcal{S}_0^{(\xi)}}{\partial\xi}\right)^2 + \frac{l^2}{2m\xi} + \alpha + \frac{F}{2}\xi^2 = P \tag{349}$$

and

$$-E\eta + \frac{2}{m}\eta\left(\frac{\partial\mathcal{S}_0^{(\eta)}}{\partial\eta}\right)^2 + \frac{l^2}{2m\eta} + \alpha + \frac{F}{2}\eta^2 = -P. \tag{350}$$

Integrating these equations, one obtains

$$\mathcal{S}_0^{(\xi)}(\xi) = \sqrt{\frac{m}{2}}\int d\xi\sqrt{E - U_\xi(\xi)}, \qquad U_\xi(\xi) \equiv \frac{\alpha - P}{\xi} + \frac{l^2}{2m\xi^2} + \frac{F}{2}\xi \tag{351}$$

and

$$\mathcal{S}_0^{(\eta)}(\eta) = \sqrt{\frac{m}{2}}\int d\eta\sqrt{E - U_\eta(\eta)}, \qquad U_\eta(\eta) \equiv \frac{\alpha + P}{\eta} + \frac{l^2}{2m\eta^2} + \frac{F}{2}\eta. \tag{352}$$

As the result, we have obtained the complete integral of Hamilton-Jacobi equations having the form

$$\mathcal{S} = \mathcal{S}(\xi,\eta,\varphi;E,l,P;t) = -Et + l\varphi + \mathcal{S}_0^{(\xi)}(\xi;E,l,P) + \mathcal{S}_0^{(\eta)}(\eta;E,l,P). \tag{353}$$

It depends on three generalized coordinates ξ,η,φ and three integration constants E,l,P. Considering $\mathcal{S}$ as generating function of a canonic transformation with E,l,P being new momenta, one defines dynamics of the system implicitly from the three equations

$$Q_E = \frac{\partial\mathcal{S}}{\partial E}, \qquad Q_l = \frac{\partial\mathcal{S}}{\partial l}, \qquad Q_P = \frac{\partial\mathcal{S}}{\partial P}, \tag{354}$$

where Q_E, Q_l, Q_P are another three constants. As the system has three degrees of freedom, its solution is specified by the total six integration constants, as it should be. Constants Q_E and Q_l can be set to zero, because reference points for the time and angle φ are arbitrary. The third equation yields the relation between ξ and η. Together with the second equation, this yields the trajectory without time dependence. The latter is obtained using the first equation. Doing this analytically is hardly possible because of complicated integrals.

10.2 Generalized harmonic oscillator

The motion of a particle in one dimension is described by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left(p^2 + \omega_0^2 q^2\right) + \frac{\lambda}{4}\left(p^2 + \omega_0^2 q^2\right)^2. \tag{355}$$

a) Using the generating function $F(q, Q) = \left(\omega_0 q^2/2\right) \cot Q$ define new canonical variables Q and P and find the transformed Hamiltonian $\mathcal{H}(Q, P)$.

b) Set up Hamilton-Jacobi equation for the Hamiltonian $\mathcal{H}(Q, P)$ and the action $\mathcal{S}(Q, \alpha, t)$, where α is an integration constant.

c) Find $q(t)$ and $p(t)$ using Hamilton-Jacobi method and the function $\mathcal{S}(Q, \alpha, t)$. Use initial conditions $q(0) = q_0$ and $p = 0$.

Solution.

a) Transformation formulas have the form

$$p = \frac{\partial F}{\partial q}, \qquad P = -\frac{\partial F}{\partial Q}. \tag{356}$$

From this one obtains

$$p = \omega_0 q \cot Q, \qquad P = \frac{\omega_0 q^2}{2} \frac{1}{\sin^2 Q}. \tag{357}$$

Resolving the second equation for q and substituting the result into the first equation, one obtains

$$q = \sqrt{\frac{2P}{\omega_0}} \sin Q, \qquad p = \sqrt{2\omega_0 P} \cos Q. \tag{358}$$

Inserting this into the Hamiltonian, one obtains

$$\mathcal{H} = \omega_0 P + \lambda \left(\omega_0 P\right)^2. \tag{359}$$

b) Using the formula

$$d\mathcal{S} = P dQ - \mathcal{H} dt \tag{360}$$

along the trajectory, thus

$$P = \frac{\partial \mathcal{S}}{\partial Q}, \qquad \mathcal{H} = -\frac{\partial \mathcal{S}}{\partial t}, \tag{361}$$

one obtains Hamilton-Jacobi equation

$$\frac{\partial \mathcal{S}}{\partial t} + \mathcal{H}\left(Q, \frac{\partial \mathcal{S}}{\partial Q}\right) = 0. \tag{362}$$

With the transformed Hamiltonian above, this becomes

$$\frac{\partial \mathcal{S}}{\partial t} + \omega_0 \frac{\partial \mathcal{S}}{\partial Q} + \lambda \left(\omega_0 \frac{\partial \mathcal{S}}{\partial Q}\right)^2 = 0. \tag{363}$$

The complete integral of this equation can be searched for in the form

$$\mathcal{S} = -Et + \mathcal{S}_0(Q). \tag{364}$$

For $\mathcal{S}_0(Q)$ one obtains the equation

$$\omega_0 \frac{\partial \mathcal{S}_0}{\partial Q} + \lambda \left(\omega_0 \frac{\partial \mathcal{S}}{\partial Q}\right)^2 = E. \tag{365}$$

Its solution can be searched for in the form

$$\mathcal{S}_0(Q) = \alpha Q, \tag{366}$$

where the irrelevant integration constant has been discarded and constant α can be expressed via the conserved energy E using

$$\omega_0 \alpha + \lambda \left(\omega_0 \alpha\right)^2 = E. \tag{367}$$

However, it is more convenient to work with α than with E. The full action reads

$$\mathcal{S}\left(Q, \alpha, t\right) = -\left[\omega_0 \alpha + \lambda \left(\omega_0 \alpha\right)^2\right] t + \alpha Q. \tag{368}$$

c) Using $\mathcal{S}\left(Q, \alpha, t\right)$ as generating function of another canonical transformation that nullifies the transformed Hamiltonian, one obtains the solution for the dynamics in the form

$$\beta = \frac{\partial \mathcal{S}}{\partial \alpha}, \tag{369}$$

where β is another integration constant. Thus one obtains

$$\beta = -\left[\omega_0 + 2\lambda \omega_0^2 \alpha\right] t + Q \tag{370}$$

and

$$Q = \tilde{\omega}_0 t + \beta, \qquad \tilde{\omega}_0 \equiv \omega_0 + 2\lambda \omega_0^2 \alpha, \tag{371}$$

$\tilde{\omega}_0$ being renormalized frequency. Generalized momentum P is obtained from the transformation formula

$$P = \frac{\partial \mathcal{S}}{\partial Q} = \alpha \tag{372}$$

and it is constant. Now the dynamics of original variables q, p can be obtained as

$$q = \sqrt{\frac{2P}{\omega_0}} \sin Q = \sqrt{\frac{2\alpha}{\omega_0}} \sin\left(\tilde{\omega}_0 t + \beta\right) \tag{373}$$

and

$$p = \sqrt{2\omega_0 P} \cos Q = \sqrt{2\omega_0 \alpha} \cos\left(\tilde{\omega}_0 t + \beta\right). \tag{374}$$