Analytical Dynamics - Graduate Center CUNY - Fall 2008

Professor Dmitry Garanin

Problem set 03 (with solution for problem 3)

1 Hamiltonian formalism for the double pendulum

(10 points) Consider a double pendulum that consists of two massless rods of the same length l with the same masses m attached to their ends. The first pendulum is attached to a fixed point and can freely swing about it. The second pendulum is attached to the end of the first one and can freely swing, too. The motion of both pendulums is confined to a plane, so that it can be described in terms of their angles with respect to the vertical, φ_1 and φ_2.

(a) Write down the Lagrange function for this system.

(b) Introduce generalized momenta p_1 and p_2 and change to the Hamiltonian description. Find the transformation matrix that yields the velocities φ_1 and φ_2 in terms of the momenta p_1 and p_2: Write down the Hamilton function H(φ_1; p_1; φ_2; p_2) using the transformation matrix.

(c) Obtain the Hamilton equations.

2 Canonical transformations

(10 points) a) The canonical transformation between two sets of variables is given by

$$Q = \ln\left(1 + \sqrt{q}\cos p\right), \qquad P = 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p.$$

Show directly that this transformation is canonical. Show that

$$F_{pQ}(p, Q) = -\left(e^Q - 1\right)^2 \tan p$$

is the generating function of this transformation.

b) For what values of α and β the transformation

$$Q = q^\alpha \cos(\beta p), \qquad P = q^\alpha \sin(\beta p)$$

is canonical? What is the form of the generating function $F_{pQ}(p, Q)$ in this case?

3 Hamilton-Jacoby equation

(10 points) The motion of the particle in one dimension is described by the Hamiltonian

$$\mathcal{H}(q, p) = \frac{p^2}{2} + \frac{\omega^2 q^2}{2} + \lambda\left(\frac{p^2}{2} + \frac{\omega^2 q^2}{2}\right)^2. \tag{1}$$

a) Using the generating function

$$F_{qQ}(q, Q) = \frac{\omega q^2}{2}\cot Q$$

define new canonical variables Q and P and find the transformed Hamiltonian $\mathcal{H}(Q, P)$

Solution: From the above generating function one obtains

$$p = \frac{\partial F}{\partial q} = \omega q \cot Q, \qquad P = -\frac{\partial F}{\partial Q} = \frac{\omega q^2}{2}\frac{1}{\sin^2 Q}. \tag{2}$$

Resolving the second of these equations for q and then substituting the result into the first equation one obtains

$$q = \sqrt{\frac{2P}{\omega}}\sin Q, \qquad p = \sqrt{2P\omega}\cos Q. \tag{3}$$

Substituting this into the Hamiltonian one obtains

$$\mathcal{H}(Q,P) = \omega P + \lambda\omega^2 P^2. \tag{4}$$

One can see that Q is a cyclic variable, thus $P = const$. The Hamiltonian equation for Q has the form

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = \omega + 2\lambda\omega^2 P. \tag{5}$$

The solution of this equation is

$$Q = \omega_{\text{eff}} t + \varphi_0, \qquad \omega_{\text{eff}} \equiv \omega\left(1 + 2\lambda\omega P\right). \tag{6}$$

One can relate P to the conserved energy E of the system.

$$E = \omega P + \lambda\omega^2 P^2$$

thus

$$\omega P = \frac{1}{2\lambda}\left(-1 + \sqrt{1 + 4\lambda E}\right) \tag{7}$$

and

$$\omega_{\text{eff}} = \omega\sqrt{1 + 4\lambda E}. \tag{8}$$

Substituting the results into Eq. (3), one obtains

$$q(t) = q_0 \sin\left(\omega_{\text{eff}} t + \varphi_0\right), \qquad p(t) = p_0 \cos\left(\omega_{\text{eff}} t + \varphi_0\right), \tag{9}$$

where

$$q_0 = \frac{1}{\omega}\sqrt{\frac{1}{\lambda}\left(-1 + \sqrt{1 + 4\lambda E}\right)}, \qquad p_0 = \sqrt{\frac{1}{\lambda}\left(-1 + \sqrt{1 + 4\lambda E}\right)}.$$

One can see that for small energy E, precisely for $\lambda E \ll 1$, the amplitude of the oscillations becomes small and independent of λ,

$$q_0 \cong \frac{1}{\omega}\sqrt{2E}, \qquad p_0 = \sqrt{2E},$$

whereas $\omega_{\text{eff}} \cong \omega$.

b) Set up the Hamilton-Jacobi equation for the Hamiltonian $\mathcal{H}(Q,P)$ and find the Hamilton principal function (action) $\mathcal{S}(Q,\alpha,t)$, where $\alpha = const$.

