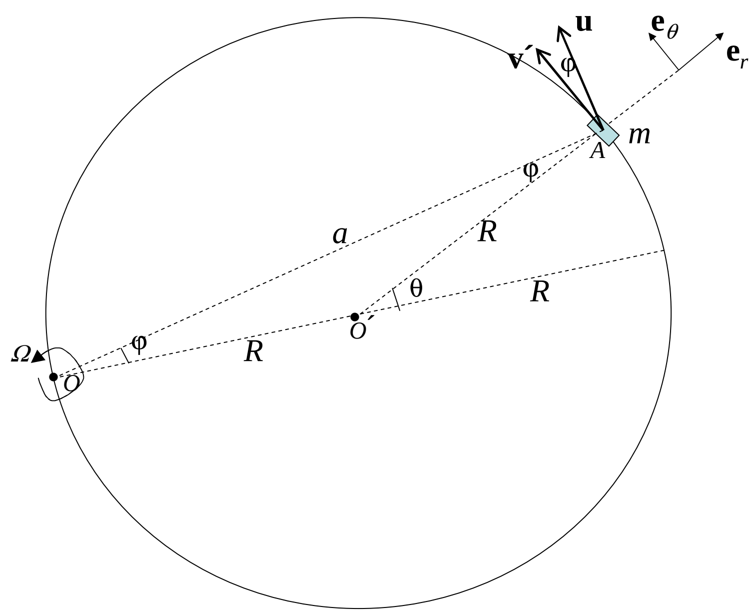
Bead sliding along a rotating ring

A ring of radius R is rotating in its plane with the constant angular velocity Ω around a point O. A bead of mass m can slide along the ring without friction.



Describing the position of the bead on the ring with the angle θ,

a) Construct the Lagrange function and obtain the equation of motion,

b) Find the effective kinetic, potential and total energies

c) Find the force $\mathbf{F}$ acting on the bead.

Solution: **a)** In this problem the potential energy is absent, thus the Lagrange function has the form

$$\mathcal{L} = \frac{m\mathbf{v}^2}{2}, \tag{1}$$

wher $\mathbf{v}$ is the bead's velocity that consists of two contribution, sliding of the bead and rotating of the ring, respectively,

$$\mathbf{v} = \mathbf{v}' + \mathbf{u}. \tag{2}$$

Thus one can write

$$\mathcal{L} = \frac{m}{2}\left(\mathbf{v}' + \mathbf{u}\right)^2 = \frac{m}{2}\left(v'^2 + u^2 + 2\mathbf{v}' \cdot \mathbf{u}\right). \tag{3}$$

Here

$$v' = R\dot{\theta} \tag{4}$$

and, from the triangles,

$$u = a\Omega = 2R\Omega\cos\varphi = 2R\Omega\cos\frac{\theta}{2}. \tag{5}$$

The angle between $\mathbf{v}'$ and $\mathbf{u}$ is also $\varphi = \theta/2$, so that the Lagrange function becomes

$$\begin{aligned}
\mathcal{L} &= \frac{m}{2}\left(v'^2 + u^2 + 2v'u\cos\frac{\theta}{2}\right) \\
&= \frac{mR^2}{2}\left(\dot{\theta}^2 + 4\Omega^2\cos^2\frac{\theta}{2} + 4\Omega\dot{\theta}\cos^2\frac{\theta}{2}\right) \\
&= mR^2\left[\frac{1}{2}\dot{\theta}^2 + \Omega^2\left(1+\cos\theta\right) + \Omega\dot{\theta}\left(1+\cos\theta\right)\right] \\
&\Rightarrow mR^2\left[\frac{1}{2}\dot{\theta}^2 + \Omega^2\left(1+\cos\theta\right)\right]. \qquad (6)
\end{aligned}$$

The last term in the above expression has been dropped since it is a full time derivative

$$\Omega\dot{\theta}\left(1+\cos\theta\right) = \frac{d}{dt}\Omega\left[\theta + \sin\theta\right]$$

that does not make a contribution into the Lagrange equation that can be checked directly. The Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \qquad (7)$$

has the form

$$\ddot{\theta} + \Omega^2\sin\theta = 0, \qquad (8)$$

the equation of motion for the pendulum.
b) Already from the final expression for the Lagrangian, Eq. (6), it is clear that the problem is equivalent to that of a pendulum and the effective kinetic and potential energies are given by

$$T_{\text{eff}} = \frac{1}{2}mR^2\dot{\theta}^2, \qquad U_{\text{eff}} = -mR^2\Omega^2\left(1+\cos\theta\right). \qquad (9)$$

