Analytical Dynamics - Graduate Center CUNY - Fall 2008

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Problem set 03 Due: Mo, Dec 15

1 Hamiltonian formalism for the double pendulum

(10 points) Consider a double pendulum that consists of two massless rods of the same length l with the same masses m attached to their ends. The first pendulum is attached to a fixed point and can freely swing about it. The second pendulum is attached to the end of the first one and can freely swing, too. The motion of both pendulums is confined to a plane, so that it can be described in terms of their angles with respect to the vertical, φ_1 and φ_2.

(a) Write down the Lagrange function for this system.

(b) Introduce generalized momenta p_1 and p_2 and change to the Hamiltonian description. Find the transformation matrix that yields the velocities φ_1 and φ_2 in terms of the momenta p_1 and p_2: Write down the Hamilton function $\mathrm{H}(\varphi_1; p_1; \varphi_2; p_2)$ using the transformation matrix.

(c) Obtain the Hamilton equations.

2 Canonical transformations

(10 points) a) The canonical transformation between two sets of variables is given by

$$Q = \ln\left(1 + \sqrt{q}\cos p\right), \qquad P = 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p.$$

Show directly that this transformation is canonical. Show that

$$F_{pQ}(p, Q) = -\left(e^Q - 1\right)^2 \tan p$$

is the generating function of this transformation.

b) For what values of α and β the transformation

$$Q = q^\alpha \cos(\beta p), \qquad P = q^\alpha \sin(\beta p)$$

is canonical? What is the form of the generating function $F_{pQ}(p, Q)$ in this case?

3 Hamilton-Jacoby equation

(10 points) The motion of the particle in one dimension is described by the Hamiltonian

$$\mathcal{H}(q, p) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + \lambda\left(\frac{p^2}{2} + \frac{\omega^2 q^2}{2}\right)^2.$$

a) Using the generating function

$$F_{qQ}(q, Q) = \frac{\omega q^2}{2}\cot Q$$

define new canonical variables Q and P and find the transformed Hamiltonian $\mathcal{H}(Q, P)$

b) Set up the Hamilton-Jacoby equation for the Hamiltonian $\mathcal{H}(Q, P)$ and find the Hamilton principal function (action) $S(Q, \alpha, t)$, where $\alpha = const.$

c) Find $q(t)$ and $p(t)$ using the Hamilton-Jacoby method. Use initial conditions $q = \alpha$ and $p = 0$ at $t = 0$.