Solution: With the transformed Hamiltonian of Eq. (4), the Hamilton-Jacobi equation reads

$$\frac{\partial \mathcal{S}}{\partial t} + \omega\frac{\partial \mathcal{S}}{\partial Q} + \lambda\left(\omega\frac{\partial \mathcal{S}}{\partial Q}\right)^2 = 0.$$

The solution can be searched for in the form

$$\mathcal{S}(Q,t) = -Et + S_0(Q).$$

Here $S_0(Q)$ satisfies the equation

$$\omega\frac{\partial \mathcal{S}_0}{\partial Q} + \lambda\left(\omega\frac{\partial \mathcal{S}_0}{\partial Q}\right)^2 = E.$$

One finds

$$\omega\frac{\partial \mathcal{S}_0}{\partial Q} = \frac{1}{2\lambda}\left(-1 + \sqrt{1 + 4\lambda E}\right)$$

and thus

$$\mathcal{S}_0 = \frac{Q}{2\lambda\omega}\left(-1 + \sqrt{1 + 4\lambda E}\right) + \alpha,$$

where α is an integration constant, and further

$$\mathcal{S}(Q,E,t) = -Et + \frac{Q}{2\lambda\omega}\left(-1 + \sqrt{1 + 4\lambda E}\right) + \alpha.$$

Now one can consider E as a new momentum and Q as an old coordinate. Then the equation of motion for Q follows from

$$\beta = \frac{\partial \mathcal{S}}{\partial E} = -t + \frac{Q}{\omega}\frac{1}{\sqrt{1 + 4\lambda E}} = -t + \frac{Q}{\omega_{\text{eff}}},$$

thus Q is given by Eq. (6) with ω_{eff} given by Eq. (8). This further leads to the final solution of the problem, Eq. (9). Of course, this method of the solution is not natural.

c) Find $q(t)$ and $p(t)$ using the Hamilton-Jacoby method. Use initial conditions $q = \alpha$ and $p = 0$ at $t = 0$.

Solution: The Hamilton-Jacobi equation for the original Hamiltonian (1) has the form

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{1}{2}\left(\frac{\partial \mathcal{S}}{\partial q}\right)^2 + \frac{\omega^2 q^2}{2} + \lambda \left[\frac{1}{2}\left(\frac{\partial \mathcal{S}}{\partial q}\right)^2 + \frac{\omega^2 q^2}{2}\right]^2 = 0.$$

We search for the solution in the form

$$\mathcal{S}(q,t) = -Et + S_0(q).$$

Substitution yieds

$$\frac{1}{2}\left(\frac{\partial \mathcal{S}_0}{\partial q}\right)^2 + \frac{\omega^2 q^2}{2} + \lambda \left[\frac{1}{2}\left(\frac{\partial \mathcal{S}_0}{\partial q}\right)^2 + \frac{\omega^2 q^2}{2}\right]^2 = E.$$

Resolving this quadratic equation results in

$$\frac{1}{2}\left(\frac{\partial \mathcal{S}_0}{\partial q}\right)^2 + \frac{\omega^2 q^2}{2} = \frac{1}{2\lambda}\left(-1 + \sqrt{1 + 4\lambda E}\right) \equiv E_{\text{eff}}.$$

Integrating over q, one obtains the action

$$\mathcal{S}\left(q, E, t\right) = \int^q dq' \sqrt{2\left[E_{\text{eff}} - U(q')\right]} - Et \tag{10}$$

with $U(q) = \omega^2 q^2/2$. Using this as the generating function $\Phi\left(q, P, t\right)$ with $P = E$, one obtains the implicit formula for $q(t)$

$$Q = \frac{\partial \mathcal{S}}{\partial E} = \frac{\partial E_{\text{eff}}}{\partial E} \int^q dq' \sqrt{\frac{1}{2\left[E_{\text{eff}} - U(q')\right]}} - t, \tag{11}$$

where

$$\frac{\partial E_{\text{eff}}}{\partial E} = \frac{1}{\sqrt{1 + 4\lambda E}}.$$

For the harmonic oscillator one can calculate the integral analytically as follows

$$Q = \frac{1}{\sqrt{1 + 4\lambda E}} \int^q dq' \sqrt{\frac{1}{2E_{\text{eff}} - \omega^2 q'^2}} - t = \frac{1}{\omega_{\text{eff}}} \int^{\tilde{q}} \frac{d\tilde{q}'}{\sqrt{1 - \tilde{q}'^2}} - t = \frac{1}{\omega_{\text{eff}}} \arcsin \tilde{q} - t, \tag{12}$$

where

$$\tilde{q} \equiv \sqrt{\frac{\omega^2}{2E_{\text{eff}}}} q. \tag{13}$$

Inverting Eq. (12) one obtains the well-known solution

$$\tilde{q} = \sin\left(\omega_{\text{eff}} t + \omega Q\right) = \sin\left(\omega t + \varphi_0\right) \tag{14}$$

or

$$q = \sqrt{\frac{2E_{\text{eff}}}{\omega^2}} \sin\left(\omega_{\text{eff}} t + \varphi_0\right) = \frac{1}{\omega}\sqrt{\frac{1}{\lambda}\left(-1 + \sqrt{1 + 4\lambda E}\right)} \sin\left(\omega_{\text{eff}} t + \varphi_0\right). \tag{15}$$

After that one finds p as

$$p = \frac{\partial \mathcal{S}}{\partial q} = \sqrt{2\left[E_{\text{eff}} - U(q)\right]} \tag{16}$$

that for the harmonic oscillator yields the well-known expression in the form modified for our problem:

$$p = \sqrt{2E_{\text{eff}}}\sqrt{1 - \tilde{q}^2} = \sqrt{2E_{\text{eff}}} \cos\left(\omega_{\text{eff}} t + \varphi_0\right) = \sqrt{\frac{1}{\lambda}\left(-1 + \sqrt{1 + 4\lambda E}\right)} \cos\left(\omega_{\text{eff}} t + \varphi_0\right). \tag{17}$$