The total effective energy

$$E_{\text{eff}} = T_{\text{eff}} + U_{\text{eff}} = \frac{1}{2}mR^2\dot{\theta}^2 - mR^2\Omega^2\left(1+\cos\theta\right) \qquad (10)$$

is conserved. Note that the true total energy is just $\mathcal{L}$ and it does not conserve.
c) The force $\mathbf{F}$ acting on the bead is the reaction force from the ring. Since the friction is absent, this force is directed radially, there is no component of $\mathbf{F}$ in the direction tangential to the ring. Since $\mathbf{F}$ is a force due to a holonomic constraint, and in the Lagrangian formalism holonomic constraints are eliminated, there is no way to find $\mathbf{F}$ within the Lagrangian formalism. On the other hand, the Newtonean formalism yields

$$\mathbf{F} = m\dot{\mathbf{v}}, \qquad (11)$$

i.e., it is sufficient to calculate the acceleration. It is convenient to project the vectors onto the frame vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ (see Figure). One has thus

$$\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta. \qquad (12)$$

Differentiation yields

$$\dot{\mathbf{v}} = \dot{v}_r\mathbf{e}_r + v_r\dot{\mathbf{e}}_r + \dot{v}_\theta\mathbf{e}_\theta + v_\theta\dot{\mathbf{e}}_\theta. \qquad (13)$$

The time dependences of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are due to the double rotation of the bead, *along* the ring and *with* the ring. One elementarily obtains

$$\dot{\mathbf{e}}_r = \left(\dot{\theta} + \Omega\right)\mathbf{e}_\theta, \qquad \dot{\mathbf{e}}_\theta = -\left(\dot{\theta} + \Omega\right)\mathbf{e}_r. \qquad (14)$$

Thus the acceleration takes the form

$$\mathbf{a} = \dot{\mathbf{v}} = \left[\dot{v}_r - \left(\dot{\theta} + \Omega\right)v_\theta\right]\mathbf{e}_r + \left[\dot{v}_\theta + \left(\dot{\theta} + \Omega\right)v_r\right]\mathbf{e}_\theta. \qquad (15)$$

For the velocity components using Eqs. (4) and (5) one has

$$\begin{aligned} v_r &= u \sin\varphi = 2R\Omega \cos\frac{\theta}{2} \sin\frac{\theta}{2} = R\Omega \sin\theta \\ v_\theta &= v' + u\cos\varphi = R\dot{\theta} + 2R\Omega\cos^2\frac{\theta}{2} = R\left[\dot{\theta} + \Omega\left(1+\cos\theta\right)\right] \end{aligned} \tag{16}$$

and

$$\begin{aligned} \dot{v}_r &= R\Omega\cos\theta\,\dot{\theta} \\ \dot{v}_\theta &= R\left[\ddot{\theta} - \Omega\sin\theta\,\dot{\theta}\right]. \end{aligned} \tag{17}$$

Thus one obtains

$$a_\theta = \dot{v}_\theta + \left(\dot{\theta}+\Omega\right) v_r = R\left[\ddot{\theta} - \Omega\sin\theta\,\dot{\theta} + \left(\dot{\theta}+\Omega\right)\Omega\sin\theta\right] = 0, \tag{18}$$

where Eq. (8) has been used. Now Eq. (11) yields $F_\theta = 0$, as expected. Next one obtains

$$\begin{aligned} a_r &= \dot{v}_r - \left(\dot{\theta}+\Omega\right) v_\theta = R\left[\Omega\cos\theta\,\dot{\theta} - \left(\dot{\theta}+\Omega\right)\left(\dot{\theta}+\Omega\left(1+\cos\theta\right)\right)\right] \\ &= R\left[\Omega\cos\theta\,\dot{\theta} - \left(\dot{\theta}+\Omega\right)^2 - \left(\dot{\theta}+\Omega\right)\Omega\cos\theta\right] \\ &= -R\left[\left(\dot{\theta}+\Omega\right)^2 + \Omega^2\cos\theta\right]. \end{aligned} \tag{19}$$

This yields

$$F_r = -mR\left[\left(\dot{\theta}+\Omega\right)^2 + \Omega^2\cos\theta\right]. \tag{20}$$

For $\theta = \dot{\theta} = 0$ this reduces to $F_r = -m\left(2R\right)\Omega^2$ that is a known expression for the centrifugal force.